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DALLAS. E. H. Jones, Mahoney.
FORT WORTH. Alexander.
GALVESTON. A. A. Bennett, Underwood.
GEORGETOWN. Wunder.
HOUSTON. Daniell, Lovett.
RUSK. P. C. Porter.
SAN ANGELO. Hagelstein.
SAN ANTONIO. Roach.
STEPHENVILLE. J. L. Riley.

UTAH. (2)

LOGAN. Saxer.
SALT LAKE CITY. J. L. Gibson.

VERMONT. (6)

BENNINGTON. Mabrey.
BURLINGTON. Swift, E. Thomas.
ESSEX JUNCTION. Donahue.
MIDDLEBURY. Perkins.
ST. JOHNSBURY. Torrey.

VIRGINIA. (18)

BLACKSBURG. J. E. Williams.
EMORY. J. S. Miller.
CLIFTON STATION. Stone.
HAMPDEN-SIDNEY. J. B. Smith.
HOLLINS. Dickinson.
LEXINGTON. L. W. Smith, Watts.
LYNCHBURG. Larew.
MONTEREY. Colaw.
RICHMOND. Duke.
RICHMOND COLLEGE. Gaines.
SALEM. Carpenter.
SWEET BRIAR. Morenus.
UNIVERSITY. W. H. Echols, Luck.
WILLIAMSBURG. A. Davis, Oglesby.
WOODBURY FOREST. W. L. Lord.

WASHINGTON. (14)

PULLMAN. Hix.
SEATTLE. E. T. Bell, Boothroyd, Gavett, Hays, Neikirk, Moritz, Raynor, Stanwick, Wear.
SPOKANE. Guy.
TACOMA. Hanawalt.
WALLA WALLA. Bratton, Eells.

WEST VIRGINIA. (6)

BETHANY. Balch.
BUCKHANNON. C. E. White.
MORGANTOWN. Eiesland, Hodgson.
WHEELING. Flanagan, Githens.

WISCONSIN. (24)

ASHLAND. Cortes E. Smith.
BELOIT. W. A. Hamilton, Haynes, G. W. Smith.
LA CROSSE. Adkins, Johnston.
MADISON. F. E. Allen, H. T. Burgess, Dowling, Dresden, T. M. Simpson, Slichter, F. Wood.
MILTON. A. E. Whitford.
MILWAUKEE. K. S. Arnold, Boren, Ericson, Frumveller.
PLATTEVILLE. Warner, W. H. Williams.
RIPON. Woodmansee.
RIVER FALLS. McMillan.
SINSINAWA. Sr. Dobbin.
SUPERIOR. C. W. Smith.

WYOMING. (3)

LARAMIE. Fitterer, Ridgaway, Stromquist.

FOREIGN MEMBERS. (Other than Canada.)

CHINA. (8)

CHANGSHA. Leavens.
PEKING. Chang Shen-Fu, Heinz, Hsia.
SHANGHAI. Kuo Chiu Liu, Patten, E. L. Sanford.
TANGSHAN. Yen.

ENGLAND. (2)

CAMBRIDGE. Richmond.
LONDON. Greenhill.

INDIA. (2)

CALCUTTA. Bose.
SOJITRA. Pandya.

ITALY. (1)

TURIN. Prompt.

SOUTH AFRICA. (1)

RONDEBOSCH. Muir.

TURKEY. (1)

CONSTANTINOPLE. Mourad.

RECAPITULATION OF MEMBERSHIP.

Individual members January 1, 1918	1,056	
Institutional members January 1, 1918	84	
Total membership January 1, 1918		1,140

CHARTER MEMBERSHIP.

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	11	
Net gain in institutional members	32	
Total net gain		43

Constitution and By-Laws of the Mathematical Association of America

(as amended December 28, 1918)

ARTICLE I—NAME AND PURPOSE.

1. This organization shall be known as THE MATHEMATICAL ASSOCIATION OF AMERICA.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

ARTICLE III—OFFICERS.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President and Vice-Presidents shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms. The Secretary-Treasurer, and the Committee on Publications, consisting of the Manager, the Editor, and one other member, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad intermim.*

ARTICLE IV—MEETINGS.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

ARTICLE V—SECTIONS.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have full control of the publication and sale of the official journal.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who were admitted to membership before April 1, 1916, constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

3. *Committees.* The official journal shall be under the general management of the Committee on Publications. There shall also be appointed by the Council a Board of Associate Editors who shall give assistance in connection with the official journal under the direction of the Committee on Publications.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

VOLUME XXV

JANUARY–DECEMBER, 1918

PUBLISHED BY THE ASSOCIATION

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ISSUED MONTHLY EXCEPT IN JULY AND AUGUST

LANCASTER, PA., AND URBANA, ILL.

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CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, R. D. CARMICHAEL, University of Illinois, Urbana, Ill.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the ASSOCIATION, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

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VOLUME XXV JANUARY, 1918 NUMBER 1

BIBLIOGRAPHICAL NOTE ON THE USE OF THE WORD MASS IN CURRENT TEXTBOOKS.[1]

By EDWARD V. HUNTINGTON, Harvard University.

CONTENTS.

I. INTRODUCTION.

The history of dynamics from before the time of Newton to the present day has been a history of continually increasing precision in the use of terms. Such words as power, energy, force, which only a few generations ago were hopelessly confused and ambiguous, are now well defined and accurately used by all careful writers.

Not quite all the ambiguities have been removed, however. In particular, the term "mass" is one which is still defined in a variety of ways—with much resulting confusion.

[1] Presented to the American Mathematical Society at its Cleveland meeting, September 5, 1917.

With a view to ascertaining exactly what meaning present-day students are being taught to attach to the word "mass," I have looked up the definition of the term in a large number of current books on mechanics and physics, and I venture to present here the result of this survey, in the hope that it may be not without interest to teachers and other mature students. Let me state most emphatically, however, that this bibliographical report is not intended for beginners, or for class-room use. *The chief end of a course in dynamics is not a definition of "mass," but the development of power in the solution of fundamental dynamical problems;* if too much time is spent on controversial matters, too little energy will be available for the mastery of essential principles.

II. Preliminary Remarks on Measurable Quantities in General.

Since we are not here concerned with metaphysical speculations as to the ultimate nature of matter, but only with a certain measurable property of matter which may be used in the equations of mechanics, it is important to recall briefly under what circumstances any given phenomenon can be treated as a measurable quantity. The whole progress of modern science has been a constant effort to reduce merely qualitative data to a basis of accurate quantitative measurement, and no concept (like the "food value" of a bag of flour, or the "amount of matter" in a lump of metal) can be regarded as having any scientific standing until some method of reducing that concept to measurement has been agreed upon.

A set of elements, $a, b, c, \cdots$, is said to be measurable, in terms of a unit element, u, of the set, when the following conditions A, B, C, are satisfied:

(A) Given any two elements a, b, it must be possible to decide whether $a = b$, $a < b$, or $a > b$.

This condition implies the existence, among the elements, of a rule of comparison, $<$, which is supposed to obey all the ordinary laws of serial order.

(B) It must be possible to form multiples and sub-multiples of each element, or at least of the unit element, to any agreed-upon extent.

This condition implies the existence, among the elements, of a rule of combination, $+$, which is supposed to obey all the ordinary laws of addition in algebra.

(C) Given any element, a, it must be possible to find at least one positive integer, n, such that $(1/n)u \leqq a$; and for every such value of n (within some prescribed range) it must be possible to find two consecutive positive integers, m and $m + 1$, such that

$$m\frac{u}{n} \leqq a < (m+1)\frac{u}{n}.$$

This condition implies that the rule of addition, $+$, and the rule of comparison, $<$, obey the law that if $x < y$ then $a + x < a + y$.

The wider the range of integers over which these conditions hold, the greater the accuracy attainable in the measurement.[1]

[1] We may note, in passing, that many sets of entities satisfy condition A but not condition B. For example, in considering sensations of color, we say: this color is "less red" than that; but

Now it is obvious that *before the idea of "mass" can be used in the equations of dynamics, the "masses" of bodies must be defined as measurable quantities;* and all the different definitions of the word mass are in reality different definitions of the way in which the measurement of mass is supposed to be effected.

III. Current Definitions of Mass.

We now take up, in order, the four principal types of definitions of mass contained in the current textbooks. The books referred to are listed at the end of the article.

1. *Mass as Measured by Force per Acceleration (Inertia).*

In the opinion of J. Clerk Maxwell (p. 40), "the only definition of equal masses which can be admitted in dynamics" is the following: "Any two bodies are of equal mass if equal forces applied to these bodies produce, in equal times, equal changes in velocity"; and this definition is in fact the one adopted by the largest number of authors. Thus:

Lamb (p. 20): "It is observed . . . that the same force applied in succession to different bodies produces in general different degrees of acceleration. This is described as due to differences in the inertia or mass of the respective bodies. Two bodies which acquire equal velocities in equal times, when acted on by the same force, are regarded as dynamically equivalent, and their masses are said to be equal."

Poincaré (p. 13, tr.): "The quotient of force divided by acceleration is what we mean by the mass of a body."

E. H. Hall (p. 142): "Equal masses are, by definition, quantities of matter which, whatever their inequalities in other respects, are alike in this, that they require equal forces to give them equal velocities in equal times."

Similarly in Routh (p. 23), Poynting & Thomson (p. 4), Rankine (p. 482), Loney (p. 5), Cox (p. 118), Clifford (p. 59), Macgregor (p. 191), Garnett (p. 22), Williamson & Tarleton (p. 32), H. Adams (p. 10), Encyclopædia Britannica, Art. Mechanics (p. 974), New International Encyclopædia, Art. Matter (p. 1033), Lanza (p. 11), Trowbridge (p. 61), DuBois (p. 3), Wright (p. 46), Carhart (p. 42), Kimball (p. 18), Duff (p. 28), Slocum (p. 70), Ferry (p. 77), Mann & Twiss (p. 36), Thwing (p. 12), Trautwine (p. 336), Hayward (p. 194), Chwolson (p. 72), Deschanel (p. 44), Delauney (p. 119), Jamin (p. 62), Massau (p. 111), Ringelmann (p. 99), Violle (p. 98), La Grande Encyclopédie, Art. Masse, sub-heading Mécanique (p. 371), Autenrieth (p. 194), Hamel (p. 46), Streintz (p. 100), Voigt (p. 39), Voss (p. 49), Wüllner (p. 56), "Hütte" (p. 148), Weber & Wellstein (p. 93).

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we hardly say: this color is exactly "twice as red" as that. Such entities may be said to be comparable, but not additive. On the other hand, many sets of entities satisfy B but not A. For example, two points in n-dimensional space have a readily definable "sum," but we do not speak of either of such points as being "less than" the other. Such entities may be said to be additive, but not comparable. No entities can be said to be strictly measurable unless they are both comparable and additive.

Also in the following books, if, as is clearly intended, we understand by "inertia" the ratio of force over acceleration: Tait (p. 28), Gray (p. 109), R. W. Stewart (p. 93), Morley (p. 27), Martin (p. 4), Slate (p. 224), Fuller & Johnston (p. 5), Culler (pp. 4, 20), Ames (1897, p. 31), Crew & Jones (p. 25), Carhart & Chute (p. 6), Duhem (p. 105).

This definition of mass as force ÷ acceleration is obviously unintelligible without some previous definition of the equality of two forces, and this is expressly pointed out by a number of writers. Thus:

Loney (p. 5): "It is here assumed that it is possible to create forces of equal intensity on different occasions, *e. g.*, that the force necessary to keep a given spiral spring stretched through the same distance is always the same when other conditions are unaltered."

R. W. Stewart (p. 92): "Equal forces may be defined as forces which extend the same spiral spring to the same extent."

Similarly, Voss (p. 51), Cox (p. 118), Chwolson (p. 70), Larmor (p. 278), and others.

Or, if preferred, equal forces may be defined as forces which give to the same lump of metal equal accelerations. In any case, it must be remembered that forces can be specified and compared without involving in any way the idea of the mass of a body.[1]

(1*a*) A variation of this definition (by force over acceleration) is adopted by some writers who replace the general expression F/a by W/g, where W is the special force due to gravity, and g the corresponding acceleration. (Older writers use G/g for W/g.)

Thus I. P. Church (p. 53): "Since [for any given body] the quotient G/g is invariable, it will be used as the measure of the mass M in the body."

C. M. Woodward (p. 176): "The ratio W/g is quite universally represented by one letter m, which stands numerically for the mass of the body."

Similarly, Winkelmann (p. 4), Merriman (p. 148), Moulan (p. 129), Kent (p. 488), Larousse (p. 1310), Wood (p. 16), Worthington (p. 9), Perry (Calc., p. 26, Mech., p. 40), and probably Weisbach (p. 158), Hancock (p. 4), and Hudson (p. 74).

(1*b*) Another variation, expressed in terms of kinetic energy, is found in J. J. Thomson (pp. 28–30), Larmor (p. 181), Sanford (p. 27), Cheston, Gibson & Timmerman (p. 94), and Young (p. 108). Compare Holman, p. 61.

2. *Mass as Measured by a Beam Balance* (*Standard Weight*).

In spite of Maxwell's opinion that force divided by acceleration is the only admissible definition of mass, many writers in both pure and applied mechanics prefer a second definition, which we shall call the beam balance definition. Thus:

Franklin & MacNutt (pp. 25, 14): "The mass of a body, as a quantity, is

[1] Daniell's contention (p. 20) that "Force can never be measured until we know . . . the mass acted upon and the acceleration actually imparted to it" would appear to be refuted by the whole history of the science of statics.

defined by the operation of weighing by a balance. . . . The only proper definition of a quantity is the definition which corresponds to the fundamental method which is actually used in measuring that quantity. . . . It is all very well to *talk* about defining the mass of a body in accordance with the above utterly impracticable method of measuring its mass [by force and acceleration]; but sensible men always *define* things in physics in the way they *do* them."

Ganot (p. 15): "Two bodies are said to have equal masses when, if placed in a perfect balance in vacuo, they counterpoise each other."

Pender (p. 930): "Two bodies are said to have equal masses if, when they are suspended simultaneously in a vacuum, one from each end of an equal-armed balance, there is no tipping of the beam."

Capito (p. 170): "This is the only way in which the mass of a body can be found direct."

Similarly, in Love (p. 68), Maurer (p. 142), Stewart & Gee (p. 63), Daniell (p. 12), Nichols & Franklin (p. 8), Gage (p. 2), Henderson & Woodhull (p. 39), Hoadley (p. 15), Müller-Pouillets (p. 94).

Also in the following books, where "weighing" is clearly to be understood as "weighing on a beam balance": Thomson & Tait (p. 220), Jeans (p. 29), Lorentz (p. 118), Planck (p. 2), Minchin (p. 2), Newcomb (p. 554), Wentworth & Hill (p. 7), Dolbear (p. 19), C. F. Adams (p. 10), Everett (p. 16), Ziwet & Field (p. 120), Hastings & Beach (p. 5); and probably Miller & Lilly (p. 2), Black & Davis (pp. 154–155), and Föppl (p. 30).

The essential equivalence between the force ÷ acceleration definition and the beam balance definition is maintained by many writers. Thus

Kennelly (p. 8): "Mass [of a body] is estimated either by its inertia or by its weight."

Rowland & Ames (p. 46): "It is possible to define two objects as having equal masses if, when set in motion by the same cause, they have identical motions. [This is the force ÷ acceleration definition.] . . . Or, two bodies might be defined as being of equal mass if they have the same weight. [This is the beam balance definition.] . . . There is no a priori reason why there should be any connection between these two definitions; but it is found by experiment . . . that two bodies which have the same inertia also have the same weight. Consequently, it is immaterial which of these two properties is taken as the basis of comparison. . . . In practice masses are compared and measured by means of a balance, which is an instrument to measure weight."

It should not be forgotten, however, that the equivalence between these two definitions is merely an assumption, and indeed one which seems likely to be called into question by modern researches into the behavior of bodies moving with very high velocities. For a further discussion of the contrast between these two definitions, see § VI, *F*, below.

3. *Mass as Measured by Mutual Acceleration* (*Interaction*).

The following definition was first proposed by Mach in 1868,[1] and is associated in England with the name of Karl Pearson.

Mach (p. 218): "Those bodies are bodies of equal mass which, mutually acting on each other, produce in each other equal and opposite accelerations."

Pearson (p. 329): "We conceive a standard corpuscle, Q; . . . then

$$\text{Mass of } A = \frac{\text{Acceleration of } Q \text{ due to } A}{\text{Acceleration of } A \text{ due to } Q}.$$

We have here a perfectly clear and intelligible definition of the mass of A relative to Q. It is in this manner that mass is invariably determined scientifically."

Appell (p. 89): "The ratio of the masses of two points is, by definition, the inverse of the ratio of the accelerations which each of them determines in the other; the numerical value of one mass having been chosen arbitrarily, the values of all the others are determined. The word force does not enter into the principles of dynamics as here developed."

Hoskins (p. 227): "The masses of two particles are in the inverse ratio of the accelerations which they give each other. This is a definition of mass."

Similarly, in Goodwill (p. 48), Encyclopædia Britannica, Art. Motion (p. 907), Barton (p. 196), Roberts (p. 72), Crew (Mech., p. 66, Physics, pp. 57–59), Ames (1904, p. 60), Lecornu (p. 207), Andrade (p. 54).

Also, as an alternative definition, in Auerbach (p. 36), and others.

The most obvious case of interaction between two bodies is the case of impact or collision, and hence most of the writers who use the mutual acceleration definition regard a "*collision balance*," that is, an apparatus for comparing velocities before and after collision, as the fundamental instrument for measuring masses. See, for example, Goodwill, pp. 50–53.

As to the relation between this definition and the preceding definitions: Love (p. 168) makes a distinction between the "mass-ratio" of two bodies, as determined by the mutual acceleration definition, and the "ratio of the masses" of the bodies, as determined by the beam balance definition, and asserts the equality of the two. Pearson (pp. 333–337) shows how, on certain assumptions, the mutual acceleration definition may be regarded as including both the beam balance definition and the force-per-acceleration definition as special cases. (Compare Love, pp. 168, 340.) On the other hand, if we accept the principle of action and reaction, the collision balance definition is obviously a special case of the force-per-acceleration definition, since the bodies are acted upon, during the brief interval of impact, by equal forces.

(3*a*) The purely analytic definitions of mass given by Kirchhoff (p. 23), Boltzmann (p. 21), Timerding (p. 287), and Webster (p. 23) may also be classed under this heading (of mutual acceleration).

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[1] E. Mach, " Ueber die Definition der Masse," *Carl's Repertorium für Experimental-Physik*, Vol. 4, pp. 355–359 (1868).

These three types of definitions are the ones which the student is most likely to meet in his ordinary reading. (Compare The Century Dictionary, Art. Mass.)

4. *Mass as Measured by the Number of Identical Corpuscles in a Body.*

The following definition depends on a certain assumption in regard to the structure of matter.

Johnson's Cyclopædia, Art. Dynamics (p. 875): "If we suppose . . . that the ultimate particles or molecules of all substances are the same, and that we may designate by the term density the degree of proximity of the particles of any body to each other, then the number of particles in a given volume may be taken to denote the mass of the body."

Hertz (p. 46): "The number of material particles in any space, compared with the number of material particles in some chosen space at a fixed time, is called the mass contained in the first space."

La Grande Encyclopédie, Art. Masse, subheading Astronomie (p. 371, tr.): "One can then define the mass of a body as the number of identical particles (points matériels) of which it is composed."

Houston & Seal (p. 30): "The mass of a body is proportional to the number of its molecules."

This definition of mass, though seldom explicitly stated, is probably the idea that most authors have in mind when they speak vaguely of mass as "quantity of matter."

Granting the basic assumption, the definition is logically quite defensible. Unfortunately, however, there is not a shred of evidence in support of this assumption, and the student who has been brought up on any such notion of the ultimate structure of matter finds it difficult to adjust himself to the modern theories, which make the structure of the atom as complex as that of the solar system. And of course, as a practical method of determining the mass number of any given body, the definition fails completely, and resort must be had, sooner or later, to one of the other methods of measuring mass. (Compare Barton, p. 195.)

IV. Incomplete Definitions of Mass.

All the definitions so far mentioned satisfy the fundamental condition that masses must be measurable quantities. But a number of widely used text-books still retain "definitions" of mass which are no definitions at all, because they fail to explain how mass is to be measured.

5. *"Quantity of Matter."*

Almost every book on the subject pays its respects, in some form or other, to the time-honored "definition" of mass as "quantity of matter."

In compiling the present report, whenever I found this "definition" supplemented by a statement of how the "quantity of matter" is to be measured, I

ignored the words "quantity of matter" and accepted this supplementary statement as the author's real definition of mass. But where the words "quantity of matter" stand alone (as in Ziwet, p. 129), they must be regarded as a totally inadequate definition of the mass of a body as a quantitative concept. The situation is well described by many authors. For example:

Encyclopædia Britannica, Art. Motion (p. 907): "The mass of a body is often loosely defined as the measure of the quantity of matter in it. This definition correctly indicates that the mass of any portion of matter is equal to the sum of the masses of its parts, . . . but gives no test for comparison of the masses of bodies of different substances."

Jamin (p. 63, tr.): "Some authors define mass as the quantity of matter which the body contains. . . . This idea is vague and illusory, for matter is not a thing which one can measure. . . . Those who wish to preserve this idea of mass are obliged, at the start, to define what they propose to mean by equal quantities of matter."

Chwolson (p. 73, tr.): "The primitive definition of the term mass as quantity of matter is not admissible, for . . . for heterogeneous materials, the notion of equal or unequal quantities of matter is entirely lacking."

Hoskins (p. 2): "The mass of a body is often briefly defined as its quantity of matter. These words, however, convey no definite idea of the meaning of mass as a factor in the determination of motion."

It cannot be emphasized too strongly that without a definite statement of the method of measurement, the phrase "quantity of matter" is empty and useless in dynamics. *Merely calling a thing a quantity does not make it a quantity.* It is just as absurd to speak of "quantity" of matter without defining the method of measurement, as it would be to speak of "quantity" of beauty, or "quantity" of temperature, without defining some method of estimating the values of these "quantities."

There are, of course, familiar cases in which we do have a direct intuitive perception of comparative "quantity." For example, any one can tell that one speed is many times faster than another speed, or that one bulk is many times larger than another bulk. Speed and bulk are perfectly definite intuitive magnitudes, regardless of the method of measurement which may be used to determine their numerical value. *But in the case of matter, it is precisely this intuitive perception of quantitative relations which is entirely lacking.*

Suppose, for example, that a load of coal and a load of wood come to the door, and let us ask the question, which has more "matter" in it? It will be found, on reflection, that no answer to this question is forthcoming, *except* by reference to one or other of the measurable properties described above. We can tell at once which load has more bulk; we can hazard a guess as to which has more weight, or as to which would require more force to set it in moton; we can even amuse ourselves by pretending to count the number of identical particles of which we may imagine the two loads to consist; but when we try to think of any direct comparison between the mere "matter" in one load with the mere

"matter" in the other load, without making even subconscious use of any one of these methods of measurement, the mind becomes absolutely blank. No such direct comparison is possible, even in theory (except, of course, in the often cited but quite trivial and irrelevant case of the comparison of bodies of the same homogeneous material).

Fortunately for the student, the number of textbook writers who still persist in defining mass as *merely* "quantity of matter" is rapidly decreasing (since the publication of Pearson's *Grammar of Science*), and is already negligibly small.

6. *Circular Definitions.*

The definition of mass in terms of density, followed by a definition of density in terms of mass, still appears in Tait and Steele (p. 42), Bowser (p. 6), Andrews & Howland (p. 11). (Compare Thomson & Tait, p. 220.) The obvious circularity of this definition has been pointed out by many critics. Similarly, Dadourian (p. 103) defines mass in terms of "kinetic reaction," and on the same page defines "kinetic reaction" in terms of mass.

7. *Definition Lacking.*

A number of books in which one would expect to find a definition of mass use the term freely without formal definition of any kind. For example, Jamieson (p. 2), Millikan (p. 13), Smith & Longley (p. 90), Hedrick & Kellogg (p. 1), Watson (p. 68), Silberstein (p. 50).

(7*a*) On the other hand, some very successful writers prefer to develop the whole theory without using the *word* mass at all; so Cotterill & Slade (*loc. cit.*), Merriman (p. 148), Black & Davis (p. 154), and especially Sir George Greenhill (*loc. cit.*).

8. *Mass as Lump of Matter.*

One further use of the word mass may be mentioned, although it hardly pretends to be a definition of the mass of a body in any quantitative sense. When one speaks of "the impact of two masses," or when one solves a problem about "two masses suspended by a cord over a pulley," one obviously means by *a mass* simply *a body*, or *lump of matter*. This is a very common and convenient usage which ought not to be likely to create any misunderstanding.

This completes the record of all the important uses of the word mass in the current textbooks.

It remains to add a brief report on the various names that are given to the units of mass.

V. Remarks on the Names of the Units of Mass.

(1) If the mass of a body is defined by the force-per-acceleration definition, the name of the unit of mass is naturally derived from the names of the units of force and acceleration, just as the name of the unit of acceleration is itself

derived from the names of the units of length and time. Thus we speak of a mass of 1 lb. per ft.-per-sec.^{2}, or 1 kg. per cm.-per-sec.^{2}.

Worthington (p. 9) abbreviates 1 lb. per ft.-per-sec.^{2} into 1 "slug," while Maurer (p. 143) calls this unit a "gee-pound."

On the other hand, some writers prefer to leave the unit unnamed altogether.

Church (p. 53): "No name will be given to the unit of mass, it being always understood that the fraction G/g will be put for M before any numerical substitution is made."

Similarly, Perry (Calc., p. 26), Sanford (p. 28).

(2) If the beam balance definition or the mutual acceleration definition is adopted, the name of the unit of mass is simply the name of the lump of metal which is used as the standard; as the standard pound body, the standard kilogram body, etc.

It must be remembered, however, that the words pound, kilogram, etc., are used also (quite properly) to denote units of force. Hence, in books in which forces and masses appear in the same equations, it is necessary to distinguish between the *pound force* and the *pound mass*, and between the *kilogram force* and the *kilogram mass*.

This distinction is emphasized by many writers, for example, Kennelly (p. 10). The attempt sometimes made, however, to symbolize this distinction by writing "lb." for one of the units and "pd." for the other seems foredoomed to failure, since no one can remember which is which. For example, Worthington (p. 9) uses "pd." for force and "lb." for mass, while Dadourian (p. 109) uses "pd." for mass and "lb." for force. Similarly, the two new notations recently proposed by Hudson (p. 74)—"pounds (abs.)" for mass as measured by weight, and "pounds (grav.)" for mass as measured by weight divided by g—would seem to serve only to increase the existing confusion.[1]

In books in which forces are compared only with forces, and masses only with masses—so that no equation contains both forces and masses together—the necessity for this distinction disappears. This is the real secret of the simplicity, as regards units, of the method advocated by the present writer (see § VI, *H*, below).

VI. Conclusions.

In conclusion, I venture to express my own personal preference, in regard to the use of the word mass, by offering to defend the following propositions:

(*A*) It is impossible to assume, at the beginning of a course in mechanics,

[1] As an illustration of the confusion which arises when units of force and mass are introduced into the same equation, it may be amusing as well as instructive to reproduce the following statements from two books which have had a wider circulation perhaps than any others in their respective fields of theoretical and applied mechanics.

Routh (*Dynamics of a Particle*, p. 25): "The equation $W = mg$ shows that the weight of a unit mass is g."(!) (One had always supposed that W was a force, and g an acceleration!)

Weisbach (*Mechanics of Engineering*, p. 159): "Hence the mass of a body whose weight is 20 pounds is 0.62 pounds; and inversely the weight of a mass of 20 pounds is 644 pounds."(!)

Quotations almost as distressing as these might be made from many more modern textbooks.

that a student has any intuitive idea of the meaning of the expression "the mass of a body." Before this term is introduced, its meaning should be defined (§ 7). The use of the word "mass" as a synonym for "lump of matter" is often convenient, and need not be discouraged (§ 8); but this gives, of course, no quantitative idea of the "mass of the body."

(*B*) The definition of "the mass of a body" as the number of identical corpuscles in it (§ 4), and the pseudo-definition of "mass" as "quantity of matter" (§ 5), are unsatisfactory, and should be abandoned.

(*C*) The definition of "mass" by mutual acceleration, at least in its general form (§ 3), is too abstract for beginners.[1]

There remain to be considered, therefore, only the inertia definition (§ 1) and the beam balance definition (§ 2).

(*D*) The idea of the *inertia of a body*, as measured by force divided by acceleration, is important, and the Newtonian hypothesis that the inertia of a body is constant (that is, independent of the body's velocity) should be thoroughly understood. Any teacher who wishes to introduce the term "mass of a body" to mean the inertia of the body has excellent authority for doing so (§ 1).

(*E*) The idea of the *standard weight of a body*, as measured by a beam balance, is also important, and any teacher who wishes to introduce the term "mass of a body" to mean the standard weight of the body has excellent authority for doing so (§ 2).

(*F*) *It is by no means a matter of indifference, however, which of these two definitions of "mass" is adopted*, since many passages in standard literature presuppose one of these definitions to the absolute exclusion of the other.

As a first example, consider the following quotation from Poincaré (p. 13, tr.): "In the new mechanics, the mass of a body increases enormously with the velocity, and becomes infinite when the velocity approaches the velocity of light." Here the word "mass" clearly must mean inertia; the passage would become quite unintelligible if "mass" were interpreted as standard weight.—Secondly, consider the following proposition, cited by many authors as a fundamental law of nature: "Force = mass × acceleration." Here the word "mass," whatever else it may mean, certainly does *not* mean inertia (that is, force ÷ acceleration), since, if it did, the proposition in question would be not a law of nature, but merely a trivial algebraic transformation of the very definition of inertia. To give the proposition any significance as a law of nature, the word mass must be interpreted as something other than inertia, for example, as standard weight.—Thirdly, consider the contrast between the two ways of naming the units. If the inertia definition is adopted, the natural units of mass are the lb. per ft.-per-sec.^{2}, the kg. per cm.-per-sec.^{2}, etc. (where "lb." and "kg." are to be understood as units of force); while if the beam balance definition is adopted, the natural units of mass are the pound-mass, the kilogram-mass, etc. (which are the names

[1] The collision balance, which is a very instructive apparatus, is best interpreted as an instrument for comparing the inertias of two bodies, the equality of the forces acting on the two bodies during the impact being insured by the principle of action and reaction (§ 3).

of certain lumps of metal); moreover, the latter units differ from the former by the numerical factors 32.1740 and 980.665 respectively.

These three illustrations may suffice to show that the choice between the two definitions of mass is a serious matter, which affects the whole development of the course.

(*G*) *In view of these considerations, it would appear desirable, at least at the beginning of the course, to employ only the separate, well-established terms "inertia of a body" and "standard weight of a body" to denote these two closely related but still quite distinct conceptions.* Later in the course, the student should be told, as an important matter of general information, that the word "mass" is used by some writers to denote inertia and by other writers to denote standard weight. *There is at any rate nothing to be lost by following this plan; and there is certainly much to be gained in the way of clear thinking.*

All these recommendations, it will be noticed, leave entirely open the question whether $F/a = F'/a'$ or $F = ma$ is the better form of the fundamental equation of mechanics, and I hope that a general acceptance of the truth of the propositions (*A*)–(*G*) may serve to clear the ground for a more satisfactory discussion of that question.

(*H*) To avoid possible misunderstanding, I should like to add that while I have constantly advocated the use of the equation $F/a = F'/a'$ as the fundamental equation of mechanics, I have no objection whatever to the *use of the letter m as an abbreviation for F/a, or W/g.* It is obvious, however, that the letter m, when so used, denotes an inertia, that is, a force-per-acceleration, and not a number derived from a beam balance. Hence, when numerical computation is involved, I like to replace such an m by the more explicit W/g before inserting numerical values, thus avoiding the possibility of confusion between the units of inertia and the units of standard weight. By following this plan, nothing is lost in the way of algebraic compactness, and much is gained in the way of sureness and comfort in numerical computation.[1]—The alternative plan of regarding the "m" in $F = ma$ as a fundamental concept (usually vaguely defined as "quantity of matter") appears to me to offer no advantages, and to lead, in practice, to many disturbing and quite superfluous complications.

VII. List of Books Examined.

This list includes merely those books on mechanics or physics which happened to come readily to hand, and is not intended to be in any sense exhaustive. The numbers in [ ] refer to sections of the present paper.

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J. S. Ames, 1897, *Theory of Physics*, Harpers, 1897 [1].

[1] For a systematic presentation of the working principles of mechanics, as developed from the fundamental equation $F/a = F'/a'$, see E. V. Huntington, *The Logical Skeleton of Elementary Dynamics*, American Mathematical Monthly, Vol. 24 (1917), pp. 1–16. Reprints of this article, which a number of teachers have found useful in the class room, may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, O., at ten cents a copy.

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Information in regard to errors of classification or omission will be gratefully received.

BOOK REVIEW.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Projective Geometry. By L. WAYLAND DOWLING. McGraw-Hill Book Company, New York, 1917. 215 pages. $2.00.

If long and expectant waiting for the guest's arrival insures a hearty welcome, this little book first sees the light under most auspicious circumstances. For many years those who have had the good fortune to teach projective geometry have been wishing for a text in English that should lay sufficient emphasis on its non-metrical character and at the same time should be adapted to the powers of the average college junior or even the exceptional younger student. Cremona, excellent book though it is, does not satisfy the first condition; Holgate's translation of Reye has long been out of print; and Veblen and Young's masterful treatise has seemed to many too heavy for the purpose. Here we have a book, small and compact, of pleasing external appearance, well printed in good type, with clear and attractive page broken by figures of reasonable size. Its point of view is non-metrical and yet it does not neglect the metrical applications. It merits careful consideration by all who are interested in this beautiful field of mathematical thought and sympathetic trial in many of the institutions where the subject is taught.

Perhaps the first question confronting the author of an elementary text on projective geometry is whether the conic shall be studied first as generated by projective ranges and pencils or as made up of the self-conjugate elements of a polar field. Shall he ally himself with the school of which we may take Reye as an example or with that typified by Enriques? The logical advantages of the latter have been clearly pointed out,[1] its wisdom from a pedagogical standpoint is not so clear. It takes much longer to reach what to the student seems worth while.

The teacher finds himself, perhaps, in the position of the guide who must choose between two paths up the mountain. There is on the one hand a long and rugged climb, which leads directly to the top, and on which the wonderful landscape bursts suddenly into view in all its beauty; and on the other is an easier path, which half-way up the mountain affords a fine view, if not the equal of that at the higher level. The most hardy of his party would not stop short of the top in any event. Of those who, exhausted, would fail to finish the longer ascent, some, encouraged by the lesser view, will go on to the greater; others, stopping at the half-way house, will at least have something for their labor. Must it not be agreed that unless the party is made up entirely of hardy climbers the wise guide will choose the second route? But is this a true analogy? We confess doubt.

Our author frankly admits in the preface that he chooses Reye for his pattern, and yet, if we may continue the figure, he does not fail to point the way to the higher level. After viewing the beautiful landscape embracing the conic sections and their properties from the level of generation by projective forms, he concludes the book with chapters on projectively related primitive forms of the second kind and polarities in a plane and in a bundle. The question might be raised whether it would have been possible and desirable in one book to give the teacher his choice of paths. Our author has not attempted this. Much of the book would have to be entirely recast if the teacher wished to make his first approach to the conics through the polar field. In particular the involution on a line could not be derived from the involution on a conic.

With what simple means and how quickly are we led to the conics! Only four chapters, 36 pages, are necessary to introduce the student to the fundamental notions, the primitive forms, the theorem of Desargues, the principle of duality, the theorem of perspective quadrangles, the harmonic set, as a preparation for projective one-dimensional primitive forms, the generation of the conic, the theorems of Pascal and Brianchon, and the polar theory of the conic. These are treated in four chapters covering 57 pages. To be sure, the involution is not yet known, nor those properties that depend on it. Three chapters intervene, Chapter IX on the Diameters, Axes, and Algebraic Equations of Curves of the Second Order, Chapter X on Ruled Surfaces of the Second Order, and Chapter XI on Projectively Related Elementary Forms, before in Chapter XII we have the involution defined as a cyclic projectivity of order two. This chapter deals

[1] Charlotte Angas Scott, *Annals of Mathematics*, 2d Series, Vol. 2, pp. 64–72.

with the theory of the involution and imaginary elements, and the following Chapter XIII with the foci and focal properties. This closes the discussion of one-dimensional forms, the remaining fifty pages being occupied with two-dimensional forms, the two chapters mentioned above.

In general outline these first thirteen chapters handle the subject in about the manner that one familiar with Reye would expect. A few features deserve special mention.

In considering the fundamental theorem the author constructs a harmonic scale, calls attention to the fact that harmonic constructions from three points can never yield all the points of a line, though "theoretically we may arrive at a point-row whose points are everywhere dense," and gives a formulation of the Dedekind postulate as here applied. He confesses that the treatment is meager, but it furnishes a starting point for the teacher who considers it advisable to lay stress on the logical foundation of the fundamental theorem.

Defining the involution on a conic as a cyclic projectivity of order two furnishes the author with an occasion for a short discussion of cyclic projectivities in general. The definition of imaginary elements and the solution of certain problems involving their use follow closely the latest edition of Reye's first volume.

Though nearly a quarter of the book is devoted to metrical matters the double ratio receives scant attention.

A more detailed outline of the last two chapters may be in place, since in the selection of material here there is a wider range of possibilities. The definition and determination of perspective and projective transformations of two-dimensional forms are followed by the plane perspectivity, affinity, similitude, and congruence, with a short discussion of the double elements of a collineation. In the last chapter we have the construction and classification of polarities in the plane and bundle; orthogonal and absolute polarity and antipolarity; two polarities in the same plane or bundle with resulting collineation and with application to the cyclic planes and focal axes of cones; quadratic transformations, inversion, circular transformations.

In handling these two chapters the teacher is afforded ample opportunity to leave his impress upon his course. He will wish to amplify the treatment at many points and readjust the emphasis, in order to bring out the advantages of the polar field approach to the conic. One must delay long enough on the summit for the mists to roll away. He will find it necessary even to correct misleading statements such as,[1] "If two planes are collinearly related and have a self-corresponding line, they are in perspective position, or else they are superposed and have in common a sheaf of rays." Evidently what the author means is a line of self-corresponding points rather than a self-corresponding line. Again, in the demonstration of the theorem,[2] "In affinately related planes, the ratio between the areas of corresponding figures is constant," it is implied that corresponding segments bear to each other a ratio that is constant throughout the plane.

[1] Page 165.
[2] Page 170.

Scattered through the book there are some fifty sets of exercises averaging six exercises to the set. For the formulation of these the teacher will be grateful. We have noticed two of them that are faulty in statement:[1] "Show that in the configuration of Desargues *any* line may be taken as an axis of perspectivity. The two triangles and the center of perspectivity will then be uniquely determined," is not clear;[2] "If a hexagon whose vertices are not coplanar nor its three diagonals concurrent is projected from any point on a line which meets all three of the diagonals, show that the lines projecting the vertices are rays of a cone of the second order," draws the conclusion of Pascal from the hypothesis of Brianchon.

The historical notes are few and do not attempt to sketch the development of the subject.[3] "Chasles, *Géométrie Supérieure*, 1880," may be thought misleading, since the date is not that of the first edition.

While the author, as he himself says, has patterned after Reye, he has by no means given us a mere translation. The language and style are the author's, not Reye's. Some things we think might be better said in the interest of clearness and accuracy. For example,[4] "Like primitive forms are each composed of the same kind of elements," appears in its connection to be a definition. The breaks in the argument for the purpose of dualizing are sometimes disturbingly frequent. For some things we should use other names; throw of points or lines where the author uses range and pencil, two-dimensional forms for forms of the second kind, quadratic transformation for quadric transformation. We see no reason for banishing the diagonal point. In passing may we express the hope that some steps will soon be taken to standardize the nomenclature of this subject. It is certainly not an advantage pedagogically to have several different names for the same thing.

It is a pleasure to the teacher to be able to point out to his class the excellences of good figures. This pleasure is in measure denied to the user of the book we are discussing; for while the figures are in general clear and easily read, the teacher, if he calls attention to them at all, will be forced to remark on the carelessness exhibited in their construction. The inconsistency in the use of small circles surrounding designated points and in the use of dotted lines is astonishing. In each of two figures three lines that should pass through a point form a triangle of considerable size. Very unfortunate is Fig. 120, one of the most pretentious in the book. The crude approximations to ellipses here shown are an offence to even the slightly trained eye. What a pity that a book must carry a blemish of this sort, which might so easily have been avoided!

Let us conclude with the hope that this book will find wide acceptance and accomplish much in bringing the subject to the attention of a larger body of students.

JOHN W. BRADSHAW.

UNIVERSITY OF MICHIGAN.

[1] Ex. 5, page 23.
[2] Ex. 9, page 80.
[3] Ex. 6, page 53.
[4] Page 17.

PROBLEMS FOR SOLUTION.[1]

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2660. Proposed by JOSEPH E. ROWE, State College, Pa.

Prove that the distance measured along the side of a triangle, from the point of contact with the inscribed circle to the point of contact with an escribed circle, is equal to the side of the triangle between the two circles.

2661. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find a parallelopipedon whose edges, and the diagonals of its faces, are all rational whole numbers.

2662. Proposed by JOHN LOUKE, New York City.

Assume we have two piles of gold bars. The dimensions of the bars in the first pile are $2.643 \times 5.286 \times 10.573$ and the dimensions of the bars in the second pile are $2.13 \times 6.53 \times 10.573$. If possible, arrange bars from the first pile and from the second pile so as to form perfect cubes, the bars from the piles to be taken separately or in combination.

2663. Proposed by R. P. BAKER, University of Iowa.

From the identity

$$\prod_{k=0}^{\infty}\left(\frac{1}{1-x^{2k+1}}\right) \equiv \prod_{l=1}^{\infty}(1+x^l),$$

Macmahon (*Combinatorial Analysis*, vol. I, p. 10) proves the number of partitions of any integer n into odd parts is equal to the number of partitions of n into parts no two of which are equal.

Taking the classes of some cardinal number, devise an ordinal arrangement which puts each member into one-to-one correspondence with a number of the other class and of such a nature that a direct calculation determines which is the partner of any member in either class.

2664. Proposed by J. W. NICHOLSON, Louisiana State University.

Find the sum of

$$\tfrac{1}{3} - \tfrac{2}{15} + \tfrac{3}{35} - \cdots + (-1)^n \frac{n}{(2n-1)(2n+1)}.$$

2665. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A telegraph wire, which weighs 1/10 of a pound per yard, is stretched between poles on a level ground, so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

2666. Proposed by W. WOOLSEY JOHNSON, Annapolis, Maryland.

Ten equations between five quantities $x_1, x_2, \cdots, x_5$ being written as follows: $x_1 = 1 - x_3x_4$ and four others formed by cyclic interchange of the suffixes; also $x_5x_1x_2 = x_5 + x_2 - 1$ and four others formed by cyclic interchange; prove that only three of these equations are independent. In other words, the values of x_1 and x_2 being assumed at pleasure, x_3, x_4, and x_5 can be so determined as to satisfy all ten equations.

2667. Proposed by E. L. REES, The University of Kentucky.

Given one diagonal of a parallelogram and the area of the rectangle whose sides are equal to those of the parallelogram, construct the parallelogram so that the diagonal shall make a given angle, α, with a given line and so that the sum of the angles that two adjacent sides make with this line shall be equal to a given angle, β.

[1] Hereafter the problems will be numbered consecutively beginning at the next number above the total number proposed prior to this date. EDITORS.

2668. Proposed by B. F. FINKEL, Drury College.

Show that

$$v = \frac{2}{9}\frac{ga^2\sigma}{\eta},$$

where a is the radius of a droplet, σ its density, η the viscosity of the air and v the velocity under gravity g. Stokes's Law.

2669. Proposed by S. A. COREY, Albia, Iowa.

Let $A_1, A_2, \cdots, A_8$, and $-(A_1 + A_2 + \cdots + A_8)$ be the vector sides of an enneagon, plane or gauche. Also let $B_1, B_2, \cdots, B_8$, and $-(B_1 + B_2 + \cdots + B_8)$ be the vector sides of a second enneagon, where

$$\begin{aligned}
B_1 &= C_1A_1 - C_2C_5A_3 - C_3C_6A_5 + C_4C_5C_6A_7,\\
B_2 &= C_1A_2 - C_2C_5A_4 - C_3C_6A_6 + C_4C_5C_6A_8,\\
B_3 &= C_2A_1 + C_1A_3 - C_4C_6A_5 - C_3C_6A_7,\\
B_4 &= C_2A_2 + C_1A_4 - C_4C_6A_6 - C_3C_6A_8,\\
B_5 &= C_3A_1 + C_4C_5A_3 + C_1A_5 + C_2C_5A_7,\\
B_6 &= C_3A_2 + C_4C_5A_4 + C_1A_6 + C_2C_5A_8,\\
B_7 &= C_4A_1 - C_3A_3 + C_2A_5 - C_1A_7,\\
B_8 &= C_4A_2 - C_3A_4 + C_2A_6 - C_1A_8,
\end{aligned}$$

C_1, C_2, C_3, C_4, C_5, and C_6 being scalars.

Then, if a_s = tensor A_s, b_s = tensor B_s, and cos (A_rA_s) = cosine of the angle included between A_r and A_s, and cos (B_rB_s) = cosine of the angle included between B_r and B_s, establish the following relation between the sides and angles of the two enneagons:

$$\begin{aligned}
[C_1^2 + C_5C_2^2 + C_6C_3^2 + C_5C_6C_4^2]&[a_1a_2 \cos(A_1A_2) + C_5a_3a_4 \cos(A_3A_4)\\
&+ C_6a_5a_6 \cos(A_5A_6) + C_5C_6a_7a_8 \cos(A_7A_8)]\\
&= b_1b_2 \cos(B_1B_2) + C_5b_3b_4 \cos(B_3B_4) + C_6b_5b_6 \cos(B_5B_6) + C_5C_6b_7b_8 \cos(B_7B_8).
\end{aligned}$$

Show that Geometry problem 506 is a special case of the foregoing. Give illustrative example, using triangle or other simple geometric figure, by assuming that some of the sides of the first enneagon are zero.

SOLUTIONS OF PROBLEMS.

482 (Algebra). Proposed by C. F. GUMMER, Kingston, Ont.

Find a necessary and sufficient condition that the infinite sequences of positive quantities $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ may be such that the series $a_1x_1 + a_2x_2 + \cdots$ and $b_1x_1 + b_2x_2 + \cdots$ either both converge or both diverge, when the x's are any positive quantities.

Solution by the Proposer.

The condition is that $a_1/b_1, a_2/b_2, \cdots$ all lie between two positive limits m and M $(m < M)$. It is sufficient; for it makes $m < (a_rx_r)/(b_rx_r) < M$. It is necessary; for if it does not hold, either the sequence $(a_1/b_1, a_2/b_2, \cdots)$ or $(b_1/a_1, b_2/a_2, \cdots)$ has $+\infty$ for one of its limits. That is (taking the first case), there is a partial sequence $(a_{i_1}/b_{i_1}, a_{i_2}/b_{i_2}, \cdots)$ of increasing quantities tending to ∞. Hence, using an argument due to DuBois-Reymond, the series

$$\sum_{r=2}^{\infty}\left(\sqrt{\frac{b_{i_{r-1}}}{a_{i_{r-1}}}} - \sqrt{\frac{b_{i_r}}{a_{i_r}}}\right)$$

converges, and the series

$$\sum_{r=2}^{\infty}\frac{a_{i_r}}{b_{i_r}}\left(\sqrt{\frac{b_{i_{r-1}}}{a_{i_{r-1}}}} - \sqrt{\frac{b_{i_r}}{a_{i_r}}}\right)$$

diverges. Hence, if

$$x_{i_r} = \frac{1}{b_{i_r}}\left(\sqrt{\frac{b_{i_{r-1}}}{a_{i_{r-1}}}} - \sqrt{\frac{b_{i_r}}{a_{i_r}}}\right) \qquad (r = 2, 3, \cdots),$$

the partial series $\Sigma_r a_{i_r} x_{i_r}$ diverges while $\Sigma_r b_{i_r} x_{i_r}$ converges. Also the other x's may be chosen so small as to make the corresponding values of bx less than the terms of a given convergent series; so that we have made one series diverge while the other converges.

Also solved by S. BEATTY.

483 (Algebra). Proposed by C. R. DUNCAN, Amherst College.

Prove or disprove the following theorem: An infinite series $A_1 + A_2 + A_3 + \cdots + A_n + \cdots$ is convergent or divergent according as

$$\lim_{n=\infty} \frac{A_n}{1 - \dfrac{A_n}{A_{n-1}}} = 0 \quad \text{or} \quad \neq 0.$$

SOLUTION BY EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

The test when applied to the series

$$1 + \tfrac{1}{1} + 2 + \tfrac{1}{2} + 3 + \tfrac{1}{3} + \cdots + n + \frac{1}{n} + \cdots$$

indicates convergence, whereas the series clearly diverges. If this example be considered artificial consider the series

$$\frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \cdots + \frac{1}{n(\log n)^p} + \cdots.$$

Cauchy's integral test shows that this series diverges when $p \leqq 1$ and converges when $p > 1$.

$$\frac{A_n}{1 - \dfrac{A_n}{A_{n-1}}} = \frac{1}{\dfrac{1}{A_n} - \dfrac{1}{A_{n-1}}} = \frac{1}{n(\log n)^p - (n-1)[\log (n-1)]^p}.$$

Now

$$\frac{1}{(\log n)^p} < \frac{1}{n(\log n)^p - (n-1)[\log (n-1)]^p} < \frac{1}{[\log (n-1)]^p}.$$

Hence, the limit sought by the test is zero for all positive values of p, whereas it should be zero only for values greater than unity.

The suggested test for convergence is therefore no good.

If the limit exists and is greater than zero, then

$$\frac{1}{\dfrac{1}{A_n} - \dfrac{1}{A_{n-1}}} > \gamma \quad \text{or} \quad \frac{1}{A_n} - \frac{1}{A_{n-1}} < \frac{1}{\gamma}.$$

Hence

$$\frac{1}{A_{n+p}} - \frac{1}{A_{n-1}} < \frac{p+1}{\gamma},$$

$$A_{n+p} > \frac{1}{\dfrac{1}{A_{n-1}} + \dfrac{p+1}{\gamma}} > \frac{\gamma}{\dfrac{p+1+\gamma}{A_{n-1}}}.$$

(We have assumed a series of positive terms.) The series, therefore, diverges by comparison with the harmonic series. Hence, the test for divergence when applied to series of positive terms is valid.

Also solved by HORACE OLSON.

514 (Geometry). Proposed by VICENTE MILLS, Manila, P. I.

Given an equilateral triangle, the length of the sides being unknown, and a point within, the distances from which to the vertices are given, required the length of a side of the triangle and the angles subtended at the given point by the sides of the triangle.

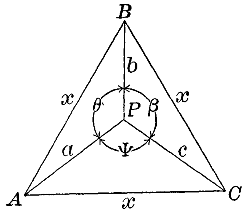
I. Solution by William W. Johnson, Cleveland, Ohio.

From the figure, we have

$$\theta + \beta + \psi = 360^\circ, \qquad \theta = 360^\circ - (\beta + \psi)$$

and

$$\cos\theta = \cos(\beta + \psi) = \cos\beta\cos\psi - \sin\beta\sin\psi, \quad \text{or} \quad \cos\theta - \cos\beta\cos\psi = -\sin\beta\sin\psi.$$



Squaring both sides, we get

$$\cos^2\theta - 2\cos\theta\cos\beta\cos\psi + \cos^2\beta\cos^2\psi = \sin^2\beta\sin^2\psi$$
$$= (1 - \cos^2\beta)(1 - \cos^2\psi)$$
$$= 1 - \cos^2\beta - \cos^2\psi + \cos^2\beta\cos^2\psi.$$

Transposing and uniting, we have

$$\cos^2\theta + \cos^2\beta + \cos^2\psi - 2\cos\theta\cos\beta\cos\psi = 1. \tag{1}$$

From the cosine formulas in trigonometry, we get

$$\cos\theta = \frac{a^2 + b^2 - x^2}{2ab} \quad (A), \qquad \text{sos } \beta = \frac{b^2 + c^2 - x^2}{2bc} \quad (B), \qquad \cos\psi = \frac{c^2 + a^2 - x^2}{2ca} \quad (C).$$

Substituting these values in (1), clearing of fractions, and arranging terms, we get

$$x^6 - (a^2 + b^2 + c^2)x^4 + (a^4 + b^4 + c^4 - a^2c^2 - b^2a^2 - c^2b^2)x^2 = 0. \tag{2}$$

Factoring, we find one root is $x^2 = 0$. Removing the factor x^2, we have

$$x^4 - (a^2 + b^2 + c^2)x^2 + a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = 0. \tag{3}$$

This is a quadratic equation in x^2. Solving equation (3) for x, we find

$$x = \pm\sqrt{\frac{a^2 + b^2 + c^2}{2} \pm \tfrac{1}{2}\sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)}}. \tag{4}$$

The angles θ, β, and ψ may now be found by inserting the values of x found in (4). The negative signs before the radicals apply when the point P lies outside the triangle.

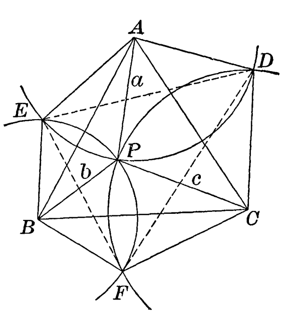
Having found x, the following formulas, adapted for logarithmic computation, may be used to replace (A), (B), and (C).

$$\cos\tfrac{1}{2}\theta = \sqrt{\frac{p(p-x)}{ab}} \quad \text{in which} \quad p = \tfrac{1}{2}(a + b + x),$$

$$\cos\tfrac{1}{2}\beta = \sqrt{\frac{q(q-x)}{bc}} \quad \text{in which} \quad q = \tfrac{1}{2}(b + c + x),$$

$$\cos\tfrac{1}{2}\psi = \sqrt{\frac{r(r-x)}{ac}} \quad \text{in which} \quad r = \tfrac{1}{2}(a + c + x).$$

II. Solution by Daniel Kreth, Wellman, Ia.



Let ABC be the given triangle, and P the given point. Put $AP = a$, $BP = b$, and $CP = c$. Let x = a side of the triangle Then

$$\text{the area} = \frac{x^2}{4}\sqrt{3}. \tag{1}$$

Construct the triangle AEB = triangle APB, triangle BFC = triangle BPC, and triangle CDA = triangle CPA. Draw DE, EF, and FD.

Since $\angle\ DAE = 2\angle\ BAC = 120^\circ$, DE is equal to a side of an equilateral triangle inscribed in a circle whose radius is a.

Hence, $DE = a\sqrt{3}$. In the same manner we find $EF = b\sqrt{3}$ and $FD = c\sqrt{3}$.

$$\text{Area of triangle } DAE = \frac{a^2}{4}\sqrt{3}; \tag{2}$$

$$\text{area of triangle } EBF = \frac{b^2}{4}\sqrt{3}; \tag{3}$$

$$\text{area of triangle } FCD = \frac{c^2}{4}\sqrt{3}; \tag{4}$$

and

$$\text{area of triangle } DEF = \tfrac{3}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}. \tag{5}$$

$$(2) + (3) + (4) + (5) = \text{the area of the polygon } AEBFCD$$
$$= 2 \times \text{area of triangle } ABC.$$

Equating the two expressions for the area of the triangle, we have

$$\frac{x^2}{4}\sqrt{3} = \tfrac{1}{8}(a^2+b^2+c^2)\sqrt{3} + \tfrac{3}{8}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)};$$

$$\therefore\ x = \sqrt{\tfrac{1}{2}\{(a^2+b^2+c^2) + \sqrt{3}[(a+b+c)(b+c-a)(a+c-b)(a+b-c)]\}}.$$

In the triangles we have all the sides given, to find the angles.

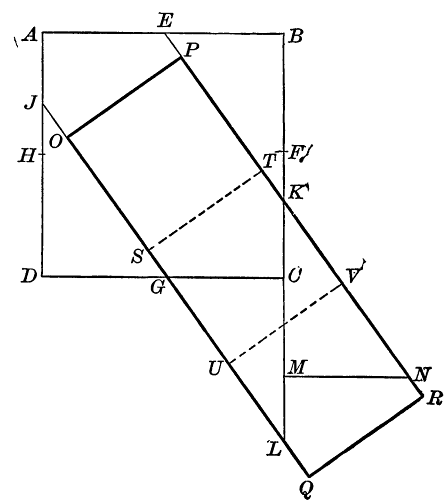
Also solved by NATHAN ALTSHILLER, NELLE L. INGELS and the PROPOSER.

515 (Geometry). Proposed by C. F. GUMMER, Kingston, Ontario.

Show how to cut up a square carpet and make it into three equal square carpets. Estimate the total length of seam in comparison with a side of the original carpet.

I. SOLUTION BY HARRY C. BRADLEY, Massachusetts Institute of Technology.

Construction. Let $ABCD$ be the original square. Take E, F, G, H as the middle points of its four sides. Lay off $BK = DJ$ equal to the distance $EF = GH$. Draw EK and JG, which are obviously parallel. Remove the triangles EBK and JDG, and place them in the positions



GCL and KMN, respectively. In any convenient position draw the line OP perpendicular to JG and EK. Remove the piece $AEPOJ$, and fit it into the position $MNRQL$. The square has now been transformed into the rectangle $OPRQ$, which is three times as long as wide, and may be divided by the lines ST and UV into three equal squares.

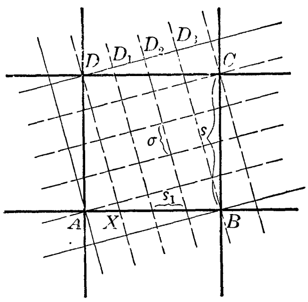
Proof. The area of the rectangle $OPRQ$ is evidently the same as that of the original square, and it is only necessary to prove that the long side, OQ, is three times the short side OP. The side $OQ = JL = JG + GL = JG + EK = 2JG$. To avoid fractions, let the side of the original square equal 6. Then $DG = DH = 3$, $JD = GH$ (by construction) $= 3\sqrt{2}$, and

$$JG = \sqrt{DG^2 + JD^2} = 3\sqrt{3}.$$

Whence, $OQ = 6\sqrt{3}$. The area of the rectangle $OPRQ$ = the area of the square $ABCD = 6 \times 6 = 36$. That is, $OP \times OQ = 36$, or $OP = 2\sqrt{3}$. Hence $OQ = 3OP$.

The Length of Seam. The length of seam, $KL + GC + MN$, is just twice the length of a side of the original square. For $GC + MN$ is the length of a side of the square, since each is equal to half a side, while $KL = KC + CL = KC + BK = BC$.

II. Solution by A. F. Frumveller, Marquette University.



Before attacking this special case, let us deduce the formulas that govern the partition of a square into N squares. Let $ABCD$ be a given square, s its side, and n a number such that $s = ns_1$; in the figure $n = 4$. Draw DX, and complete the figure as indicated. Then we have

$$(1)\quad \overline{DX}^2 = s^2 + \frac{s^2}{n^2} = \frac{s^2}{n^2}(n^2 + 1),$$

$$(2)\quad \overline{DX} = n\sigma + \frac{\sigma}{n} = \frac{\sigma}{n}(n^2 + 1);\quad \text{hence}\quad s^2 = (n^2 + 1)\sigma^2 = N\sigma^2.$$

This gives us the relation between the number of sub-squares, and the length of the fundamental segment on AB which produces this partition; $N = n^2 + 1$. The number of complete sub-squares lying within the given square is evidently the greatest integer in

$$\frac{(n-1)(n^2 - n + 1)}{n},$$

which number is the product of

$$\left(\frac{\overline{AB}}{s_1} - 1\right)\left(\frac{\overline{DX}}{\sigma} - 1\right);$$

the shorter formula conveyed by our figure, $(n^2 + 1) - 4n = (n - 1)^2$, holds only when n is integral.

The number N is an integer not only when n is integral, but likewise when $n = \sqrt{k}$, in which case $N = k + 1$, and

$$s_1 = \frac{\sqrt{k}}{k}s;$$

if $N = 3$, $n = \sqrt{2}$, and

$$s_1 = \frac{s}{2}\sqrt{2},$$

or half the diagonal of the original square. The length of all the cuts made in the given square, when n is rational, is clearly $2n$ times $\overline{DX}$, or

$$2s\sqrt{n^2 + 1};$$

when n is irrational, it will be noted that the last cut parallel to $\overline{DX}$ within the square (call it $\overline{E_rX_r}$) does not reach the baseline, but crosses the side of the square at a point X_r. By similar triangles, we then have

$$\frac{\overline{DX}}{s_1} = \frac{\overline{E_rX_r}}{s - [ns_1]},$$

where the bracket means as usual, "the greatest integer in ns_1"; hence

$$\overline{E_rX_r} = \overline{DX}\left(\frac{s}{s_1} - \frac{[ns_1]}{s_1}\right) = \overline{DX}(n - [n]),$$

and

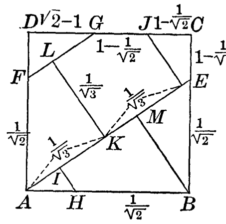
$$\Sigma(\overline{D_iX_i}) = \overline{DX}([n] + (n - [n])) = n \cdot \overline{DX}.$$

Since the construction of the cross-lines is similar, the sum of all the cuts is as before $2s\sqrt{n^2+1}$.

The skew lattice-work of our figure can be readjusted by slipping downwards each of the columns lying between D and C, until the points $D_1, D_2, \cdots, D_r$ rest on $\overline{DC}$; this is optional for n integral; but for n irrational it is *necessary*, in order to have congruent fragments along the sides of the given square, out of which our N sub-squares may be patched up. Thus we find the actual lines on which the original square must be partitioned. (This problem, for $N = 3$, is solved as above in Sundara Row's "Geom. Exercises in Paper Folding," but without any hint as to how the solution was arrived at.)

The patched squares have "seams" across them, whose total length is $2s$, since two equal segments from the sides combine to form one seam.

III. Solution by the Proposer.



Join AF to BE, GJ to AH, HB to JC and DG, CE to DF.

$$\text{Total seam} = AF + AH + HB + CE$$

$$= \left(\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}\right) AB = 2AB.$$

Also solved by E. B. Escott.

426 (Calculus). Proposed by C. N. SCHMALL, New York City.

If A be the area of a plane triangle constructed with the sides a, b, c, such that

$$a^3 + b^3 + c^3 = 3k^3,$$

show that the maximum value of A is $\frac{1}{4}k^2$.

Solution by Elijah Swift, University of Vermont.

Using the Heronian formula for the area of a triangle in terms of the sides, we have

$$16A^2 = -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2.$$

Clearly we may make $16A^2$ a maximum instead of A.

By Euler's method of solving an isoperimetric problem, we must equate to zero the partial derivatives with respect to a, b, c, of $16A^2 - 4\lambda(a^3 + b^3 + c^3 - 3k^3)$. (For convenience I use -4λ instead of λ.) In this way we get the equation

$$-4a^3 + 4ab^2 + 4ac^2 - 4\lambda(3a^2) = 0,$$

and two more similar to it. Dividing by $4a$, which cannot be zero,

$$(1) \qquad a^2 + 3\lambda a = b^2 + c^2$$

and two equations, (2) and (3), obtained by permuting cyclically the letters a, b, c in (1).

But from (1) a must satisfy the equation

$$(4) \qquad 2x^2 + 3\lambda x = a^2 + b^2 + c^2,$$

and (2) and (3) show that b and c are roots of the same equation. Since this equation has only one positive root, a, b, c must all be equal, and the triangle equilateral, whence it appears that its area must be $\frac{1}{4}k^2\sqrt{3}$, which is not the value given in the statement.

The same method used here shows that this result holds when the auxiliary condition is $a^n + b^n + c^n = 3k^n$, $n > 1$.

It is evident from the conditions of the problem that the values found give a maximum.

Also solved by R. A. Johnson, S. A. Corey, and J. B. Reynolds.

427 (Calculus). Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, Ohio.

Of all ellipses circumscribed about a given parallelogram, the minimum [maximum], with regard to area, has as conjugate diameters the diagonals of the parallelogram.

SOLUTION BY HENRY D. THOMPSON, Princeton, N. J.

Let the given parallelogram be $PQP'Q'$, with $QP = 2h$, and $Q'P = 2k$, let the diagonals $P'P$, $Q'Q$ intersect at O, and through O take the x-axis parallel to, and positive in, the direction of QP and $P'Q'$, and the oblique y-axis parallel to, and positive in, the direction $Q'P$ and $P'Q$, and let the angle of the axes be ω. Take a representative ellipse through $PQP'Q'$ cutting the positive x-axis in the point U, with the coördinates (u, o), and the y-axis in the point $V(o, v)$. Then the equation of this ellipse is

$$\frac{x^2}{u^2} + \frac{y^2}{v^2} = 1, \tag{1}$$

and since it passes through $P(h, k)$ the relation holds

$$\frac{h^2}{u^2} + \frac{k^2}{v^2} = 1. \tag{2}$$

The area for the ellipse is

$$uv\pi \sin \omega, \tag{3}$$

and this is a minimum when

$$M = uv \tag{4}$$

is a minimum. Differentiation of (4) and (2) gives for the minimum $dM/du = v + udv/du = 0$, with $-h^2/u^3 - (k^2/v^3)dv/du = 0$, and elimination of dv/du gives $k^2/v^2 = h^2/u^2 = \frac{1}{2}$ from (2), whence $u = h\sqrt{2}$, $v = k\sqrt{2}$. These values set in d^2M/du^2 give $4k/h$, a positive magnitude, and the area of the ellipse is a minimum. This ellipse is $x^2/2h^2 + y^2/2k^2 = 1$, and the tangents at the vertices evidently intersect on the diagonals $P'P$ and $Q'Q$, which are thus conjugate, since OU and OV are conjugate.

The theorem can be proved without the use of the calculus as follows: The coördinates of a point on (1) can be expressed by means of a parameter φ by the equations $x = u \cos \varphi$, $y = v \sin \varphi$ which for the point $P(h, k)$ become $h = u \cos \varphi_1$, $k = v \sin \varphi_1$, where φ_1 changes with u and v. Then (4) becomes $M = hk/\sin \varphi_1 \cos \varphi_1 = 2hk/\sin 2\varphi_1$, which has the single minimum when $\sin 2\varphi_1$ is a maximum, that is unity, or when $\varphi_1 = \pi/4$, or when $u = h\sqrt{2}$, $v = k\sqrt{2}$, as before.

Also solved by MARIE WHELAN, J. B. REYNOLDS, and FLORENCE P. LEWIS.

Several solutions of Mechanics problem 339 will be printed in an early issue. EDITORS.

340 (Mechanics). Proposed by PAUL CAPRON, U. S. Naval Academy.

A rigid straight line l passes through a fixed point O, but is otherwise free to move in a plane. If C is the instantaneous center of rotation for l, prove that CO is always perpendicular to l and that, if (O being used as pole) $\rho = f(\theta)$ represents the locus of any point P on l, OC is always equal to $(d/d\theta)f(\theta)$.

SOLUTION BY S. W. REAVES, University of Oklahoma.

If we know the directions of motion at a given instant of two points of the moving line, then the center of rotation for that instant is to be found at the intersection of the normals drawn at these points to their directions of motion. (See Ziwet, *Theoretical Mechanics*, Art. 23; Demartres, *Cours de Géométrie infinitésimale*, Art. 20.)

In the problem before us the direction of motion of the point of l, which at the given instant coincides with O, is along l. Hence the normal to l at O must pass through C.

Again, the normal at P to the path of P, $\rho = f(\theta)$, must also pass through C. Hence the angle OPC is the complement of the angle ψ between OP and the tangent at P. Hence,

$$\tan OPC = \cot \psi = \frac{d\rho}{\rho d\theta}.$$

(See any book on Calculus.) Therefore, from the right triangle COP, we have

$$OC = \rho \tan OPC = \frac{d\rho}{d\theta} = \frac{d}{d\theta} f(\theta).$$

Also solved by J. B. Reynolds and W. E. Cederberg.

257 (Number Theory). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Find a general expression for the number of positive integers from 1 to 10^t, inclusive, every one of which contains the figure 9 exactly r times $(0 \leqq r \leqq t)$.

Solution by the Proposer.

In the case of the integers from 1 to 10, we have nine which do not contain the figure 9 and one which contains one 9. This shall be indicated by the expression $9 + 1$.

In the case of 10^2, the number of integers, which do not contain 9, is $9 \cdot 9$, or 9^2; which contain one 9, is $9 \cdot 1 + 9$, or $2 \cdot 9$; which contain two 9's, is 1, and we have the expansion of

$$(9 + 1)^2 = 9^2 + 2 \cdot 9 + 1.$$

For 10^3, we have $9 \cdot 9^2$, $9 \cdot 2 \cdot 9 + 9^2$, $9 \cdot 1 + 2 \cdot 9$, and 1, or $9^3 + 3 \cdot 9^2 + 3 \cdot 9 + 1$.

Then, for 10^k, assume the expansion of $(9 + 1)^k$, or

$$9^k + \binom{k}{1} 9^{k-1} + \binom{k}{2} 9^{k-2} + \cdots + \binom{k}{n-1} 9^{k-(n-1)} + \binom{k}{n} 9^{k-n} + \cdots + \binom{k}{k-1} 9 + 1.$$

For 10^{k+1} we reason as follows: The number of integers which do not contain 9 is $9 \cdot 9^k$, or 9^{k+1}; which contain one 9, is $9 \cdot \binom{k}{1} 9^{k-1} + 9^k$, or $\binom{k+1}{1} 9^k$; which contain two 9's, is $9 \cdot \binom{k}{2} 9^{k-2} + \binom{k}{1} 9^{k-1}$ or $\binom{k+1}{2} 9^{k-1}$, and which contain n 9's, is

$$9 \cdot \binom{k}{n} 9^{k-n} + \binom{k}{n-1} 9^{k-(n-1)} = \left[\binom{k}{n} + \binom{k}{n-1}\right] 9^{k-n+1} = \binom{k+1}{n} 9^{k+1-n}.$$

Hence, we have, for 10^{k+1}, the expansion of $(9 + 1)^{k+1}$, or

$$9^{k+1} + \binom{k+1}{1} 9^k + \cdots + \binom{k+1}{n} 9^{k+1-n} + \cdots + \binom{k+1}{k} 9 + 1.$$

Now, the derived expression holds for $k = 2$ and for $k = 3$; hence it holds for all positive integral values of k.

Therefore, the general expression required is $\binom{t}{r} 9^{t-r}$.

Also solved by Horace Olson, H. C. Feemster, C. C. Yen, and N. P. Pandya.

258 (Number Theory). Proposed by A. A. BENNETT, University of Texas.

Find a recursion formula in terms of binomial coefficients for a_n, where the a's are defined by the condition that the persymmetric determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot \\ a_2 & a_3 & \cdot & \cdot & \cdot \\ a_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

are each equal to unity for every positive integer n.

Solution by C. F. Gummer, Queen's University, Kingston, Ont.

Though this solution does not directly involve binomial coefficients, yet by finding the value of a_n it may be considered to dispose of the problem sufficiently.

The given conditions show that $a_0 = a_1 = 1$, and that the other a's may be found in succession uniquely from equations in which they appear with the coefficient unity. The a's being determinate there exists a sequence $x_0, x_1, \cdots$ such that

$$(1) \qquad a_n = a_{n-1}x_0 + a_{n-2}x_1 + \cdots + a_0x_{n-1}, \qquad n = 1, \ , \cdots;$$

for the first n equations of (1) have a determinant equal to unity.

If we apply the substitutions (1) to the last row of

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n+1} & \cdot & \cdot & a_{2n} \end{vmatrix},$$

and simplify by means of the other rows, the last row becomes

$$0,\ a_0x_n,\ a_0x_{n+1} + a_1x_n,\ a_0x_{n+2} + a_1x_{n+1} + a_2x_n,\ \cdots.$$

With similar treatment, the preceding row becomes

$$0,\ a_0x_{n-1},\ a_0x_n + a_1x_{n-1},\ \cdots,$$

and so for all but the first row. On simplifying by columns we get, since $a_0 = 1$,

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdot & \cdot \\ x_2 & x_3 & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n-1} \end{vmatrix} = 1.$$

A like treatment of the other determinant gives

$$\begin{vmatrix} x_0 & x_1 & x_2 & \cdot & \cdot \\ x_1 & x_2 & \cdot & \cdot & \cdot \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n} \end{vmatrix} = 1.$$

Hence, $x_2, x_3, \cdots$ are defined in terms of x_0, x_1 in the same way as $a_2, a_3, \cdots$ in terms of a_0, a_1.

Also $a_0 = a_1 = 1$, $a_2 = 2$.

Hence, $x_0 = x_1 = 1$, $x_2 = 2$, by direct calculation.

Hence, $x_n = a_n$. Hence, (1) becomes

$$(2) \qquad a_n = a_{n-1}a_0 + a_{n-2}a_1 + \cdots + a_0a_{n-1}, \qquad n = 1, 2, \cdots.$$

To calculate a_n, we infer from (2) that the coefficient of t^n in $u \equiv a_0 + a_1t + a_2t^2 + \cdots$ is equal to the coefficient of t^{n-1} in u^2, when $n = 1, 2, \cdots$.

Hence, $(u - 1)/t = u^2$.

Hence, $u = 1/2t - \sqrt{1 - 4t}/(2t)$, the minus sign being necessary to make u a series in positive powers of t.

Hence, the coefficient a_n of t^n in u equals $\dfrac{|\underline{2n}}{|\underline{n}\ |\underline{n+1}}$.

the first row of (A)]; $1^2 + 6^2 + 10^2 + 18^2 = 461$, $2^2 + 6^2 + 12^2 + 15^2 = 409$, $3^2 + 7^2 + 13^2 + 17^2 = 516$, $4^2 + 8^2 + 14^2 + 16^2 = 532$ [461, 409, 516 and 532 being the numbers in the first row of (B)].

We note above that the sum of the natural numbers not used in (A) is 25 and that 10 is used more than the usual number of times. Similarly in (B) the numbers 25 and 6 are exceptional numbers. Were we inclined to be superstitious we might therefore say that October 25 (10–25) and June 25 (6–25) are dates of transcendent importance in 1918, say, the date of the signing of the treaty of peace, and the date of cessation of fighting in the world war.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

It is designed that this new department of the MONTHLY shall record the details of organization and activity of each undergraduate mathematics club in America, shall serve as a medium of communication between such clubs, and shall contain information helpful and suggestive for their guidance, and for the development of their usefulness.

As soon as the information can be collected a list of the clubs will be published. To 35 clubs already discovered appeal has been made for details, a portion of which will be given in connection with this list. It is earnestly desired that every other club shall make itself known to the editor, reporting on such things as: (1) the club's exact name, (2) the date (year and month) of organization, (3) the club's object and those eligible for membership, (4) the names of the officers of the club for 1917–18, (5) the number of members of the club, the average attendance, and the number of meetings held each year, (6) the dates of meetings in 1917–18, the titles of papers read and the names of the speakers. The editor would like to receive also copies of all printed programs for the current and earlier years.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, 1917–18.

Membership is open: (1) to those who have had or are taking a course in analytic geometry; (2) to the members of the freshman class who win the prizes awarded in connection with the competitive examination on original problems in entrance mathematics; and (3) to mathematical students in the second semester of the freshman year who have attained to the higher honor grade in a mathematical course of the first semester. Average attendance: 52.

Chairman: Professor N. F. Davis.

Committee on Program: Professor R. G. D. Richardson, Marion R. Luther Gr., Mary I. Briggs '18, Clarence R. Adams '18, James B. Hobbs '18.

Committee on Arrangements: Doctor T. H. Brown, Lydia L. Cooper Gr., Esther E. Brintzenhoff '19, Albert S. Pratt '18, Chauncey D. Wentworth '20.

The program for the year 1917–18 is as follows:

November 16: "Geometric Exercises in Paper Folding" by May Sperry '18; "Mathematical Fallacies" by Albert S. Pratt '18;

December 14: "Growth and Form" by Mary I. Briggs '18; "Secrets of Lightning Calculators" by James B. Hobbs '18;

January 11: "Nomography" by Professor E. V. Huntington of Harvard University;

February 15: "Mathematics of Warfare" by Mary E. Carroll '19; "Probability Curves" by Clarence R. Adams '18; "The Development of Mathematical Symbolism" by Marion R. Luther Gr.;

March 15: "The Russian Peasant Method of Multiplication and the Binary Scale of Notation" by Elsie M. Flint '18; "Non-Euclidean Geometry" by Professor H. P. Manning;

April 19: "Some Philosophical and Psychological Implications of the Game of Nim" by E. B. Delabarre, Professor of Psychology in Brown University;

May: Picnic.

The Junior Mathematical Club of the University of Chicago.

Professor Slaught has kindly furnished most of the following notes:

"This club was organized by students in the autumn of 1905 chiefly through the instrumentality of N. J. Lennes who was then a graduate student at the university. Its purpose was to afford an opportunity to all students primarily interested in mathematics, whether undergraduates or graduates, to present before a sympathetic audience the results of reading and investigation, especially along lines not likely to be included in regular class work. The intention, for the most part well carried out, has been, and is, to keep these papers strictly elementary as compared with the papers presented in the Graduate Mathematical Club of the university where only the results of original research are given. In fact, the chief incentive to organizing this Junior Club was the great discouragement experienced by even the younger graduate students (not to mention the undergraduates) in trying to be interested and to look wise while attending the meetings of the Graduate Club.

"The meetings of the Junior Club occur every second week during the year from October to June, alternating with those of the Graduate Club. They rarely last for more than an hour and a quarter, the half hour preceding the presentation and discussion of papers being usually devoted to friendly intercourse encouraged by a cup of tea.

"The chief results of these meetings, looked at in perspective over a decade, are: (1) Accumulation of information by the members in many lines that they might otherwise not find time or inclination to look up for themselves; (2) stimulation to activity on the part of individuals who might otherwise be content to do the required class-room work and nothing more; (3) cultivation of independence in study, and ease and clearness in presenting the results of study to an audience; (4) preparation of students either for teaching or for further advanced study,

in many ways which would not otherwise be realized in the regular college and university work.

"The average number of members is somewhere between fifteen and twenty. There is no constitution and no requirement for membership other than a devotion to mathematics and a desire to commune with others having the same devotion. The minutes of the first meeting record that 'Professor Slaught was made an ex officio member with the power of veto' to make sure, I suppose, that things would go straight. I have almost always given the opening address at the autumn meeting.

"It may be of interest to add that the first president of the Junior Club was W. R. Longley, now professor at Yale, and that the first program committee included G. D. Birkhoff, now professor at Harvard, and Dr. H. F. McNeish, now of the De Witt Clinton High School, New York."

Officers, 1917–18: President, E. P. Lane; vice-president, C. Gouwens; secretary, Minna Schick.

The program committee consists of the officers and of I. A. Barnett, W. G. Simon and Gladys Gibbons.

All officers and members of the committee are graduate students.

The following is the program for the fall quarter, 1917:

October 12: Election of officers. "Purposes and Aims of the Club" by Professor Slaught.

October 24: "G. Cantor" by Minna Schick; "Notion of Number" by I. A. Barnett;

November 7: "Unique Factorization in the Quadratic Realm" by C. Gouwens;

November 21: "Finite Projective Geometries" by Cyril A. Nelson, Jr.;

December 5: "History of the Teaching of Collegiate Mathematics in the United States" by W. G. Simon.

The Mathematics Club of the University of Kansas, 1917–18.

Membership is open only to those who have had a three-hour year course in calculus. Average attendance about 19.

Officers: President, Mildred Abel '18; vice-president, Eran Burgert '18; secretary, Frances Adams '18; faculty adviser, Professor U. G. Mitchell.

Program Committee: Helen Garman '18, Mary Smith '18, and Edward Buffington '18.

The program for 1917–18 is as follows:

October 8: "Air Planes" by Professor Lefschetz;

October 22: "Poincaré's Non-euclidean World" by Mildred Abel '18;

October 31: "Fortunes told mathematically" by Professor Van der Vries;

November 12: "The Number Zero" by Helen Wedd '18; "The Number π" by Beulah Armstrong Gr.;

November 26: "Feuerbach's Circle" by Mary Smith '18; "A Probability Machine" by Anna Marm Gr.;

December 10: "Mathematics of the War" by Eran Burgert '18;

January 7: "Numerical Properties of Color and Sound" by Edward Buffington '18;

January 21: "Mathematical Fallacies" by Sarah Bingham '18; "Card Tricks" by Edith Whitcher '19;

February 11: "Mathematics as a Fine Art" by Georgia Beebe '18; "Mathematical Accomplishments of Women" by Irma Leon '19;

February 25: "History of Time Pieces" by Wealthea Babcock '19; "History of the Metric System" by Sadie Horsley Gr.;

March 11: "Zeno's Paradoxes" by Frances Adams '18;

March 25: "Addition and Subtraction by Logarithms" by E. B. Miller (Instructor);

April 8: "Mathematicians who became Famous in other Fields" by J. H. Hoover Gr.;

April 22: "Regular Polygons inscribable in a Circle" by Goldie Piper '19;

May 12: "The Beginnings of Higher Mathematics in England" by Faye Doddridge '19; "The Beginnings of Mathematics in the United States" by Helen Garman '18;

May 26: Annual Picnic.

TOPICS FOR CLUB PROGRAMS.

1. The Oldest Mathematical Work Extant.

This is the heiratic papyrus said to have been one of a number found at Thebes in the ruins of a small building near the Ramesseum. It was purchased at Luxor in 1858 by A. Henry Rhind and after his death it passed into the hands of a gentleman from whom it was purchased by the trustees of the British Museum in 1864.

The papyrus was copied by a scribe named Ahmes, between 2000 B. C. and 1700 B. C., from an older work. The text comprises a series of propositions or problems in arithmetic, mensuration, trigonometry, and in the various branches of practical geometry, sometimes accompanied by diagrams, representing the class of practical mathematical knowledge which an overseer of royal farms, or a revenue officer or the master-mason employed in building a pyramid or temple, would be expected to possess.

Any approximately complete bibliography of the papyrus should contain more than forty titles. Three may be regarded as referring to works which are fundamental in connection with its study. These are: (1) The facsimile in original colors published by the British Museum in 1898 (21 plates, 15 x 25 in.) with a preface from which most of the details mentioned above have been taken; (2) the translation into German, and commentary, prepared by the Heidelberg aegyptologist August Eisenlohr with the aid of two mathematicians, his brother Friedrich and Moritz Cantor; the second edition, without plates (a work in quarto format of nearly 300 pages), published by Hinrich's Buchhandlung in Leipzig under the title: *Ein mathematisches Handbuch der alten Aegypter* (*Papyrus Rhind des British Museum*); (3) the discussion of Eisenlohr's

work by L. Rodet, "Les prétendus problèmes d'algèbre du Manuel du Calculateur égyptien," in *Journal asiatique*, Paris, 1881–82, série 7, tome 18; these articles were reprinted in 1882 (122 pages).

Three references may be given to historical works: (1) M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Band I, 3. Auflage, Leipzig, Teubner, 1907, pages 57 ff.; (2) S. Günther, *Geschichte der Mathematik*, I. Teil, Leipzig, Göschen, 1908, pages 24–35; (3) W. T. Sedgwick and H. W. Tyler, *A Short History of Science*, New York, Macmillan, 1917, pp. 30–34.

In English there are also the papers by (1) F. L. Griffith, in *Proceedings of the Society of Biblical Archaeology*, London, volumes 13 and 16, 1891, and 1894; and (2) G. A. Miller, in *School Science and Mathematics*, Chicago, volume 5, 1905.

In French: (1) V. V. Bobynin, (a) "Sur le procédé employé dans le papyrus Rhind pour réduire les fractions en quantièmes," *Bibliotheca Mathematica*, Leipzig, 2. Reihe, Band 4, 1890; (b) "Méthode expérimentale dans la science des nombres et principaux résultats obtenus," *L'Enseignement Mathématique*, Paris-Genève, tome 8, 1906; (2) G. Milhaud, *Nouvelles études sur l'histoire de la pensée scientifique*, Paris, Alcan, 1911, pp. 58–66; (3) L. Brunschvicg, *Les étapes de la philosophie mathématique*, Paris, Alcan, 1912, pp. 26–32.

2. Geometrography and other Methods of Measurement of Geometrical Constructions.

As far back as 1833 Steiner wrote a passage, often quoted, regarding the desirability of an investigation as to the simplicity and exactitude of geometric constructions;[1] but it is only within the past thirty years that theories along these lines have been developed.

Geometrography may be defined, in the words of its inventor Émile Lemoine (1840–1912), as "the art of geometrical constructions." Its aim is to discover which of the various ways of solving a problem is the simplest, or, in other words, which way requires us to perform the smallest number of operations.

Lemoine's first memoir on the subject was read before the French Association for the Advancement of Science in 1888 and during the next twenty years he published more than thirty papers and notes on the subject. The theory is pretty well summed up in his little book: *Géométrographie ou art des constructions géométriques* (Scientia no. 18). Paris, Gauthier-Villars, 1902.

Reference may also be given to: (1) Lemoine's papers on a geometrographic comparison of twelve constructions deduced from eleven solutions of the same problem in *Comptes rendus de l'Association Française pour l'Avancement des Sciences*, 1899 and 1900; to (2) Lemoine's revision of Note IV, "Sur la géométrographie," pages 517–548 of Rouché et Comberousse, *Traité de géométrie*, tome I, 8e éd., Paris, Gauthier-Villars, 1912; to (3) E. Haentzschel, "De l'exactitude des constructions géométriques," *L'Enseignement Mathématique*, tome 9, 1907,

[1] *Die geometrischen Constructionen ausgeführt mittelst der Geraden Linie und eines festen Kreises*, Berlin, 1833, § 19.

pp. 45–51; and to (4) recent volumes of *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht,* which had a department, under the editorship of K. Hagge, devoted to the discussion of problems in geometrography.

For extension of the considerations to space, see Lemoine's papers: (1) "La géométrographie dans l'espace," *Comptes rendus de l'Academie des Sciences,* Paris, vol. 131, 1900; (2) "Géométrographie dans l'espace ou stéréométrographie," *Comptes rendus de l'Association Française pour l'Avancement des Sciences,* 1900.

Among English writings the following may be noted: (1) J. S. MACKAY, "The Geometrography of Euclid's Problems," *Proceedings of the Edinburgh Mathematical Society,* vol. 12, 1894, pp. 2–16; (2) J. L. COOLIDGE, *A Treatise on the Circle and the Sphere,* Oxford, Clarendon Press, 1916, pp. 166–179; (3) H. P. HUDSON, *Ruler and Compasses,* London, Longmans, 1916, pp. 112–117; (4) C. E. YOUNGMAN, "On Two Constructions for the Regular 17-Side," *Mathematical Questions and Solutions from "The Educational Times,"* new series, vol. 10, 1906, pp. 55–56; and (5) R. F. MUIRHEAD, "Constructions with Straight-edge and Dividers," *Mathematical Gazette,* London, 1905, vol. 3, pp. 209–211.

Other systems of measurement are described in A. GRÜTTNER, *Die Grundlagen der Geometrographie,* Leipzig, Quelle und Meyer, 1912; in K. ROHN und E. PAPPERITZ, *Lehrbuch der darstellenden Geometrie,* Band I, 4. Auflage, Leipzig, Veit, 1913, pp. 486–493 and 501–502; and in *Encyklopädie der mathematischen Wissenschaften,* Band III1, Heft 4, Leipzig, Teubner, 1910, pp. 528–531.

The issue of the MONTHLY for February, 1896, contained a portrait of Lemoine and a biographical sketch by D. E. Smith. Reference is made to the influence which he exerted in the realm of music through his celebrated soirées, "La Trompette." (*Cf.* L. AUGÉ DE LASSUS, *La Trompette. Un demi-siècle de musique de chambre.* Paris, Delagrave, 1911.)

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Mr. D. R. BELCHER has been appointed instructor in mathematics at Adelbert College, Western Reserve University.

At Syracuse University, Associate Professor F. F. DECKER has been promoted to a professorship of mathematics, and Dr. J. L. JONES has been promoted to an assistant professorship.

Following the death of President P. W. MCREYNOLDS, of Defiance College, Defiance, O., Professor A. G. CARIS, of the department of mathematics, was made acting president of the College.

At the University of Maine, Mr. M. F. JORDAN and Mr. Q. STAUFFER have been appointed instructors in mathematics; the department of mathematics has united with the military department in offering courses in navigation.

Large numbers of teachers of mathematics in secondary schools and colleges have enlisted in the military service since the opening of the present academic year. It has been the plan of the editors of the MONTHLY to record as many of these enlistments as possible, especially the members of the Association. The editor of the "Notes and News" columns of the MONTHLY welcomes all notices sent him concerning the military service of any member of the Association, or other mathematicians.

Mr. GILBERT THAYER, a charter member of the Association, has been commissioned first lieutenant in the aviation section of the Signal Officers' Reserve Corps, in California. At the University of Maine, Associate Professor L. J. REED has been granted leave of absence to serve as statistician to the War Trade Board in Washington. Dr. J. M. STETSON, instructor in mathematics at Adelbert College of Western Reserve University, has resigned to enter the national army. Dr. A. H. NORTON, professor of mathematics and vice-president of Elmira College, N. Y., and president of the Syracuse section of the Mathematics Teachers of the Middle States and Maryland, has been granted leave of absence by Elmira College and sailed for France in December to engage in Y. M. C. A. work. Professor W. MARSHALL, of Purdue University, is serving under the Food Administration as chief statistician of the International Sugar Committee with headquarters in New York City.

The following mathematical papers were presented at the October meeting of the Cambridge Philosophical Society: "The convergence of certain multiple series," by G. H. HARDY; "Bessel functions of large order," by G. N. WATSON; "A particular case of a theorem of Dirichlet;" "Ramanujan's empirical expansion of modular functions," by L. J. MORDELL; "Extension of Abel's theorem and its converses," by A. KIENAST.

Parts 3 and 4 of Vol. 16, *Proceedings of the London Mathematical Society*, recently issued, contain the following: "Bessel functions and Kapteyn series" (continued from part 2), by G. N. WATSON; "On non-absolutely convergent, not necessarily continuous, integrals" and "On multiple integration by parts and the second theorem of the mean," by W. H. YOUNG; "On the classification of the integrals of linear partial differential equations of the first order," by M. J. M. HILL; "A problem on Diophantine approximation," by H. T. J. NORTON; "A symmetrical relation between Legendre's functions with parameters cosh α, sinh α," by F. J. W. WHIPPLE; "On two theorems of combinatory analysis and some allied identities," by L. J. ROGERS. At the regular meeting of the London Mathematical Society held on November 1 under the presidency of Professor H. M. MACDONALD, the following papers were presented: "Tetrahedra in relation to spheres and quadrics," by J. H. GRACE; "The continuation of the hypergeometric series," by M. J. M. HILL; "Restricted Fourier series and the convergence of power series," by W. H. YOUNG; "Invariants and covariants of linear homogeneous differential equations," by E. B. STOUFFER; "The simultaneous system of two quaternary quadratic forms," by H. W. TURNBULL.

Among the multitude of scientific papers presented at the meetings of the Paris Academy of Science during the past six months are the following on mathematical subjects: "Surfaces such that the Laplace equation of the network formed by the lines of curvature is integrable," by C. GUICHARD; "A generalization of Taylor's series," by G. D. BIRKHOFF; "Binary indeterminate conjugate forms remaining invariant under a group of linear substitutions," by G. JULIA; "Fourier-Bessel transcendentals with several variables," by M. AKIMOFF; "The convergence of conjugate trigonometrical series," by J. PRIWALOFF; "The development in a continued fraction of a quadratic irrational," by M. AMSLER; "The measurement of linear ensembles," by M. LEAU; "The continued fraction of Stephen Smith" and "Some properties of binary indefinite quadratic forms," by G. HUMBERT; "The classification of the transcendental points of the inverse of integral or meromorphic functions," by G. REMOUNDOS; "Orthogonal surfaces," by H. DUPORT; "The pisiform surface," by P. HUMBERT; "The notion of neighborhood in abstract ensembles," by M. FRÉCHET; "A new method in the numerical evaluation of the coefficients of a series," by M. PETROVITCH; "A new method in the summation of trigonometrical series," by M. ANGELESCO; "The theory of trigonometrical series," by W. H. YOUNG; "Abelian functions," by G. SCORZA; "The integration of certain systems of differential equations," by E. GOURSAT; "The triple cyclic systems of Steiner," by S. BAYS; "A functional equation and spherical unicursal curves," by W. DETAUNENBERG.

The volumes of the *Proceedings of the Edinburgh Mathematical Society* are issued annually in two parts, the first in May and the second in November. The recently completed volume 35, issued by the aid of a grant from the Carnegie Trustees, contains the following papers extending over 106 pages: By H. DATTA, (1) "On the theory of continued fractions"; (2) "On the failure of Heilemann's theorem"; by L. R. FORD, (1) "On a class of continued fractions"; (2) "A geometrical proof of a theorem of Hurwitz"; (3) "The effect of a rise in prices upon the amount of small money used"; by G. B. JEFFERY, "Transformations of axes for Whittaker's solution of Laplace's equation"; by W. P. MILNE, (1) "The apolar locus of two tetrads of points"; (2) "The co-apolars of a cubic curve"; by G. D. TAYLOR, "Geometrical illustrations of cyclant substitutions"; by E. T. WHITTAKER, "On the latent roots of compound determinants and Brill's determinants"; by A. W. YOUNG, "On the computation of a Lagrangian interpolation."

In addition to the *Proceedings* the Edinburgh Mathematical Society has published also 21 numbers of *Mathematical Notes, a Review of Elementary Mathematics and Science.* The last number (pages 239–254) was published in December, 1916.

Professor F. R. MOULTON, University of Chicago, is Secretary of Section A (Mathematics and Astronomy) of the American Association for the Advancement of Science. His absence from the holiday meetings at Chicago was due to his official duties at Pittsburgh.

The twenty-fourth annual meeting of the American Mathematical Society was held in New York on December 27–28, 1917. There were eleven papers on the printed program, the authors representing seven colleges and universities.

The ninth regular meeting of the American Mathematical Society at Chicago, being the fortieth meeting of the Chicago Section, was held at the University of Chicago on December 28–29, 1917, in conjunction with the third annual meeting of the Mathematical Association of America. There were eighteen papers on the printed program, the authors representing nine colleges and universities. Two additional papers were also read.

Former Captain STUART C. GODFREY, Corps of Engineers, West Point, New York, is now Major Godfrey and is located at Fort Leavenworth, Kan.

A booklet on the "History of Limits and Fluxions in England during the Eighteenth Century," by Professor FLORIAN CAJORI, is now in process of publication by the Oxford University Press, having been delayed since 1915 by the exigencies of the war.

A series of public addresses in the state of Montana on the nations concerned in the great war was given during the Red Cross campaign by members of the faculty of the University of Montana, and the proceeds were devoted to the Red Cross. The address on Scandinavia was given by Professor N. J. LENNES, head of the department of mathematics.

The Central Association of Science and Mathematics Teachers was organized in Chicago in 1902. The membership has steadily grown from 324 in 1908 to 973 in 1917. The last annual meeting was held in Columbus, Ohio. An important feature of the program was the report of the Committee on Mathematical Requirements by the chairman, Mr. J. A. FOBERG, of Crane Junior College, Chicago. Mr. Foberg is the representative of the Central Association on the National Committee of the Mathematical Association of America.

The secretary has had specific inquiries for the following back numbers of the AMERICAN MATHEMATICAL MONTHLY. It will confer a favor upon various subscribers who are seeking to complete their files of this journal if any who have these issues for sale will inform the secretary of the ASSOCIATION, giving the price desired, including the necessary postage. The secretary's office will be glad to effect the exchange in such cases if the price is satisfactory to the subscribers. (The numbers called for by the University of Wisconsin in the last issue of the MONTHLY have been supplied.)

Vol. I, 1894—Jan., Feb., Mar., June, and July.
Vol. II, 1895—June and Nov.
Vol. III, 1896 (at least all issues except Jan., Feb., and June).
Vol. IV, 1897—Jan., Feb., Apr., June, and July.
Vol. V, 1898—March and May.
Vol. VI, 1899—May, September, and November.
Vol. VII, 1900—April and May.

Vol. VIII, 1901—November.
Vol. IX, 1902—May and June–July.
Vol. XI, 1909—April.
Vol. XIV, 1907—Jan. and Oct.
Vol. XIX, 1912—June and July.
Vol. XX, 1913—Oct.

Professor W. D. MacMillan, who spent the autumn in Texas, has now returned to his work at the University of Chicago in the department of astronomy. Professors E. J. Wilczynski and H. E. Slaught are out of residence during the winter quarter, and Professor L. E. Dickson has returned from his sojourn at the University of California.

NOTES ON THE THIRD ANNUAL MEETING OF THE ASSOCIATION.

By H. E. SLAUGHT, University of Chicago.

The attendance at the third annual meeting was most gratifying, in view of war conditions. The total number present at the various sessions was 119, including 93 members. Members from a great distance were warmly welcomed, including Professors Cajori from Colorado, Dodd and Ettlinger from Texas, Huntington, Phillips, and Ransom from Massachusetts, Archibald from Rhode Island, and Kingston from Winnipeg.

The joint dinner on Thursday evening with the Chicago Section of the Society was attended by seventy-three persons and proved most enjoyable. The Quadrangle Club is admirably adapted for such occasions, providing ample room and complete seclusion, with full opportunity for social intercourse both before and after the dinner and, indeed, throughout the progress of the meetings. President Cajori presided and short talks were given by members representing various parts of the country. A matter of deep interest was the Secretary's roll-call of mathematicians in the national service, a manifestly incomplete list, which, however, was supplemented by numerous additions on the spot, and which it is desired to make as complete as possible for early publication in the Monthly.

As was the case at the New York meeting a year ago, there was much sustained enthusiasm over the election of officers which was closed at the business meeting late Friday afternoon. The contest was very close in some cases, especially in connection with the Council. The detailed figures will be given in the Secretary's official report. The result was as follows: E. V. Huntington, President; D. N. Lehmer and J. W. Young, Vice-Presidents; W. D. Cairns, Secretary-Treasurer; Florian Cajori, Elizabeth B. Cowley, G. A. Miller, and E. J. Wilczynski, members of the Council to serve till January 1920. The Secretary's figures will show that the total vote was not as large as one would suppose it should be. Probably

many have not yet fully realized that these are *real* elections and that each and every member has a voice in the determination of the affairs of the Association. Why should we not have at least 1,000 votes at these annual elections?

The program was pronounced on every hand to be excellent. It was the first occasion when voluntary papers were called for by the Committee, but it is understood that henceforth this is to be an integral part of all programs. This announcement is made thus early in order that all may be aware of the privilege and that many may take advantage of the coming months to prepare such papers for submission to the next program committee for approval long in advance of the next meeting. On this occasion the time was very short, too short, but evidently those who responded to the call had been preparing for emergencies—at any rate, the result was most gratifying. The programs of all the sectional meetings are evidently conducted entirely on this basis and with great success—indeed, the sectional meetings should prove to be a fruitful source for the discovery of good papers for the national programs.

The address of Professor Roever proved to be inspiring and the prolonged discussion which it aroused led to the expression of a desire to organize the college teachers of descriptive geometry as a special division of the Association—at present they have no organization and are laboring under the dilemma that they are recognized neither by the drawing departments nor by the mathematical departments. The feeling prevailed that we in the Association should welcome them into full mathematical fellowship and steps will soon be taken to invite them into the membership of the Association.

The financial report of the Secretary-Treasurer showed a safe balance on the 1917 business of the Association. Whether this will be maintained during 1918 depends upon the loyalty and support of the present membership. There will be the temptation—even the pressure—in the case of many to economize by dropping memberships in some of the scientific associations, but now is the time, of all times, when such organizations should be supported at all hazards—even at personal sacrifice on the part of individuals. Many such bodies are proposing to hold on the rolls of active membership all who are in the military or naval service of the country without their payment of dues if they so request. Our Council voted to put our Association among those who gladly perform this patriotic service, believing that our membership will back up this action by holding our numbers not only up to the present total but by actually increasing this total through still further accretions from the ranks of present non-members. It was surely gratifying to have twenty-two individuals and two institutions elected to membership at this meeting.

In spite of the heavy burden (over $400) for increased cost of paper for the MONTHLY during 1917, the Association carried on the work of its standing committees and provided the subvention agreed upon for the *Annals;* and we look forward with confidence to 1918 feeling assured that no backward step will be necessary on account of the lessening of individual support.

An important step forward was taken in connection with the amendment to the constitution providing for a further subdivision of the editorial responsibilities. Professor Carmichael now becomes editor-in-chief of the MONTHLY and a feeling of absolute confidence prevails throughout the Association that the MONTHLY could not be in better hands, and that we are entering upon an epoch of distinct advancement. Already, improvements and new departments are in process of development which will commend themselves to all readers of the journal. The personnel of the editorial committee remains unchanged except for the promotion of Professor Carmichael to the editorship. The business management of the MONTHLY remains as heretofore. In accordance with the amended by-laws, the associate editors are now appointed annually by the Council on the recommendation of the editorial committee. These appointments will appear in the official report of the Council actions. The Council held three sessions, ten members being present at each session.

In response to a call sent out previous to the Chicago meeting, some twenty-five Illinois members of the Association met on Thursday afternoon to consider the question of forming a section of the Association. The secretary's report of this meeting will appear in the February number of the MONTHLY.

An effort was made to compile a more serviceable index to the last volume of the MONTHLY than those in previous volumes, and it is hoped that this effort will appeal to our readers as worth while and justifying the added labor and expense involved. The work was done under the supervision of Professor Helen A. Merrill of the editorial staff.

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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PUBLISHED BY THE ASSOCIATION

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THIRD ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The third annual meeting of the Association was held at the University of Chicago on Thursday and Friday, December 27–28, 1917, in conjunction with the Chicago Section of the American Mathematical Society, which met on Friday and Saturday of that week. There were 119 in attendance at the various sessions, including the following 93 members of the Association and one institutional delegate:

L. D. AMES, University of Missouri.
R. C. ARCHIBALD, Brown University.

I. A. BARNETT, Chicago, Ill.
MRS. W. E. BECKWITH, College for Women, Western Reserve University.
G. A. BLISS, University of Chicago.
P. P. BOYD, University of Kentucky.
H. T. BURGESS, University of Wisconsin.
W. H. BUSSEY, University of Minnesota.

W. D. CAIRNS, Oberlin College.
FLORIAN CAJORI, Colorado College.
D. F. CAMPBELL, Armour Institute of Technology.
R. D. CARMICHAEL, University of Illinois.
W. E. CEDERBERG, Augustana College.
H. E. COBB, Lewis Institute.
L. M. COFFIN, Coe College.
C. E. COMSTOCK, Bradley Polytechnic Institute.
M. W. COULTRAP, North Western College.
A. R. CRATHORNE, University of Illinois.
G. H. CRESSE, University of Michigan.
D. R. CURTISS, Northwestern University.

ALFRED DAVIS, Parker High School, Chicago.
E. W. DAVIS, University of Nebraska.
L. E. DICKSON, University of Chicago.
E. L. DODD, University of Texas.
ARNOLD DRESDEN, University of Wisconsin.
OTTO DUNKEL, Washington University.

ARNOLD EMCH, University of Illinois.
L. C. EMMONS, Michigan Agricultural College.
H. T. ETTLINGER, University of Texas.

J. A. FOBERG, Crane Junior College, Chicago.
W. B. FORD, University of Michigan.
A. F. FRUMVELLER, Marquette University.

CORNELIUS GOUWENS, University of Iowa.
M. E. GRABER, Heidelberg University.

LAURENCE HADLEY, Earlham College.
W. A. HAMILTON, Beloit College.
HARRIS HANCOCK, University of Cincinnati.
J. O. HASSLER, Crane Junior College, Chicago.
E. R. HEDRICK, University of Missouri.
T. H. HILDEBRANDT, University of Michigan.
F. H. HODGE, Franklin College.
E. V. HUNTINGTON, Harvard University.

Nelle L. Ingels, Greenville College.

Jessie M. Jacobs, Urbana, Ill.

A. M. Kenyon, Purdue University.
J. M. Kinney, Hyde Park High School, Chicago.
H. R. Kingston, University of Manitoba.
Daniel Kreth, Wellman, Ia.

Kurt Laves, University of Chicago.
D. A. Lehman, Goshen College.
Flora Le Stourgeon, Leggett School, Detroit.
E. B. Lytle, University of Illinois.

W. D. MacMillan, University of Chicago.
Malcolm McNeill, Lake Forest College.
W. O. Mendenhall, Earlham College.
G. A. Miller, University of Illinois.
G. R. Mirick, New Castle (Pa.) High School.
U. G. Mitchell, University of Kansas.
C. N. Moore, University of Cincinnati.
E. H. Moore, University of Chicago.
C. C. Morris, Ohio State University.
E. J. Moulton, Northwestern University.
G. W. Myers, University of Chicago.

M. J. Newell, Evanston (Ill.) High School.

H. L. Olson, Chicago, Ill.

C. I. Palmer, Lewis Institute.
H. B. Phillips, Mass. Institute of Technology.
L. C. Plant, Michigan Agricultural College.

Jessie G. Quigley, College of Saint Teresa.

O. J. Ramler, Catholic University of America.
W. R. Ransom, Tufts College.
S. E. Rasor, Ohio State University.
H. L. Rietz, University of Illinois.
W. H. Roever, Washington University.

Ida M. Schottenfels, Chicago.
G. H. Scott, Benzonia Academy.
J. B. Shaw, University of Illinois.
L. S. Shively, Mount Morris College.
C. H. Sisam, University of Illinois.
H. E. Slaught, University of Chicago.
A. W. Smith, Colgate University.
G. W. Smith, Beloit College.
C. C. Spooner, Northern Michigan State Normal School.

E. H. Taylor, Eastern Illinois Normal School.
E. J. Townsend, University of Illinois.
H. W. Tyler, Mass. Institute of Technology.

J. N. Van der Vries, University of Kansas.

L. G. Weld, Pullman (Ill.) Manual Training School.
A. E. Whitford, Milton College.
J. A. Whitted, Hedding College.
E. J. Wilczynski, University of Chicago.

J. W. A. Young, University of Chicago.

John Zimmerman, Dubuque German College.

Cliff Guild, Illinois Wesleyan University (institutional delegate).

Some were doubtless kept from the meetings because of the inconveniences of railway travel and because of considerations of economy based on patriotic grounds;* yet it is worthy of note that so large a number showed by their presence and participation their conviction that the torch of learning must be kept bright in this as well as in other ways in the present troublous times. Twelve came from institutions more than five hundred miles from Chicago, including four from Massachusetts, three from Texas, one each from Colorado, Washington, D. C., Manitoba, New York and Rhode Island.

As announced in the program the opening session on Thursday morning was devoted to papers presented by members on their own initiative. It has been recognized from the outset that the Association programs must not merely include papers and discussions on topics selected by a committee from among the most important subjects in collegiate mathematics, but that there shall also be afforded an open forum for the presentation of papers not thus limited. The

* See letter by Professor T. W. Richards, in *Science*, Dec. 28, 1917.

latter might well include reports on special methods of problem solutions, pedagogical discussions, accounts of original investigations, the organization of a particular topic or chapter in some collegiate subject, etc. That such a combined program has elements of strength greater than one composed either entirely of selected papers on specified topics or papers entirely unrestricted, seems unquestionably answered in the affirmative by the result at Chicago. In anticipation of similar programs at future meetings, the officers of the Association earnestly urge this opportunity upon the attention of our members, so that when a call comes for either type of papers, there may be a ready and hearty response. The "Questions and Discussions" in recent numbers of the MONTHLY indicate that our members appreciate an open forum, and it seems evident that a fuller opportunity should be provided than the crowded pages of the MONTHLY allow.

On Thursday evening the joint dinner of the Society and the Association was held at the Quadrangle Club with an attendance of 73. The toastmaster, President Cajori, called upon the following members, who responded with brief remarks: Professor E. H. Moore, University of Chicago; Professor W. B. Ford, University of Michigan; Professor E. V. Huntington, Harvard University; Professor L. E. Dickson, University of Chicago; Professor E. L. Dodd, University of Texas; and Professor E. R. Hedrick, University of Missouri. Much interest was expressed in various phases of war activity and in the share which the Association's members have in the country's service. By request, the secretary of the Chicago Section of the Society and the secretary of the Association collated rather hastily and read a list of 48 American mathematicians known to be in national service at present. About forty members took advantage of the invitation of the American Association of University Professors to attend the dinner on Friday evening at the Quadrangle Club. As in the case of earlier meetings, so at Chicago admirable accommodations and plans made by the local members of the Committee on Arrangements helped greatly to further social intercourse and to render everyone's enjoyment the more complete. The members of this Committee and those of the Program Committee merit much praise for the thoughtful planning which made the meetings a success.

President Cajori presided except on Thursday afternoon, when he called Professor E. W. Davis to the chair. Professor L. E. Dickson, president of the American Mathematical Society, presided at the joint session on Friday afternoon. At this session Professor W. B. Ford gave his retiring address as chairman of the Chicago Section of the Society, on the subject "A conspectus of the modern theory of divergent series." On behalf of the Association, Professor L. D. Ames of the University of Missouri gave an address "On a definition of the real number system by means of infinite decimals." Professor Ford's paper has been offered to the *Bulletin* of the Society for publication. The following is an abstract of Professor Ames's paper:

The purpose of this paper was to present a definition of the real number system which, while sound logically, shall be pedagogically simpler than the

well-known methods. Pedagogically the number system is built up by successive steps; with one exception each step follows rather closely our intuitive notions; but in the definition of irrational numbers all the commonly accepted methods of approach break rather sharply with our intuitions and are rather complicated. The mode of procedure suggested in this paper is as follows: It is first shown that all rational numbers can be expressed as terminating or repeating decimals, and that conversely all terminating or repeating decimals are rational. Then examples of individual infinite decimals which do not repeat are given; these are not rational, but they make as vivid an appeal to the student as do the rationals. A real number is then defined as an expression in the decimal notation, this being justified pedagogically and logically. A logical treatment of the properties of real numbers follows closely the natural suggestions of the decimal notation. The notion of a continuum is approached as follows: The rational number system has certain properties of order which are carefully stated. The student knows some infinite decimals which are not rational; these can be inserted one at a time between rationals, and the enlarged system will have at each stage the properties just mentioned. To speak crudely, there seem to be vacant spaces where new individual numbers can be inserted between those already given. On the other hand it is not possible to insert in the set of all infinite decimals a single new element so that the enlarged set shall have the properties stated. To speak crudely again, all the vacant spaces seem to be filled. This idea is the basis of the definition given of a continuum.

The other parts of the program are grouped under three leading heads, and abstracts are given corresponding to the numbers in each list, together with reports of some further informal discussions.

CONTRIBUTED PAPERS.

(1) "The Graph of $f(x)$ in Line-Coördinates for Complex Numbers." PROFESSOR A. F. FRUMVELLER, Marquette University.

(2) "Note on the Generalization of the Witch and the Cissoid." PROFESSOR F. H. HODGE, Franklin College.

(3) "Fermat's Method of Infinite Descent." PROFESSOR W. H. BUSSEY, University of Minnesota.

(4) "On the Disciplinary and Applied Values of Mathematical Study." PROFESSOR C. N. MOORE, University of Cincinnati.

(5) "On the Content of a Second Course in Calculus." PROFESSOR E. J. MOULTON, Northwestern University.

(6) "Comments on Mathematics in the High School." PROFESSOR HARRIS HANCOCK, University of Cincinnati.

(1) The graph of $f(x)$ in ordinary point-coördinates leads to a 3-dimensional diagram when x or y is taken as complex; and to a 4-dimensional diagram when both are complex. Professor Frumveller's paper was described as an attempt to escape from this difficulty by employing line-coördinates. A base-line

OO_1 and two lines, OX, OY, perpendicular to it and in one plane, are the framework used. The "locus" of the moving line is that part of the plane over which the line is free to range. Lacunary regions exist, bounded by the "envelope" of the line.

When x, or y, or both, become complex, the lines OX, OY are replaced by planes. The moving line now ranges over a 3-space, and generates ruled surfaces, the lacunary regions being now bounded by the curves called respectively the "edge of regression," and the "line of striction." A method of handling this diagram conveniently was indicated.

(2) In this note Professor Hodge presented two simple constructions, each leading to a family of quartic curves involving a parameter k. As k becomes infinite the first construction merges into the classic construction for the witch and the equation becomes the equation of the witch. Under the same conditions the second construction becomes the construction for the cissoid and its equation becomes the equation of the cissoid.

(3) Professor Bussey's introduction to Fermat's famous method of infinite descent and discussion of its range of usefulness will appear in full in a forthcoming issue of the MONTHLY.

(4) In this paper the disciplinary value of mathematical study as a training in reasoning was exhibited by Professor Moore in the following fashion: Certain well-known examples of deductive reasoning in the fields of public debate and natural science are analyzed into their logical elements. Parallel with each example are given one or more examples of deductive reasoning in elementary algebra and geometry, which are built up from precisely the same logical elements. Thus the medium for the transfer of training is shown in explicit form. The applied value of mathematical study is shown by pointing out that a knowledge of elementary mathematics is essential for a proper understanding of the scientific discussion of almost any subject that can be studied scientifically. Thus, for example, some knowledge of mathematical notation and procedure is necessary, not only in all the natural sciences, but also in psychology and the social sciences.

(5) In his discussion Professor Moulton commented on the obvious general need of a second course in calculus, and on the diversity of ways in which the need is met. One source of difficulty in outlining a course for general consideration lies in the differences in the preparation of students in various institutions due mainly to the different lengths of the first course in calculus. An outline was presented of a course to follow a first course of about a hundred lectures. It is the hope of the editors that a fuller account of this paper may be given in an early number of the MONTHLY.

(6) Professor Hancock said that the arguments given in recent years by various non-mathematicians in support of the classics and English apply with equal force to mathematics. Noteworthy are a book on "The value of the

classics" by Dean Andrew F. West of Princeton University, a speech by Dean Roscoe Pound of Harvard Law School, the speech by Ex-President Grover Cleveland at the inauguration of Woodrow Wilson as president of Princeton University, and the answering comments of President Wilson. Professor Hancock urged also the desirability of having an experimental high school which shall be entirely antipodal to the Flexner "Modern School," and reported on various investigations of the past few years, chiefly of one in Cincinnati in which business men and a general group of college and university teachers voted predominatingly for a rigorous course which should include mathematics.

Session on Descriptive Geometry.

(1) Address: "Descriptive Geometry and its Merits as a Collegiate as well as an Engineering Subject." Professor W. H. Roever, Washington University.

Discussions by

(2) Professor F. Higbee, Department of Descriptive Geometry and Drawing, State University of Iowa.

(3) Professor A. V. Millar, Department of Drawing, University of Wisconsin.

(4) Professor Arnold Emch, Department of Mathematics, University of Illinois.

(5) Mr. Willard W. Ermeling, Instructor in Descriptive Geometry, Crane Junior College, Chicago.

(6) Mr. W. F. Willard, Instructor in Drawing, Carl Schurz High School, Chicago.

General Discussion.

(1) In his address Professor Roever traced the development from the early attempts to solve the problems encountered in building and stone-cutting to the descriptive geometry of Monge, and showed how the needs of the artist were responsible for perspective, which in turn brought forth projective geometry, axonometry and photogrammetry and thus completed the work begun by Monge. He emphasized the two purposes: (1) to represent, by figures which lie in a plane, the objects of space; (2) to solve, by geometric constructions which can be executed in a plane, the problems of space; and then showed that the Mongean method fulfills best the second purpose and perspective the first, but that the axonometric method of making parallel projections satisfies simultaneously both. He called attention to the fact that the development of descriptive geometry in Europe has been confined almost exclusively to the technical schools and that in this country also it is confined to the engineering schools, being taught there to a limited extent only. He closed by saying that the hope for its development in this country depends on its introduction into the colleges and universities, that for this there is sufficient justification, that the requirements demanded of teachers of descriptive geometry should be raised and that mathematicians should familiarize themselves with it more fully.

Professor Roever's address will appear in full in an early number of the MONTHLY.

(2) "The study of descriptive geometry," writes August Comte, "possesses an important philosophical peculiarity, quite independent of its high industrial utility. This is the advantage which it so preëminently offers in habituating the mind to consider very complicated geometrical combinations in space, and to follow with precision their continual correspondence with the figures which are actually traced—of exercising to the utmost, in the most certain and precise manner, that important faculty of the human mind which is properly called imagination, and which consists, in its elementary and positive acceptation, in representing to ourselves, clearly and easily, a vast and variable collection of ideal objects, as if they were really before us. . . ." The more often one reads this statement and considers its true meaning, the more one becomes convinced that it contains the first and last word on the value of descriptive geometry as a college subject.

This "high industrial utility" of which Comte writes has quite generally been conceded. As evidence of this fact one might point out that from the beginning of technical schools down to the present descriptive geometry has held an important place in their courses of study, and has held it, one is tempted to add, in spite of the revolutions through which these same courses of study have passed with changing administrations and times. The relation which exists between descriptive geometry and designing is fundamental; it is the bed-rock upon which may be based a knowledge of the art of drafting. As a fundamental part of the training of a designer, descriptive geometry should aim to accomplish three things: First, it should teach orthographic projection, and teach it most thoroughly. A designer must not only be able to make drawings but he must read them as well, and orthographic projection may truly be called the grammar of that language which is the means of communication between designers and builders. Secondly, descriptive geometry should develop the ability to solve graphically problems concerning the relations of points, lines and planes. These are, of course, but the elements in the representation of all engineering structures and it is essential that designers and draftsmen be trained to solve problems relating to them directly on the drawing-board. Thirdly, and perhaps most important of all, descriptive geometry should promote the ability to analyze a problem into its component parts, to reason logically and clearly from a given set of conditions to a required set of conclusions, to build up from a drawing a mental picture of what is there represented, or the reverse of this process; for without the ability to analyze, to reason, to visualize, and to translate from one language to another, a draftsman is lacking in the essential qualifications of his calling.

But since descriptive geometry has become established as an essential part of a technical course of study, perhaps we may assume its value in such a course and pass on to consider, without further comment here, whether or not the

subject might with profit be seriously considered as deserving of a place in a non-technical college course. Let us turn again to our quotation from Comte. From this it must be evident that descriptive geometry, quite apart from its recognized practical value, is entitled to serious consideration for its educational advantages. If it has such advantages, then undoubtedly the subject deserves a place in a non-technical course of study.

Being a form of geometry, it should need no defense before an association such as this. If mathematics in any form possesses educational value, apart from the unquestioned usefulness of such subjects, then it should be agreed that descriptive geometry likewise has an educational value and is therefore deserving of recognition as a proper subject for study. But in addition to this perhaps recognized fact, it should be kept in mind that in descriptive geometry analysis and logical deduction are verified by graphical solution. Thus, not only does descriptive geometry possess all the merits of other forms of geometry but in addition it insists upon a mental attitude which leads one to visualize the problem, to see it in all its parts, and to work with conditions which are actually represented by a drawing. "A mathematical problem," to quote from another writer on the subject, "may usually be attacked by what is termed in military parlance the method of systematic approach; that is to say, its solution may be gradually felt for, even though successive steps leading to that solution cannot be clearly foreseen. But a descriptive geometry problem must be seen through and through before it can be attempted. The entire scope of its conditions, as well as each step toward its solution, must be grasped by the imagination. It must be taken by assault." Even to the sceptic it must be obvious that such a form of mental training is valuable; and when to this is added the usefulness of descriptive geometry as a real means of solving difficult problems with a considerable degree of precision, it should be granted that the subject might properly be placed in every college course.

(3) In descriptive geometry, objects of three dimensions are represented in outline on a plane sheet. This method of conveying an idea from one person to another by means of a drawing is widely used and is one of the most accurate and concise methods employed for that purpose. It is often superior to a written or oral description. The fundamental principles involved in making such a representation of an object are the principles included in descriptive geometry. The subject therefore has great practical importance and the principles are so universally employed that many times the user does not realize he is employing descriptive geometry. Those who have done much designing, either of buildings or of machines, and who have not made a formal study of descriptive geometry, use its principles; and if they take up the study of the subject later they find it very easy. For those who are preparing themselves to do design work, the study of the subject develops the reasoning and visualizing powers and enables them to progress more rapidly and intelligently with design.

The visualizing process, so often spoken of in connection with descriptive

geometry, depends upon reasoning. The student reasons from the given conditions that the objects or magnitudes must be in a certain position in space and by his reasoning power builds up a mental picture, the clearness of which depends upon the clearness of his reasoning. The subject, therefore, in addition to its practical application, gives excellent training in reasoning—a particular kind of reasoning which results in a definite mental picture. Students who have had thorough training in mathematics, particularly calculus, find descriptive geometry much easier than those who have not had such training. The subject is peculiar in that the problems are usually seen as a whole. The student first analyzes the problem, which involves seeing it in complete form and as well the various fundamental steps which must be taken to reach the conclusion, and then he makes the construction which involves taking these steps in order until the conclusion is reached. There can be no better training given than by this process of grasping the problem as a whole and then testing the accuracy of the conclusion by an actual mechanical construction.

There are only a few fundamental principles in the subject, but these principles must be thoroughly learned if the student is to be master of the subject. He must know these principles as he knows the fundamental operations in arithmetic if he expects to solve successfully original problems. The fact that there are few fundamental principles and that these can be thoroughly mastered makes the subject just that much more valuable. With the laboratory practice which usually accompanies descriptive geometry comes the value of motor training and the emphasizing of principles studied in the class-room. This part of the subject should by no means be omitted and a fair degree of accuracy should be demanded. If the students in our colleges are to become proficient in the use of English, instructors in all subjects must help. Frequently a student must be told that he appears to understand the problem under discussion but that his statements are inaccurate and would be unintelligible to a person not familiar with the subject. He must repeat one or more times and may need the help of other students before the answer is satisfactory. There is no better subject in the college curriculum for emphasizing the need of clear and concise English than descriptive geometry.

The subject then is not merely of practical value but gives also excellent opportunity for training in reasoning, thoroughness, motor activity, and English.

(4) Professor Emch found himself in agreement with most of the views and arguments advanced by Professor Roever. A course in descriptive geometry, or constructive geometry, is certainly of great benefit to the student of mathematics. As in other branches of science, it is desirable that the student should first acquire a certain amount of geometric knowledge of forms and their representation before he is asked to argue about the logical foundations of science. This principle should be observed earlier in the teaching of geometry in the high school, *i. e.*, much more attention should be paid to the constructive side of geometrical instruction in its elementary stages. Much complaint about poor

results in many classes in geometry is probably due to this neglect. At the University of Illinois this want is met by a course in constructive geometry, which is open to students who have had analytic geometry. As a rule this course is very well attended; it serves as an introduction to projective geometry and other more advanced courses in geometry. It comprises chapters on the various possibilities of geometric construction, accuracy and simplicity of construction, inversion in the plane and in space (including stereographic projection), isometric projection, orthographic projection, perspective, the analytic form and mathematical principles of these projections, collineation, cross-ratio and its invariance in projection, elementary synthetic theory of conics and quadrics, description of curves by linkages.

It would seem that a course of this or a similar sort, as suggested by Professor Roever, should be taken by every prospective teacher of mathematics or writer of mathematical text-books along geometric lines. The great number of faulty drawings in many text-books on geometry would thereby be avoided. Moreover, no course can be made more inspiring for future geometrical research than constructive, or descriptive, geometry for beginners who intend to specialize in mathematics.

(5) Professor Roever's address suggests to Mr. Ermeling several questions to which attention may well be drawn. Perhaps the first that occurs is this: Should the subject of descriptive geometry be confined to technical schools? Prof. T. E. French of Ohio State University has said that descriptive geometry has three important functions: (1) to train the student to think in terms of three dimensions; (2) to visualize quickly and accurately; (3) to build up a clear mental image by a training in constructive imagination. These are of primary importance to the engineer, who deals with three-dimensional things. Are they any less important to the general student? Is not one of the needs of a course in mathematics this very training in visualization? We have all seen the pleasure of the students of analytical geometry in making graphs of their work. Are we doing our duty by the student of mathematics if we give him the symbols, the values, the equations and omit the other benefits so easily obtained by a course of mathematics worked out over the drawing-board? Is it not like trying to teach chemistry or physics without laboratory work?

It is undoubtedly true as Prof. V. T. Wilson says in the preface to his text-book, "that in the application of the principles of descriptive geometry to the making of working drawings, the necessary modifications have often caused the loss of sight of the fact that descriptive geometry is a mathematical subject." This raises the questions of (*a*) what should be the qualifications of the teacher of descriptive geometry, and (*b*) whether the subject should be taught by the department of drawing or by the department of mathematics. A little consideration would probably lead us to see that the teacher of the subject should have not only a mathematical training, but also a practical experience in the drafting room; or, to state it more broadly, not only engineering training but

engineering experience. It does not then greatly matter whether the subject be considered a part of the drawing department or of the department of mathematics. The important thing is to have descriptive geometry related directly to the other branches of mathematics.

With all the foregoing facts in mind, it would seem that descriptive geometry has a value such that the subject could well be required in a general collegiate training in mathematics and not confined to the engineering courses as at present.

(6) In attempting to discuss the merits of Professor Roever's excellent paper, it might be well to answer first some of the questions which he submitted. (1) It is Mr. Willard's opinion that the preparation of a teacher of descriptive geometry should include an engineering education, inasmuch as the subject finds its greatest usefulness in that field. If a student majors in mathematics in a purely academic college course, he should likewise have a course in descriptive geometry. (2) It would seem from the meager data available that most of the smaller colleges present the subject in connection with the mathematics department, whereas, in the engineering schools, it is presented in conjunction with the drawing department. This is so because engineers emphasize the drawing phase for its practical application to many problems in industrial life. The colleges do not emphasize the drawing part of the subject perhaps because the department of engineering has not been established or developed, the student on that account giving the analytical part of the subject his exclusive attention. (3) The old texts, which have become fossilized, are in a measure responsible for a great deal of the distaste which students have for the subject. In the speaker's experience, where the subject was presented in the freshman year of an engineering course in college, many students were predisposed to failure, through the reputation of descriptive geometry which was passed on by those who had completed the course, a majority of these students having come from schools where not all the shops and drawing departments were yet established.

The advisability of extending this subject into the secondary schools rests largely with the school itself. It is obviously undesirable to require boys and girls alike to take descriptive geometry in a purely academic or college preparatory course in high school; but, where our city schools have included technical courses with a view to engineering preparation, it is quite possible for such students to grasp the subject, and highly desirable as well, especially in the third or fourth year. It is also well to observe that boys who have had a thorough course in plane and solid geometry have very little difficulty in descriptive geometry. In the Chicago high schools, technical course students are concerned in their sophomore year exclusively with developments and penetrations of solids. In these exercises are many problems involving the theorems of descriptive geometry. Why would it not be easy for these boys to cover a satisfactory course in descriptive drawing immediately following their sophomore year? Our universities and engineering schools, as a rule, are loath to accept the work of the secondary schools for entrance in this particular, and perhaps justly so, because of the

varying success of our high schools in carrying on this work. Boys who leave from our best technical high schools for the universities and engineering schools very often are required to duplicate this work, much to their chagrin. It might be well for our higher institutions, through their examiners, to differentiate such institutions as are proficient and do offer a course in descriptive drawing, and credit this toward advancement in their college course. When this is done, there will certainly exist a happier relationship between our secondary schools and universities.

In the ensuing general discussion, Professor Van der Vries and Mr. Ermeling brought out the fact by a show of hands that in most of the institutions represented at the meeting the subject of descriptive geometry is taught not in the department of mathematics but in the department of drawing, and by technically trained instructors, although Mr. Willard and Professor Archibald were inclined to believe that a fuller investigation would show the contrary to be the case in the country at large. Professor E. W. Davis described the method as carried out at the University of Turin, where figures are sketched upon the blackboard in free-hand; he stressed the great help which descriptive geometry affords the students of solid geometry and on this ground urged this or the study of perspective as a part of the regular course in high-school geometry.

As an instance where the methods of descriptive geometry lend themselves readily to the satisfactory solution of a problem often treated unscientifically even by mathematically trained persons, Professor Roever described the method for drawing the representation of a line perpendicular to a given plane. A lively interchange of views between Professors Roever and Millar brought out the fact that the principles of representation (*e. g.*, in isometric projection, which was at the time under discussion) are not always applied consistently. In response to Professor Millar's doubt as to the ability of high-school pupils to handle descriptive geometry readily, Mr. Willard expressed his decided judgment based on his teaching experience that they do show a sufficient ability. Professors Dodd and Hadley pointed out that the course is sometimes given successfully, paying only secondary consideration to the extreme accuracy of the constructions. Following Professor Burgess's remark as to his unhappy memory of the course made up as it was of a long succession of puzzling drawings carrying out meaningless rules, Professor Wilczynski summed up the discussion by saying that it is altogether evident that there has been too much purely mechanical following of particular methods without a proper presentation of the mathematical principles involved, and that the leading colleges and universities must do their share in bettering this instruction and making more certain the undoubted benefits which come from the study of descriptive geometry.

Reports of Standing Committees.

(1) National Committee on Mathematical Requirements.

(*a*) "Scientific Investigations of the Committee." Professor A. R. Crathorne, University of Illinois.

(*b*) "The Work of a Committee representing the Central Association of Science and Mathematics Teachers." MR. J. A. FOBERG, Crane Junior College, Chicago, Chairman Coöperating Committee.

(2) Committee on Libraries.

(A report of this committee was published in the October MONTHLY in the form of a list of 160 mathematical books for schools and colleges.)

Discussion opened by PROFESSOR H. E. SLAUGHT, University of Chicago.

(3) Committee on Mathematical Dictionary. Preliminary Report by the Chairman, PROFESSOR E. R. HEDRICK, University of Missouri.

(4) Committee on Annals of Mathematics. Report by the Chairman, PROFESSOR E. H. MOORE, University of Chicago.

(5) Committee on Bureau of Information. Report by the Chairman, PROFESSOR J. B. SHAW, University of Illinois.

(1*a*) The subcommittee on scientific investigations is at present studying the application of the theory of correlation to educational data. Several thousand complete records of high-school pupils in different parts of the country are being collected to form a statistical basis for the investigation. The correlation coefficients not only for mathematics and other subjects, but, for comparison purposes, the coefficients for many other pairs of subjects will be discussed. The question to be answered is this: "If subject A is considered by educators to be a fundamental part of all secondary curricula, does proficiency in mathematics count more or less than proficiency, say, in stenography, or civics, towards an increase of proficiency in A?"

There is a fairly well-defined theory believed in by most psychologists to the effect that training gained in the study of one subject is of use in the study of some other subjects. The technical word "transfer" is used in this connection. The real question at issue in connection with formal discipline is the amount and method of transfer. A commonly accepted statement of this principle of formal discipline (Bagley's) is as follows:

> "The present interpretation of the doctrine of formal discipline is based upon the belief that specific habits may be generalized into ideals and prejudices, which in turn make possible the acquisition of similar habits in new fields, as when from the specific habits of accuracy and close reasoning developed in the school exercises in mathematics one comes gradually to idealize accuracy and close thinking as methods of procedure that will bring desirable results in other fields."

One of the first steps in answering the question "Should mathematics be required of all high-school pupils," would be then to investigate the "spread," "transfer" or "generalizing power" of mathematics. We all believe that a boy or a girl who has taken a thorough course in algebra will do some things better than one who has not taken such a course. What these things are and how much better the first pupil will do them are things to be found out. The number which represents the "how much better" we might call the "factor of transfer" or "coefficient of spread." The object of this subcommittee is to find out as far as possible the connection between this "factor of transfer" and the correla-

tion coefficient. The object is not primarily to find arguments in defense of mathematics, but an open-minded investigation of correlation theory applied to educational measurements, with enough data behind it to make the results conclusive.

The Council of the Mathematical Association has voted the sum of one hundred dollars to take care of the great amount of purely clerical work involved in gathering the records used in this investigation.

(1*b*) At the Chicago meeting of 1916 the Mathematics Section of the Central Association of Science and Mathematics Teachers ordered the appointment of a committee to coöperate with the National Committee on Mathematical Requirements. Professor W. W. Hart, of the University of Wisconsin, then chairman of the Section, appointed a committee of twelve, distributed over the territory covered by the Association, and during the year past this committee has acted in cooperation with the National Committee.

At the request of Professor J. W. Young, a report was prepared by a subcommittee, with Mr. Alfred Davis of the Francis W. Parker School as chairman, on the topic "Valid Aims of Mathematics Teaching in the Secondary School." This was one of several preliminary reports submitted to Professor Young, who is soon to publish a final report on this topic. This subcommittee report[1] will appear in *School Science and Mathematics*, beginning in January, 1918.

A questionnaire addressed to prominent business folk and professional people of Chicago was prepared by another subcommittee, with the idea of investigating the opinions held by such persons concerning the value of mathematics teaching in the high school. This questionnaire was published at length in *School and Society* of November 17, 1917; discussion and comment concerning it has appeared in subsequent numbers.

Another subcommittee prepared a Report on First-Year Mathematics for the high school—a topic to be worked over more in detail in coöperation with the representative of the Association of Teachers of Mathematics in New England during the coming year.

After submitting its report at the Columbus meeting of the Central Association, the Coöperating Committee was continued for a year, with instructions to carry on the work in cooperation with the National Committee.

In the discussion which followed these two reports, Professor Hedrick, as the one who had been responsible in 1916 for the appointment of the National Committee, stated that it was distinctly understood that this Committee on Mathematical Requirements was to enlist the coöperation of committees of secondary-school teachers all over the country, and that these would be thus grouped together in effective service through this agency of the National Association. The whole discussion and recent correspondence makes manifest the growing consciousness over the country that here is for the first time in the history

[1] Reprints of this report and the questionnaire may be had from Mr. Alfred Davis, Parker School, 330 Webster Ave., Chicago, at ten cents per copy.

of American mathematics a powerful organized body which is investigating the whole question of mathematics as a school subject, employing in this study the definite information and the scientific training of the mathematicians of the whole country and at the same time putting aside natural prejudices in favor of mathematics as fully and honestly as is humanly possible.

(2) Professor Slaught, in speaking of the work of the Library Committee, first endeavored to show its proper setting in the general scheme of activities of the Association. Keeping in mind the general purpose of the Association, namely, to advance the interests of mathematics in the collegiate field, he mentioned three great streams of influence: (1) Meetings—national, sectional, and local; (2) Committees—on mathematical requirements, on libraries, and on bureau of information; (3) Publications—the MONTHLY, the *Annals*, and the proposed Dictionary. The national meetings, at best, can bring together only a small fraction of the members; but we may hope to see the sections become so numerous that every member may attend at least one meeting a year; and there is wide range and great opportunity for local clubs in the larger cities and in normal schools and colleges. The standing committees of the Association constitute the laboratories from which, as time goes on, will emanate the deliberative pronouncements of this national body on the great questions of every character which concern the vital interests of undergraduate mathematics in America. Already, our various committees are doing foundation work, especially in the three lines mentioned above. The publications of the Association are the channels through which must flow all these streams of influence to the individuals and institutions in the membership of the Association, and by means of which the interchange of ideas necessary to progress and growth is made. The MONTHLY, in this respect, stands in the most intimate relation to every member; the *Annals*, in its relationship to the Association, supplies a need that has long been felt in this country; and the proposed Dictionary opens up a great field of possible service to the college teachers and students of mathematics.

The Library Committee consisting of Florian Cajori, E. S. Crawley, Solomon Lefschetz, W. R. Longley, R. E. Root, and W. B. Ford, Chairman, was appointed in reponse to a pressing need based upon requests from many sources for assistance both in selecting mathematical books for a small college library and in convincing local authorities of the need of such books. The committee wisely limited this list to what would seem to be an irreducible minimum, and also classified the books according to the needs of students at the various stages of progress in the collegiate course.[1] The members of the committee have expended much time and energy in selecting these titles. They make no claim of infallible wisdom, in including this particular book or excluding that one, but it is clear that, in general, they have shown excellent judgment, and that they certainly have made a most worthy first approximation. Any college library which contains this

[1] The report was published in full in this MONTHLY for October, 1917, pp. 368–376. Reprints may be had from the Secretary, Professor W. D. Cairns, 27 King Street, Oberlin, O., upon request accompanied with a three-cent stamp.

list of books will provide suitable opportunity for its mathematical students who wish to know something more of the various subjects than can be found in the text-books used in the class-room. The committee will welcome suggestions of any sort with respect to this list. At the business meeting in the afternoon, the committee was requested by formal vote to prepare a list of textbooks available for the various collegiate subjects, giving the publishers, the cost, and brief comments as to the character and scope of each. Presumably, the committee will include in this list all of the subjects mentioned in the tentative report of the Dictionary Committee immediately following this report. Such a list of text-books will be of the greatest service to collegiate teachers everywhere.

(3) Your Committee on Mathematical Dictionary, consisting of R. C. Archibald, H. L. Rietz, H. E. Slaught, D. E. Smith, and E. R. Hedrick, chairman, appointed at the Cleveland meeting, begs to report as follows:

After considerable correspondence and discussion at the Chicago meeting, the committee is agreed tentatively upon several important points, but none of these is irrevocably settled, and it is hoped that discussion here and suggestions by individuals later, either orally or preferably in writing, will assist the committee in revising and completing these tentative conclusions.

It is thought wise not to attempt any publication during the period of the war, but since the time required for the preparation of any such work would be very great, it is felt that we need not hesitate to initiate the work as soon as is practicable.

After some rough computation, we believe that a dictionary containing brief definitions of the words employed up to and including the end of collegiate work proper, or possibly including the first graduate year, could be condensed into two reasonable volumes which might be sold at approximately five dollars each. It is desirable that the committee have opinions as to the salableness of a work of this character and price, and as to its usefulness as thus limited in scope, and members are asked to advise the committee in writing.

As to the scope of the work, it is thought impracticable to include all mathematical terms used in the entire literature. Some line of demarkation seems inevitable. The committee has tentatively placed this line at the conclusion of the collegiate curriculum; that is, to include all words used in mathematical courses usually offered to undergraduates, or in some institutions to first-year graduates, and words that might occur in their assigned reading. This would cover the texts and references used in at least the following subjects: elementary algebra and geometry, trigonometry, plane and solid analytic geometry, and a first and second course in the calculus, together with elementary first courses in differential equations, higher analytic geometry (including curves and surfaces), higher algebra (including theory of equations, determinants, etc.), the theory of numbers, the theory of functions of real and of complex variables, the theory of groups, descriptive geometry, projective geometry, line geometry, differential geometry, mechanics, and the theory of probability and statistics.

It is assumed that no lengthy articles would be included, but only brief definitions, accompanied by necessary references to more complete sources, the ordinary linguistic comments, and possibly the equivalent words in French, German, and Italian.

A question exists regarding the inclusion of historical matter, since this is already reasonably covered by existing encyclopædias and by histories of mathematics. In certain cases, as in the entry "Taylor's Theorem," brief historical data might be given.

It was upon the basis of such a plan that the above estimate of size and price was made. Any enlargement of scope would entail obvious increase in size, time required for preparation, and price, and consequent decrease in probability of success in selling the dictionary widely to those who most need it.

It is felt that the venture is worth while only if it can be completed in a thoroughly scholarly manner, and so as to reflect credit upon the Association and upon American mathematics as a whole. To this end the interest and coöperation of some of our foremost scholars would be an essential, and we should hesitate to enter upon the work without the most certain assurance of such coöperation and support.

After hearing discussion here, and receiving suggestions from members personally, as a result both of this meeting and of the publication of this tentative report, the committee will try to formulate a more detailed and final report to be presented at some meeting of the Association during the ensuing year.

The above report was discussed briefly at the close of the business meeting in the afternoon. Two points were brought out clearly and evidently endorsed by the sentiment of those present: (1) That such a dictionary is greatly needed by those who are not within reach of large mathematical libraries, as is the case with most teachers and students of mathematics in the colleges of the country. This need was emphasized in most convincing manner by Professor A. F. Frumveller of Marquette University, Milwaukee, Wisconsin; (2) that such a dictionary is not greatly needed by the professors and research students of the graduate schools, and that the only justification for the Association's proposed sponsoring of such a venture is the real service which it may render in the collegiate field where at present no reference books comparable in scope and potential usefulness are available. This point of view was emphasized by Professor Slaught, who declared that his interest in the dictionary project was based entirely upon its adaptation to the needs of college teachers and students.

(4) A report of the Committee on the relations of the Association with the *Annals of Mathematics* was made by the chairman, Professor E. H. Moore, at the dinner on Thursday evening. Attention was called to the fact that the articles of expository or historical character for which the Association is supplying a subvention are now appearing in the *Annals* and that the next one will be by Professor D. R. Curtiss in the March issue on "A Chapter in the Theory of Equations." An appeal was made for contributions of articles of the character

in question, the present scarcity being attributed to the fact that this opportunity for the publication of such articles has so recently become effective. The committee will welcome any information concerning possible articles, especially those in which the exposition is made in the simplest and most direct form regardless of the space required in order to achieve this end.

(5) The Bureau of Information desires to report that it has served its purpose to some extent during the past year. It has replied to thirteen inquiries, furnishing information as to text books, history of mathematics, pedagogy of mathematics, a few problems of special interest to the proposer, and has thrown light on certain questions that puzzled the inquirers.

The chairman of the bureau, Professor J. B. Shaw of the University of Illinois, would offer the suggestion that the bureau be rather steadily advertised, as it probably has not yet come to the attention of many persons that such a bureau exists and is doing business. If the number of inquiries was greater the bureau would be able to maintain at least a page of replies in each number of the MONTHLY and it would be then advertised as a matter of course.

MEETINGS OF THE COUNCIL OF THE ASSOCIATION.

The Council held meetings on Thursday afternoon and on Friday and Saturday, ten members being present at each session.

(1) The following twenty-four persons and three institutions, on applications duly certified, were elected to membership:

To individual membership:

C. C. BRAMBLE, instructor in mathematics, U. S. Naval Academy, Annapolis, Md.
A. L. CANDY, professor of pure mathematics, University of Nebraska, Lincoln, Neb.
R. C. COLWELL, professor of mathematics, Geneva College, Beaver Falls, Pa.
C. G. CROOKS, professor of mathematics, Centre College, Danville, Ky.
P. J. DANIELL, assistant professor of applied mathematics, Rice Institute, Houston, Tex.
ALEXANDER DILLINGHAM, instructor in mathematics, U. S. Naval Academy, Annapolis, Md.
C. S. DOAN, head of mathematics department, Friends' Select School, Philadelphia, Pa.
THEODORE DOLL, fellow in mathematics, Northwestern University, Evanston, Ill.
H. H. DOWNING, assistant professor of mathematics, University of Kentucky, Lexington, Ky.
E. T. FRANKEL, statistician, Police Department, New York, N. Y.
L. M. GRAD, New York, N. Y.
W. H. HAYS, teacher, Broadway High School, Seattle, Wash.
L. S. JOHNSTON, professor of mathematics, State Normal School, La Crosse, Wis.
J. E. MCATEE, instructor in mathematics, University of Illinois, Urbana, Ill.

V. S. MALLORY, vice-principal and instructor in mathematics, High School, Dumont, N. J.

I. L. MILLER, professor of mathematics, Carthage College, Carthage, Ill.

M. J. NEWELL, Township High School, Evanston, Ill.

JESSIE G. QUIGLEY, professor of mathematics, College of Saint Teresa, Winona, Minn.

J. A. SALLADE, instructor in mathematics, Pennsylvania State College, State College, Pa.

MARION E. STORK, professor of mathematics and astronomy, Meredith College, Raleigh, N. C.

MARIAN M. TORREY, teacher of mathematics, St. Johnsbury (Vt.) Academy.

WARREN WEAVER, assistant professor of mathematics, Throop College of Technology, Pasadena, Calif.

W. H. WILSON, instructor in mathematics, Massachusetts Institute of Technology, Cambridge, Mass.

S. D. ZELDIN, instructor in mathematics, College of Hawaii, Honolulu, Hawaii.

To institutional membership:

ILLINOIS WESLEYAN UNIVERSITY, Bloomington, Ill.

COLLEGE OF SAINT TERESA, Winona, Minn.

MILLSAPS COLLEGE, Jackson, Miss.

(2) Since the enlistment of a number of our members in military service may make it impossible for them to continue as active members, the Council voted as follows:

On request, the name of any member of the Association who is in the army or navy, or is in other war service over-seas, will be retained on the roll of membership without the payment of dues. The MONTHLY will be forwarded, however, only if a definite mailing address is furnished by the member.

(3) It was voted to hold the next annual meeting in New York City in connection with the meeting of the American Mathematical Society. The mathematics faculty of Columbia University has kindly indicated its hearty approval of this plan.

(4) The Council approved the plan already announced in the December MONTHLY for a shortened register of members, containing the full list of members, including the names, academic record and addresses of those elected since the publication of the charter membership, and any corrections in names, titles or addresses that are reported to the secretary-treasurer promptly.

(5) Matters of regular business routine were referred with power to the Council's Committee on Finance, a standing committee which, in conformity with the amended constitution, is now composed of the President, Secretary-Treasurer and Manager.

(6) The Council voted to direct its future Nominating Committees, in choosing the second candidate for each office, to take into account not only the votes for each person for that particular office, but also the total vote for that person

for all the offices. (See "Comments on the Preliminary Ballot," page 67 of this issue.)

(7) Professor MARY EMILY SINCLAIR of Oberlin College was appointed Librarian of the Association.

(8) In accordance with the amendments adopted at the annual business meeting, the Council made the following appointments:

Secretary-Treasurer: W. D. CAIRNS.

Committee on Publications:

W. H. BUSSEY, Review Editor.
H. E. SLAUGHT, Manager.
R. D. CARMICHAEL, Editor-in-Chief.

Associate Editors:

R. C. ARCHIBALD,	HELEN A. MERRILL,
E. L. DODD,	U. G. MITCHELL,
OTTO DUNKEL,	R. E. MORITZ,
B. F. FINKEL,	D. A. ROTHROCK,
TOMLINSON FORT,	D. E. SMITH,
H. R. KINGSTON,	E. B. STOUFFER.

ANNUAL BUSINESS MEETING.

The secretary-treasurer announced the names of those just elected by the Council to membership. He also reported the death of the following ten members not previously reported, all charter members of the Association:

J. A. Colson, Searsport, Me.
Mrs. Elizabeth B. Davis, Nautical Almanac Office, Washington, D. C.
G. W. Hartwell, professor of mathematics, Hamline University.
C. S. Jackson, mathematical master, Royal Military Academy, Woolwich, Eng.
C. T. Levy, formerly teaching fellow, University of California.
J. W. Nicholson, professor of mathematics, Louisiana State University.
S. F. Norris, professor of mathematics, Baltimore City College.
E. W. Ponzer, assistant professor of mathematics, Stanford University.
L. E. Pratt, Tecumseh, Neb.
W. C. Wright, consulting actuary and accountant, Medford, Mass.

It was voted that the Committee on Libraries be continued and be asked to publish a list of current texts in collegiate mathematics, giving briefly the scope and character of each.

The amendments recommended by the Council (see November MONTHLY, pp. 445, 446) were adopted unanimously. It will be remembered that the main purpose of these was to divide the burdens of editorial work and to change the manner of choosing the secretary-treasurer so that this officer shall be selected by the Council rather than elected by vote of the Association.

The election of officers for the year 1918 was conducted both by mail and in person at this meeting, as provided by the constitution. The tellers (E. L. Dodd, O. J. Ramler, and H. B. Phillips, chairman) appointed by President Cajori reported the result of the balloting as follows, a total of 319 ballots having been cast, some of which were blank in whole or in part:

For President: E. V. Huntington, 215 votes; Oswald Veblen, 97 votes; scattering, seven votes.

For Vice-President: J. W. Young, 183 votes; D. N. Lehmer, 165 votes; R. C. Archibald, 154 votes; M. B. Porter, 101 votes.

For Secretary-Treasurer, W. D. Cairns, 294 votes.

For additional members of the Council to serve until January, 1921: Florian Cajori, 253 votes; G. A. Miller, 196 votes; E. J. Wilczynski, 194 votes; Elizabeth B. Cowley, 163 votes; H. E. Hawkes, 155 votes; W. H. Bussey, 130 votes; L. W. Dowling, 85 votes; A. M. Kenyon, 81 votes.

The following were accordingly declared elected:

President, E. V. HUNTINGTON, Harvard University.
Vice-Presidents, D. N. LEHMER, University of California, and
J. W. YOUNG, Dartmouth College.
Secretary-Treasurer, W. D. CAIRNS, Oberlin College.

Additional members of the Council to serve until January, 1921:

FLORIAN CAJORI, Colorado College,
ELIZABETH B. COWLEY, Vassar College,
G. A. MILLER, University of Illinois,
E. J. WILCZYNSKI, University of Chicago.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 1, 1917. The report of the auditing committee (Mary E. Sinclair, H. E. Slaught, and H. J. Ettlinger, chairman) was then made, and both reports were accepted and approved. The financial report is printed in full below.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 1, 1917.

RECEIPTS.			EXPENDITURES.		
Balance Dec. 21, 1916		$2,953.24	Publisher's bills		$1,942.37
			Charter membership list		391.47
1915 subscriptions	$ 2.00		Managing editor's office		250.87
1916 subscriptions	22.25		Other editors' postage		7.00
1916 indiv. dues	22.50		Committee on Membership		4.85
1916 instit. dues	5.40		Comm. on Math. Requirements		23.19
1917 subscriptions	399.75		Secretary-Treasurer's office:		
1917 indiv. dues	2,080.98		Postage	$137.72	
1917 instit. dues	263.00		Bond	5.00	
1917 init. fees	140.00		Office supplies	25.36	
Sale copies of MONTHLY	35.18		Express, telegrams,		
Sale reprints	14.43		freight, etc.	18.93	
Advertising	478.92		Library expenses	33.33	

Exchange	.42		Clerical work	187.25	
Interest State Savgs. Bk.	77.13		Printing	101.04	
Interest Peoples Bk.	30.04		New York meeting	89.10	
			Cleveland meeting	21.10	
Total 1917 receipts		3,572.00	Initiation fees paid to sections	4.00	
					622.83
Total receipts up to 1918 business		$6,525.24	ANNALS subvention		150.00
Total expenditures		3,392.58	Total expenditures		$3,392.58
Balance on 1917 business		$3,132.66	Cash on hand, not deposited		24.00
			Checking account		380.91
Rec'd on 1918–1920 business		352.81	State Savgs. Bk. Co. account		2,337.16
			Peoples Bkg. Co. account		643.40
			Liberty Bond		100.00
Balance Dec. 1, 1917		$3,485.47	Bank balance Dec. 1, 1917		$3,485.47

Approved by the auditing committee,
H. J. ETTLINGER, *Chairman*,
MARY EMILY SINCLAIR,
December 28, 1917. H. E. SLAUGHT.

When the accounts were closed on December 1, 1917, as was necessary in order to furnish the auditing committee a complete record, there remained on the total business for the calendar year 1917 the following items:

BILLS RECEIVABLE.		BILLS PAYABLE (either paid in December or estimated).	
Advertising	$169.00	Publisher's bills, June–Dec.	$1,560.00
1917 dues unpaid	75.00	Dec. ANNALS subvention	75.00
Back subscriptions	6.00	Init. fees due to sections	58.00
		President's office	5.00
	$250.00	Editors' postage	20.00
		Managing editor's office	50.00
		Secretary-treasurer's office	125.00
		Printing Chicago program, ballots, and application blanks	50.00
		Committee on Membership	5.00
		Comm. on Math. Requirements	100.00
		Additional postage	45.00
		Total, approximate	$2,100.00

If to the balance on 1917 business shown in this report, $3,132.66, there be added the amount of bills receivable, $250.00, and there be subtracted the estimated amount of bills payable, $2,100.00, there results an estimated final balance on 1917 business of approximately $1,280, as compared with a similarly estimated final balance a year ago of approximately $1,180. The major portion of this $1,280 is the thousand dollars which was turned over to the Association by the management of the MONTHLY when the Association was organized and which we feel must be kept intact as a reserve fund. The management is thus able to

present a gain in the year's finances of about one hundred dollars, this gain having been brought about through the increased amount of advertising secured for the MONTHLY, through the continued expenditure of much time by a goodly number of officers at no expense to the Association, and in spite of various unavoidable expenditures, particularly the cost of paper in connection with the publication of the MONTHLY, this increase amounting to about $400.

The financial management feels that this is an auspicious showing, when it is considered that we have provided for the most pressing need of the National Committee on Mathematical Requirements and have put into operation the plan of subvention of the *Annals of Mathematics* as approved by the Association, an arrangement which it is certain will redound to the mutual advantage of the *Annals* and the members of the Association. The Association seems secure on the financial side, but if we are to continue to make a positive and strong contribution to the cause of mathematics in America, the Association will need the full support of its membership and the Council will not be free from the necessity of scanning with care all expenditures from the treasury.

Five hundred dollars of the reserve fund is being invested in a Liberty Bond, the final payment being made early in January.

W. D. CAIRNS, *Secretary-Treasurer.*

COMMENTS ON THE PRELIMINARY BALLOT.

By the SECRETARY.

The Constitution and By-Laws provide that "all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year" and that for the final ballot "the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order." The spirit of the Constitution would thus seem to be that members, merely with the knowledge that a second candidate has been chosen in this manner, are expected to exercise their right of choice on the final ballot. The discretion here given to the Council enables its nominating committees to take cognizance of such considerations as uniform geographical distribution, a proper balance as between universities and colleges, large and small institutions, etc.

Three unfavorable symptoms have appeared in the working out of this system: (1) The number voting in the preliminary ballot is disappointingly small when compared with the membership of the Association. The full success of our democratic system is quite dependent on a large participation on the part of the membership. To the recurring expression of lack of knowledge of suitable candidates for nomination the obvious answer is the clear obligation resting upon each member of acquainting himself with his fellow members both within and outside his own particular region and of making representations at stated times of persons qualified for the manifold kinds of service in the Association. (2) It would seem that there is probably too concentrated a list of names voted for; the officers of the Association have continually tried, and have succeeded reason-

ably well in their endeavor, to enlist an ever increasing number of members in active service, in the conviction that there are many others well qualified to assume some of the responsibilities of the organization besides those who have as yet been chosen. Just how to accomplish this in the preliminary ballot without causing too great a scattering of votes and on the other hand without interfering with the members' free choice, is a puzzling question. A suggestion has been made that a list of names might be proposed for the various offices when the blanks for the preliminary ballot are sent to the members, this list to be large enough and representative enough so that the members would still have a real choice, yet might avoid the disadvantage mentioned under the next head, while there would also be the possibility of proposing other names than those on the suggested list. (3) The same names are often proposed for different offices, and the popular vote for a certain person is thus weaker than if concentrated upon one office. The Council has directed its nominating committees, in choosing the second candidate for each office, to take into account the total vote which each person has received for all the offices. This will materially aid in interpreting the wishes of the members; yet it would be a further improvement if this duplication of voting could be avoided.

It is to be understood that full secrecy is observed in all matters connected with the balloting. To make sure that the necessity of signing the ballots will not act as a deterrent in voting, the secretary will hereafter ask each member to return the *unsigned* ballot, putting his name on the outside of the envelope merely for the purpose of enabling the name of the member to be checked off from the membership list. Any suggestions as to improvements in the machinery of balloting will be gladly received by the secretary. The ends desired from the method adopted by the Association will be attained only if all members devote a stamp and a few moments' time, if no more, to each of the two ballots of the year.

THE SECOND ANNUAL MEETING OF THE MISSOURI SECTION.

The second annual meeting of the Missouri Section of the Mathematical Association of America was held in Kansas City on Saturday, November 17, 1917, in conjunction with the meeting of the Missouri State Teachers' Association. The session was held in the Public Library. The chairman of the Section, Professor L. D. Ames of the University of Missouri, presided.

Twenty-one visitors were present and the following fifteen members:

L. D. Ames, University of Missouri, Columbia.
Austin C. Andrews, Manual Training High School, Kansas City.
Otto Dunkel, Washington University, St. Louis.
Zoe Ferguson, Central High School and Junior College, St. Joseph.
R. R. Fleet, William Jewell College, Liberty.
B. F. Finkel, Drury College, Springfield.
Mary E. Helwig, High School, Kansas City, Kansas.

C. E. Horne, Park College, Parkville.
Thomas W. Jackson, High School, Fulton.
William A. Luby, Polytechnic Institute, Kansas City.
Paul R. Rider, Washington University, St. Louis.
William H. Roever, Washington University, St. Louis.
R. A. Wells, Park College, Parkville.
Ella Woodyard, High School, Kansas City, Kansas.
William H. Zeigel, Missouri State Normal School, Kirksville.

The officers elected for the ensuing year are Professor William H. Roever, Washington University, St. Louis, Chairman; Professor O. D. Kellogg, University of Missouri, Columbia, Vice-Chairman; Dr. Paul R. Rider, Washington University, St. Louis, Secretary-Treasurer.

The program of the meeting, with abstracts of papers, follows below. In the absence of Professors Dean and Scarborough their papers were read by title.

Abstracts of Papers.

1. *Some Properties of Plane and Spherical Triangles and their Frequent Analogies.* By Professor William H. Zeigel, Missouri State Normal School, Kirksville.

Professor Zeigel called attention to the fact that trigonometric functions and hyperbolic functions are special cases of elliptic functions, also that plane trigonometry is a special case of spherical trigonometry where the radius of the sphere becomes infinite. Formulæ of spherical trigonometry go over into corresponding formulæ of plane trigonometry. It is an interesting exercise to obtain these analogous forms by a direct method.

If T denotes the area of a plane triangle, r_1 the radius of a sphere, and $\sigma = \sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}$, it follows that

$$\lim_{r_1 \doteq \infty} (\sigma r_1^2) = \sqrt{s_1(s_1 - a)(s_1 - b)(s_1 - c)} = T,$$

where the sides are expressed in circular measure. But r_1 usually equals unity, hence it appears that σ is a form analogous to that expressing the area of a plane triangle.

For relations involving the area, sides, angles, altitudes, radii of inscribed, circumscribed, and escribed circles in the plane triangle, many analogous forms are found involving corresponding parts in the spherical triangle. Four rules pertaining to identical and co-functional relations were obtained which shorten materially the theoretical work of spherical trigonometry.

Numerous frequently analogous forms were deduced.

2. *The Value of Mathematics in Secondary Education.* By Dr. John W. Withers, Superintendent of Instruction, St. Louis.

The discussion on the foregoing paper was led by Miss Zoe Ferguson, Central High School and Junior College, St. Joseph, and Professor B. F. Finkel, Drury College, Springfield.

3. *Sundials and Skylights.* By Professor WILLIAM H. ROEVER, Washington University, St. Louis.

Professor Roever's paper began with a general discussion of sundial construction. He then showed how the methods of dialing could be used to determine the form of the skylight in the conical roof of a circular prison so that every cell would receive the direct rays of the sun either from exterior windows or through the skylight, for a portion of the day the year around.

4. *Pure and Applied Mathematics in the Nineteenth Century.* By Professor G. R. DEAN, Missouri School of Mines, Rolla.

5. *The Equal Parallax Curve for Frontal and Lateral Vision.* By Dr. PAUL R. RIDER, Washington University, St. Louis.

It seems evident that those creatures having side vision, such as birds, have an advantage over those having frontal vision, such as man, in their ability to gauge the relative distances of their surroundings, since the axis of their vision is perpendicular to the direction of their motion, and hence as they move forward the apparent displacement of objects is a maximum. The equal parallax curve is a curve showing the distances that a man and a bird must move forward to give the same apparent displacement of objects against the horizon. This paper derives parametric equations of the curve. It appeared in *Science,* new series, Vol. 46, No. 1183, pp. 213–214.

6. *A Simple Derivation of the Derivatives of the Trigonometric Functions.* By Professor OTTO DUNKEL, Washington University, St. Louis.

The formulæ for the derivatives of x and y with respect to the length of arc of a curve $dx/ds = \cos\tau$, $dy/ds = \sin\tau$ can easily be developed in the early part of the calculus and this is done in several texts. These formulæ depend upon the same fundamental limit that is used in the usual development of the derivatives of the trigonometric functions and yield these derivatives when applied to the circle of unit radius. For this special curve we have $x = \cos\theta$, $y = \sin\theta$, $s = \theta$, and $\tau = \pi/2 + \theta$, as is easily seen from a figure. On substituting these values in the formulæ above, we have at once the derivative of $\cos\theta$, and of $\sin\theta$. It is readily seen that this deduction holds for any angle θ.

7. *The Graphical Solution of a Cubic Equation having Complex Roots.* By Mr. WILLIAM A. LUBY, Polytechnic Institute, Kansas City.

Mr. Luby's paper assumed the cubic in the form $y \equiv x^3 + px + q = 0$. The three values of x which make y zero are of the form $a \pm ib$ and c. Then it follows that

$$(1) \qquad a + ib + a - ib - c = 0,$$

$$(2) \qquad (a + ib)(a - ib) + (a + ib)c + (a - ib)c = p,$$

$$(3) \qquad (a + ib)(a - ib)c = q - y.$$

From (1), (2), and (3), we have

$$y = -8a^3 - 2ap + q, \tag{4}$$

$$y = q \pm \frac{2}{3}\sqrt{\frac{b^2 - p}{3}}\,(4b^2 - p). \tag{5}$$

In (4) a and y are variables, in (5) b and y. Constructing the graphs of (4) and (5) we obtain readily the complex roots of $x^3 + px + q = 0$.

8. *Applied Mathematics for the Average Student.* By Professor J. H. SCARBOROUGH, Missouri State Normal School, Warrensburg.

9. *The Solution of Linear Differential Equations with Periodic Coefficients.* By Dr. JAMES E. MCATEE, William Jewell College, Liberty.

Dr. McAtee made a report on a paper by Professors F. R. Moulton and W. D. MacMillan of the University of Chicago on "The Solution of Certain Types of Differential Equations with Periodic Coefficients." This paper is to be found in Vol. 33 of the *American Journal of Mathematics*. PAUL R. RIDER, *Secretary.*

REPORT OF ORGANIZATION OF THE ILLINOIS SECTION.

At the call of Professor H. E. Slaught the following persons met in Room 38 of Ryerson Physical Laboratory, University of Chicago, at 4:30 P.M., on December 27, 1917, to discuss a plan for organizing an Illinois Section of the Mathematical Association of America: I. A. Barnett, G. A. Bliss, H. E. Cobb, C. E. Comstock, M. W. Coultrap, A. R. Crathorne, D. R. Curtiss, A. Emch, J. A. Foberg, J. O. Hassler, Miss Nelle Ingels, E. B. Lytle, M. McNeill, H. L. Olson, H. L. Rietz, Mr. Scheibler, J. B. Shaw, C. H. Sisam, E. J. Townsend, and E. J. Wilczynski.

Mr. J. A. Foberg, of Crane Junior College, Chicago, was unanimously elected chairman of the meeting; he appointed Mr. E. B. Lytle secretary.[1] A motion of Professor Townsend that an Illinois Section be organized was unanimously adopted. Among other plans it was suggested but not determined that the meetings of this new Illinois Section be held at the time and place of the meetings of the Illinois Academy of Science.

On motion of Professor Wilczynski the chairman was authorized to appoint four members who with the chairman shall constitute an executive committee with full power to arrange the time, place and program for the first meeting of the Illinois Section. In addition to the chairman, the members of this committee are: Nelle L. Ingels, Greenville College; C. E. Comstock, Bradley Polytechnic Institute; G. T. Sellew, Knox College; and L. S. Shively, Mount Morris College.

[1] Mr. Lytle was appointed secretary at the close of this meeting, so that these minutes are written up from memory.

On motion of Professor Bliss the executive committee just formed was directed to appoint a committee of three to report at the first meeting and to present nominations for permanent officers of the Section.

ERNEST B. LYTLE,
Secretary.

BOOK REVIEW.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

A First Course in Higher Algebra. By HELEN A. MERRILL and CLARA E. SMITH. The Macmillan Company, New York, 1917. xiv + 247 pages. $1.50.

In this work written with the idea that "Higher Algebra, to be worthy of the name, must employ advanced methods," we find the plan well carried out. It is written, too, with the conscious "hope that there is nothing to be unlearned in later work." Not only have the authors succeeded in expressing the material in form which will not have to be unlearned, but this text gives us an illustration of the fact that methods often reserved for advanced work can very profitably be applied to elementary college subjects, to the great advantage of the student for the work in hand as well as for the power gained for more advanced work, by the familiarity, secured here, with methods which will be used on more intricate material in later courses.

While this text omits very little that is given in the usual college algebra, it places before the student, in minimum space and with extreme clearness, most of the usual college algebra material and much of other material often found incidentally by the student in more advanced courses, or not found at all. As an instance of the careful presentation of a subject often slighted, and sometimes omitted, could be mentioned the chapters on the number system. The first of these, the chapter on rational numbers, precedes other work, and contains, in addition to the usual material, some of the simple theorems on integers, which are very useful for factoring; while the chapter on irrational numbers comes later where it can receive a careful treatment by the theory of limits, sufficiently detailed to secure the student a definite and clear understanding.

In the second chapter, which does not differ much from the usual somewhat condensed work on permutations, combinations, and probability, the student begins work on problems of which there is a wholesome number throughout the text. Many of the problems given in various chapters serve for the further development of the text.

While the chapter on determinants is clear and well developed on the whole, articles 48 and 50 suggest conflicting ideas of a minor. It seems to the reviewer somewhat unfortunate that the student is not given a greater familiarity with determinants of order higher than the third, since determinants have so many interesting properties that might well serve as one of the subjects "to arouse curiosity and to lead students to find for themselves what lies just beyond."

It is at this point in the text that the discussion of the theory of limits begins, with a chapter on variables and their limits, followed by a chapter on differentiation of algebraic functions. A clear geometric interpretation of both the first and the second derivative is given, showing the analytical significance of the maximum, the minimum, and the point of inflection. Since the student is here given the condition for maximum and minimum, the condition might well have been used later, on page 210, to determine whether the ordinate of the minimum point of $x^3 - 7x + 7$ is positive or negative, as a preliminary to the discussion of this function by Sturm's theorem.

There follows a chapter on series where the material is advisedly chosen, well presented, and made definite by examples. This chapter gives the student familiarity with the Cauchy test as well as other tests. On this foundation rests the development of functions in series, where undetermined coefficients are treated, and Maclaurin's expansion is used with careful study of the region of convergence. The binomial theorem is shown as a special case. The multinomial theorem is not given. In the treatment of partial fractions a sufficient number of examples is given to secure the desirable dexterity to the prospective student of calculus.

With the theory of limits, series, and convergence in hand, the student is now prepared for the chapter on irrational numbers, wherein the ideas of a dense set and a continuous set are made clear. The existence of non-terminating, non-repeating decimal numbers is shown, and it is explained how a sequence of numbers is said to define an irrational number. On this basis $\sqrt{2}$ is given and a method of finding the value of π is indicated. The sum, difference, product, and quotient of irrational numbers, the power and root of an irrational number, are also defined by sequences. The meaning of an irrational exponent is explained by means of a limit, and given with sufficient detail to make the following chapter on logarithms clear and full of meaning. While e and π are given their classification in the chapter on irrational numbers, it is in the work on logarithms that e is shown to be the $\lim_{n \doteq \infty} [1 + (1/n)]^n$, in which work the student has another opportunity to increase his familiarity with convergence tests.

The derivative of the Naperian logarithm of a function is given, allowing the student a working knowledge of that derivative, which, though all the foundational work on interchange of limits is not included, gives the student the material for securing the logarithmic series by Maclaurin's expansion. Here, again, there arises an opportunity to use the convergence tests to find the region of convergence of the series thus secured. Not only will this chapter probably fulfil the hope of the authors that the "work will help the student to appreciate the labor that has gone into the construction of tables of logarithms," but it must surely also give the student a more accurate idea of the logarithm as a function instead of merely as a mechanical convenience.

In addition to the usual work on complex numbers as to their combination and the geometrical representation of the numbers and of their combinations, the authors prove de Moivre's theorem and give the geometrical interpretation

of powers and roots of complex numbers. The graphing of a function of a complex variable is indicated, as is also the use of complex numbers and the complex variable in series, with graphical illustration of the sequence S_n of a series. Here, as at the end of most chapters, good references are given from which the student may secure further knowledge on the subject treated.

The more or less usual chapter on the theory of equations is given, with an exposition of the theory of Sturm's functions and the approximation of real roots by Horner's method.

At the end of the book there is a very brief chapter on integration as the inverse of differentiation. The reviewer would approve the suggestion of the authors that this chapter "may serve for reference in later work;" for, while the chapter is interesting, it is hardly essential or even useful to the student of algebra. It would, no doubt, be of some use to a student of science, but such student would need so much more thorough work on integration, gained only by more complete study and extended practice with that very important operation, that it might be wise to devote the time which would be required for this chapter to establishing a more thorough acquaintance with material already in hand.

The following misprints were noted:

Page *XIII*, line 15, exponent of x should be $n - 1$ instead of $m - 1$; page 116, equation (2), numerator should be $2 - 3x + 4x^3$ instead of $2 - 3x - 4x^3$; page 121, line 18, read *positive* for *position*; page 156, line 9, read $\Delta y/\Delta x$ instead of $\Delta y/\Delta$; page 234, line 29, read $a_0x^n + a_1x^{n-1}$ instead of $a_0x^n + a_1x^{n-}$.

The material of the text, somewhat out of the usual line, chosen with discrimination, is arranged in a way to render the text easily read and of easy use for reference, thus making the book a valuable addition to the library of the undergraduate for reference, as well as a workable text for a class in the early undergraduate years.

MARY E. WELLS.

VASSAR COLLEGE.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2670. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A telegraph wire, weighing one tenth pound per yard, is stretched between poles on level ground, so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

2671. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find two rectangular parallelopipedons whose edges are rational whole numbers and whose solid diagonals are also rational whole numbers and equal.

2672. Proposed by E. T. BELL, Seattle, Washington.

There is an identity in z, (1) $A(z) \equiv B(z)C(z)$; (*e. g.*, $A(z) = 1/(1 - k^2z^2)$; $B(z) = 1/(1 - kz)$; $C(z) = 1/(1 + kz)$); and the formal expansions $A(z) = \Sigma a(n)z^n$, $B(z) = \Sigma b(n)z^n$, $C(z) = \Sigma c(n)z^n$, $(n = 0, 1, \cdots, \infty)$, when substituted in (1), give, on equating coefficients, (2): $a(n) = b(n)c(0) + b(n - 1)c(1) + \cdots + b(0)c(n)$. If (2) is an identity in n, justify such

a use of non-convergent series to obtain it (*e. g.*, for $|k| \geqq 1$ in the above). This method of finding important identities (2) has been used freely by Hermite and many others without question of its validity, and without offering independent proofs of (2).

2673. Proposed by WILLIAM O. BEAL, University of Minnesota.

A plane through the center of an oblate spheroid makes an angle, i, with the plane of its equator. Express the eccentricity, e', of this section in terms of the eccentricity, e, of a meridian section and the angle, i.

2674. Proposed by J. O. MAHONEY, Forest Avenue High School, Dallas, Texas.

If two sides of a triangle differ by less than a certain length, e, the two opposite angles will differ by less than a certain quantity λ, expressed in degrees, such that $\lambda < 61e/a$, where a expresses, with a possible error e, the length of the apparently equal sides of the triangle.

2675. Proposed by E. B. ESCOTT, Kansas City, Mo.

Sum the series

$$-\frac{1}{2}\cdot\frac{1^3x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{2^3x^5}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{3^3x^7}{7}+\cdots.$$

2676. Proposed by EDWIN R. SMITH, State College, Pa.

Find the maximum term of the series

$$\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(-r,\,-sq,\,sp-r+1,\,1)$$

where s, r, sq and sp are positive integers, $r < s$, p and q are proper fractions such that $p + q = 1$, and $F(-r,\,-qs,\,sp-r+1,\,1)$ is a hypergeometric series. If s, r and $s - r$ are large, show that the maximum term is approximately equal to

$$\sqrt{\frac{s}{2\pi pqr(s-r)}}.$$

2677. Proposed by R. K. MORLEY, Worcester, Mass.

A quarter-mile track is to be constructed, having semi-circular ends and straightaway sides. It is required to have the rectangular part of enclosed field referred to in No. 12, Granville's Calculus, page 116, as large as possible. Find length of the straightaways.

Also, required to inscribe the maximum rectangle in a track of length l, with semi-circular ends and straightaway sides, assuming that two sides of the rectangle are parallel to the straightaways. Find the length of the straightaways and the dimensions of the rectangle.

2678. Proposed by C. F. GUMMER, Queen's University, Kingston, Canada.

Find necessary and sufficient conditions that the roots of the equation

$$x^{n+1}+a_1x^n+a_2x^{n-1}+\cdots+a_{n+1}=0$$

may be all real and separated by the roots of $x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0$.

2679. Proposed by J. W. LASLEY, University of North Carolina.

Show that the perpendicular from any point on a circle to any chord of the circle is a mean proportional to the perpendiculars from that point to the tangents at the ends of the chord.

SOLUTIONS OF PROBLEMS.

Note.—Prepare solutions as follows: (1) Give the number of the problem with the name of the proposer and his address; (2) restate the problem as published; (3) then give the name of the solver with address; (4) then give the solution carefully and neatly written.

Solutions of 477 (Algebra) were received from E. B. Escott and E. H. Worthington; of 478 (Algebra) from E. B. Escott; and of 481 (Algebra) from L. C. Mathewson. These solutions were received after copy had been prepared for publication. Professor Worthington should have received credit also for solving 476 (Algebra).

484 (Algebra). Proposed by E. V. HUNTINGTON, Cambridge, Mass.

Show that

$$\frac{\dfrac{1}{m^2}-\dfrac{k_1}{(m+1)^2}+\dfrac{k_2}{(m+2)^2}-\cdots+\dfrac{(-1)^k}{(m+k)^2}}{\dfrac{1}{m}-\dfrac{k_1}{m+1}+\dfrac{k_2}{m+2}-\cdots+\dfrac{(-1)^k}{m+k}}=\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+k}$$

for all positive integral values of m and k. Here

$$k_1=\frac{k}{1},\quad k_2=\frac{k(k-1)}{1\cdot 2},\quad k_3=\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3},\quad \text{etc.}$$

This equation was suggested to the proposer by a professor of chemistry who wishes to make use of the equation, if correct, in an actual problem in bacteriology.

I. Solution by David F. Barrow, Sheffield Scientific School.

The denominator may be summed as follows: Consider

$$\int_0^1 x^{m-1}(1-x)^k dx.$$

If we expand the integrand by the binomial theorem and integrate term by term, we obtain the denominator. But if we integrate by parts, taking $u=(1-x)^k$ and $dv=x^{m-1}dx$, we obtain

$$\left.\frac{x^m(1-x)^k}{m}\right]_0^1+\frac{k}{m}\int_0^1 x^m(1-x)^{k-1}dx.$$

The closed part vanishes at both limits. This furnishes a convenient reduction formula by successive applications of which the integration may be completed without difficulty. We are thus led to the result,

$$\frac{1}{m}-\frac{k_1}{m+1}+\frac{k_2}{m+2}-\cdots+\frac{(-1)^k}{m+k}=\frac{k!(m-1)!}{(m+k)!}.$$

In the formula to be proved, let us multiply up this denominator on the right side.

$$(1)\quad \frac{1}{m^2}-\frac{k_1}{(m+1)^2}+\frac{k_2}{(m+2)^2}-\cdots+\frac{(-1)^k}{(m+k)^2}=\frac{k!(m-1)!}{(m+k)!}\left(\frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{m+k}\right).$$

Now if $k=0$ or if $k=1$, this formula is easily seen to be true for all positive integral values of m. We shall complete the proof by mathematical induction if we establish that *if* (1) *is true for all values of m and any particular value of k, then it is true for all values of m and that value of k increased by unity.*

We assume, then, that (1) is an identity in m when $k=k$. We may therefore replace m by $m+1$, obtaining

$$(2)\quad \begin{aligned}&\frac{1}{(m+1)^2}-\frac{k_1}{(m+2)^2}+\frac{k_2}{(m+3)^2}-\cdots+\frac{(-1)^k}{(m+k+1)^2}\\&\qquad=\frac{k!m!}{(m+k+1)!}\left(\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+k+1}\right).\end{aligned}$$

Now $k_1, k_2, \cdots$ are the binomial coefficients. Hence they satisfy the relation $k_{i-1}+k_i=(k+1)_i$ (preserving the notation of the proposer). Remembering this, we subtract equation (2) from equation (1), and get

$$\frac{1}{m^2} - \frac{(k+1)_1}{(m+1)^2} + \frac{(k+1)_2}{(m+2)^2} - \cdots + \frac{(-1)^{k+1}}{(m+k+1)^2}$$

$$= \frac{k!(m-1)!}{(m+k)!}\left[\frac{1}{m} + \left(1 - \frac{m}{m+k+1}\right)\left(\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+k}\right) - \frac{m}{m+k+1}\cdot\frac{1}{m+k+1}\right]$$

$$= \frac{k!(m-1)!}{(m+k)!}\left[\frac{(m+k+1)^2 - m^2}{m(m+k+1)^2} + \frac{k+1}{m+k+1}\left(\frac{1}{m+1} + \cdots + \frac{1}{m+k}\right)\right]$$

$$= \frac{(k+1)!(m-1)!}{(m+k+1)!}\left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+k+1}\right).$$

But this is what we get if in (1) we replace k by $k + 1$. Hence (1) is true for all values of m and for $k + 1$; hence it is true for all values of m and k.

II. Solution by Edwin Bidwell Wilson, Massachusetts Institute of Technology.

The equation is an identity for all values of m and is merely a special case of a general theorem in logarithmic differentiation of rational fractions.

Suppose that

$$A\,\frac{(x-a_1)(x-a_2)\cdots(x-a_m)}{(x-b_1)(x-b_2)\cdots(x-b_n)}$$

is a rational fraction, and for simplicity let it be proper ($m < n$) and without multiple factors in either numerator or denominator (the result for the general rational fraction is more complicated only in notation).

The derivative of the logarithm of this fraction is

$$\frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_m} - \frac{1}{x-b_1} - \frac{1}{x-b_2} - \cdots - \frac{1}{x-b_n}.$$

The fraction may, however, be resolved into partial fractions as

$$\frac{c_1}{x-b_1} + \frac{c_2}{x-b_2} + \cdots + \frac{c_n}{x-b_n}$$

and its logarithmic derivative then has the form

$$-\frac{\dfrac{c_1}{(x-b_1)^2} + \dfrac{c_2}{(x-b_2)^2} + \cdots + \dfrac{c_n}{(x-b_n)^2}}{\dfrac{c_1}{x-b_1} + \dfrac{c_2}{x-b_2} + \cdots + \dfrac{c_n}{x-b_n}},$$

which must be identical with the former expression.

In the particular case mentioned in the problem the rational fraction is evidently of the form

$$\frac{A}{m(m+1)(m+2)\cdots(m+k)},$$

m replacing x. The expansion into partial fractions is

$$\frac{k_0}{m} + \frac{k_1}{m+1} + \frac{k_2}{m+2} + \cdots + \frac{k_k}{m+k}$$

and the coefficients k_p may readily be found by the "substitution" method. Indeed, if we clear of fractions and set $m = -p$ we have

$$A = k_p(-p)(-p+1)\cdots(-1)(1)(2)\cdots(k-p)$$

or

$$k_p = \frac{(-1)^p A}{p!(k-p)!} = (-1)^p\frac{k(k-1)\cdots(k-p+1)}{1\cdot 2\cdots p}$$

if the as yet undetermined A be taken as $k!$.

Another interesting special case is found on treating

$$\frac{A}{(m-k)(m-k+1)\cdots(m-1)m(m+1)\cdots(m+k-1)(m+k)},$$

especially if k be allowed to become infinite (cotangent series).

Also solved by FRANK IRWIN, C. F. GUMMER, S. BEALTY, O. SCHMIEDEL, OLIVE C. HAZLETT, and RALPH KEFFER.

485 (Algebra). Proposed by J. L. WALSH, Madison, Wisconsin.

Is it true that to every convergent series of positive terms, $a_1 + a_2 + a_3 + \cdots$, there corresponds a series of the type

$$\frac{M}{1^p} + \frac{M}{2^p} + \frac{M}{3^p} + \cdots,$$

such that $M/k^p > a_k$, $p > 1$?

SOLUTION BY E. H. MOORE, The University of Chicago.

A convergent series of positive terms, $a_1 + a_2 + \cdots + a_k + \cdots$, is a positive-valued function α; $\alpha(k) = a_k$ $(k = 1, 2, 3, \cdots)$, of the variable positive integer k, satisfying the condition that the corresponding series is convergent. Denote the class of all such functions α by $\mathfrak{C}_+$.

If the question proposed is answered in the affirmative, then there exists a sequence α_n $(n = 1, 2, 3, \cdots)$ of functions α_n of the class $\mathfrak{C}_+$, viz., the sequence of functions

$$\alpha_n : \alpha_n(k) = \frac{n}{k^{1+(1/n)}} \quad (k = 1, 2, 3, \cdots),$$

of such a nature that every function α of the class $\mathfrak{C}_+$ is dominated by a suitably chosen function α_n of the sequence, viz., $|\alpha(k)| \leqq |\alpha_n(k)|$ $(k = 1, 2, 3, \cdots)$,—that is, the class $\mathfrak{C}_+$ has the dominance property D_2 defined (for the general class of real-valued functions on the general range) in § 22 of my *Introduction to a Form of General Analysis* (The New Haven Mathematical Colloquium, Yale University Press, 1910, p. 41).

Now I have proved (§ 23*c* (5), *loc. cit.*) that the class $\mathfrak{M}^{\mathrm{III}_1}$ (and, a fortiori, the class $\mathfrak{C}_+$) of all absolutely convergent series of real-valued terms fails to have the dominance property D_2. Hence, the question proposed must be answered in the negative.

The proposition cited is one of a number of theorems involving various dominance properties. The present question may be answered still more luminously by citing the theorem of Hadamard (Acta Mathematica, vol. 18, 1894, p. 328, theorem (β); cf. also *loc. cit.*, p. 49) that for every sequence (α_n) of functions of the class $\mathfrak{C}_+$ such that for every k $\alpha_n(k)$ increases with n there exists a function α of the class $\mathfrak{C}_+$ of such a nature that for every n

$$\lim_{k=\infty} \frac{\alpha_n(k)}{\alpha(k)} = 0.$$

Also solved by ELIJAH SWIFT.

486 (Algebra). Proposed by FLORENCE P. LEWIS, Baltimore, Md.

Find the condition which must be satisfied by the coefficients of the quartic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

in order that the equation be solvable by successive applications of the quadratic formula.

SOLUTION BY THE PROPOSER.

The equation must be of the form $A(ax^2 + bx + c)^2 + B(ax^2 + bx + c) + C = 0$ or $Ay^2 + By + C = 0$. Let x_1 and x_1' correspond to root y_1, and x_2 and x_2' to root y_2, respectively. We then have $x_1 + x_1' = -b/a$ and $x_2 + x_2' = -b/a$. This says that the line joining x_1 and x_1' is bisected at $-b/2a$, or that x_1 and x_1' are harmonic conjugates as to $-b/2a$ and ∞. The same

is true of x_2 and x_2'. Hence, ∞ is a root of the sextic covariant, whose leading coefficient must therefore vanish. This gives $8a_0^2a_3 - 4a_0a_1a_2 + a_1^3 = 0$ as the required condition. The same result may be readily obtained by elementary methods.

Also solved by HERBERT N. CARLETON, OTTO J. RAMLER, and HORACE OLSON.

E. B. ESCOTT sent in three solutions of 515 (Geometry). The first two solutions were solutions given in E. CATALAN'S *Théorèmes et Problèmes de Géométrie Élémentaire*, pp. 237–239. The third is a neat original solution.

516 (Geometry). Proposed by R. M. MATHEWS, Riverside, California.

Through the edges of a trihedral angle planes are passed orthogonal to the opposite faces Prove them coaxial.

I. SOLUTION BY NATHAN ALTSHILLER, University of Oklahoma.

The planes x, y, z which pass through the edges OA, OB, OC of the trihedral angle O–ABC and which are perpendicular to the opposite faces a, b, c, cut these faces along lines OD, OE, OF, respectively. A plane p perpendicular to OA cuts the edges OA, OB, OC in the points A, B, C, and the lines OD, OE, OF in the points D, E, F. (The reader may readily construct the figure.)

The plane x is perpendicular to a by construction, and to the plane p, because x passes through OA; hence, x is perpendicular to the line of intersection BC of p with a, and therefore BC is perpendicular to AD.

The plane b is perpendicular to y by construction and to the plane p, because b passes through OA; hence, b is perpendicular to the line of intersection BE of y with p, and, therefore, BE is perpendicular to AC. For similar reasons, CF is perpendicular to AB.

The three altitudes AD, BE, CF of the triangle ABC concur, according to a well-known proposition, in a point H, the orthocenter of ABC; hence H is a common point of the planes x, y, z. Now these three planes have obviously the point O in common, hence they pass through the line OH.

Incidentally we have also proved: *The locus of the orthocenter of the triangle determined by three concurrent lines and a variable plane perpendicular to one of the given lines is a straight line concurrent with the given lines.*

II. SOLUTION BY W. WOOLSEY JOHNSON, Annapolis, Md.

Referred to the sphere, the problem is that of the existence of the orthocenter of the spherical triangle ABC.

Let $CD = p$ be the perpendicular from C upon AB and let the perpendicular from A upon BC cut it in O.

From the right triangle AOD we have

$$\tan OD = \tan OAD \sin AD. \tag{1}$$

From AEB, we have

$$\cot OAD = \cos c \tan B. \tag{2}$$

From ADC, we have

$$\sin AD = \tan p \cot A. \tag{3}$$

Dividing (3) by (2) and substituting in (1), we have $\tan OD = \tan p \cot A \cot B \sec c$.

Interchanging A and B, c and p remain unchanged; hence, the perpendicular from B cuts off from p the same segment OD or the three perpendiculars meet in a common point.

Also solved by HORACE OLSON, L. E. LUNN, C. J. PAYNE, WILLIAM HOOVER, and the PROPOSER.

517 (Geometry). Proposed by R. P. BAKER, University of Iowa.

The coördinates of the vertices of a regular icosahedron can be expressed rationally in terms of $(\sqrt{5}-1)/4$ and $\sqrt{(5+\sqrt{5})/8}$, that is, $\cos(2\pi/5)$ and $\sin(2\pi/5)$. Prove (1) that the cosine alone is sufficient; (2) that the irrationalities cannot be reduced further. (The theorem that they cannot be rational is proved in books on crystal theory.)

SOLUTION BY C. F. GUMMER, Queen's University, Kingston.

Since the coördinates of the center are expressible rationally in terms of the vertices, we may suppose it taken as origin. Moreover the vertices are all rational in terms of any three adjacent ones A, B, C; for a fourth is the reflection of A for the plane OBC, and so on. Suppose first that A, B, C have coördinates (o, o, r), $(r \sin\alpha, o, r\cos\alpha)$, $(r\sin\alpha\cos(2\pi/5), r\sin\alpha\sin(2\pi/5), r\cos\alpha)$. Since $AB = BC$, we find $\cos\alpha = 1/\sqrt{5}$; and α is the angle subtended at O by an edge. Hence, the general equations for A, B, C (with the origin for center) are

$$x_1^2+y_1^2+z_1^2 = x_2^2+y_2^2+z_2^2 = x_3^2+y_3^2+z_3^2 = \sqrt{5}(x_1x_2+y_1y_2+z_1z_2)$$
$$= \sqrt{5}(x_2x_3+y_2y_3+z_2z_3) = \sqrt{5}(x_3x_1+y_3y_1+z_3z_1).$$

Evidently no rational solution exists, which proves the second part of the theorem.

A simple solution is found by assuming $x_1 = y_2 = z_3 = u$, $x_2 = y_3 = z_1 = v$, $x_3 = y_1 = z_2 = w$, so that the equations become $u^2+v^2+w^2 = \sqrt{5}(uv+vw+wu)$. If $w = 0$, $u^2+v^2 = \sqrt{5}\,uv$, which is satisfied by $u = 1$, $v = 2\cos(2\pi/5)$. Therefore, A, B, C may be taken to be $(1, 0, 2\cos(2\pi/5))$, $(2\cos(2\pi/5), 1, 0)$, $(0, 2\cos(2\pi/5), 1)$.

518 (Geometry). Proposed by ROGER A. JOHNSON, Cleveland, Ohio.

If one angle of a triangle is 60°, the Euler line (the line through the circumcenter, orthocenter, and median point) is perpendicular to the bisector of that angle; and if one angle is 120°, the Euler line is parallel to the bisector of that angle.

SOLUTION BY J. L. COOLIDGE, Harvard University.

Let the vertices of the triangle be A_1, A_2, A_3, the middle points of the sides M_1, M_2, M_3, the feet of the altitudes H_1, H_2, H_3, the circumcenter O, and the orthocenter H.

Let us take A_1 as the angle in which we are interested, and assume it first to be acute.

The angle which the external bisector forms with A_2A_3 is $\frac{1}{2}(A_2-A_3)$ and its tangent

$$\tan\tfrac{1}{2}(A_2-A_3) = \frac{1-\cos(A_2-A_3)}{\sin(A_2-A_3)}.$$

The tangent of the angle which the Euler line makes with A_2A_3 is

$$\frac{OM_1 - HH_1}{A_2M_1 - A_2H_1}.$$

We have

$$OM_1 = r\cos A_1 = r[\sin A_2\sin A_3 - \cos A_2\cos A_3],$$
$$A_2M_1 = r\sin A_1 = r[\sin A_2\cos A_3 + \cos A_2\sin A_3],$$
$$HH_1 = (A_2H_1)\operatorname{ctn} A_3 = 2r\cos A_2\cos A_3,$$

and

$$A_2H_1 = a_3\cos A_2 = 2r\cos A_2\sin A_3.$$

The tangent of this angle is

$$\frac{\sin A_2\sin A_3 - 3\cos A_2\cos A_3}{\sin(A_2-A_3)}.$$

The two tangents are equal if

$$1-\cos A_2\cos A_3 - \sin A_2\sin A_3 = \sin A_2\sin A_3 - 3\cos A_1\cos A_3, \text{ or } \cos(A_2+A_3) = -\tfrac{1}{2}.$$

Hence,

$$A_2 + A_3 = 120^\circ.$$

The case when A_1 is obtuse may be treated by a similar method.

Also solved by A. M. Harding, Nathan Altshiller, C. C. Yen, J. F. Lü, K. K. Chan, Horace Olson, G. Breit, H. C. Gossard, Frank V. Morley, Louis Weisner, and Otto J. Ramler.

428 (Calculus). Proposed by J. L. Riley, Northwestern State Normal School, Tahlequah, Okla.

The loop of a lemniscate rolls in contact with the axis of x. Prove that the locus of the node is given by the equation

$$1+\left(\frac{dy}{dx}\right)^2=\left(\frac{a}{y}\right)^{4/3},$$

and that $2\rho\rho'=a^2$, if ρ, ρ' be corresponding radii of curvature of this locus and the lemniscate.

Solution by A. M. Harding, University of Arkansas.

The problem, as originally stated, is incorrect. It should read as above. The equation of the lemniscate, referred to a tangent at its center, is $r^2=a^2\sin 2\theta$. By the formulas of the elementary calculus it is easily shown that

$$s=\int_0^\theta a\sqrt{\csc 2\theta}d\theta,\qquad \psi=2\theta,\qquad \text{and}\qquad \rho=\frac{a}{3}\sqrt{\csc 2\theta},$$

where s is the length of arc from the node $A(x, y)$ to the point P, ψ is the angle between AP and the tangent at P, and ρ is the radius of curvature at P.

Let the lemniscate roll along the x-axis until the tangent at P coincides with this axis, and let O be the origin of coördinates. Then

$$x=OP-AP\cos\psi=a\int_0^\theta\sqrt{\csc 2\theta}d\theta-a\sqrt{\sin 2\theta}\cos 2\theta,\text{ and } y=AP\sin\psi=a\sqrt{\sin 2\theta}\sin 2\theta;$$

whence,

$$\frac{dy}{dx}=\cot 2\theta,$$

and

$$\frac{d^2y}{dx^2}=-\frac{2(\csc 2\theta)^{3/2}}{3a\sin^2 2\theta}.$$

Hence,

$$1+\left(\frac{dy}{dx}\right)^2=\csc^2 2\theta=\left(\frac{a}{y}\right)^{4/3}$$

and

$$\rho'=-\frac{3}{2}a\sqrt{\sin 2\theta}.$$

Hence,

$$\rho\rho'=-\frac{a^2}{2}.$$

Also solved by William Hoover.

430 (Calculus). Proposed by G. Paaswell, New York City.

Revolve a circle about a chord (not a diameter). Select a system of rectilinear coördinates with this chord as one axis and the origin as the intersection of the chord and the circumference. Term this axis the z-axis and pass a plane through the x- (or y-) axis. Find the area of this surface intercepted by this plane and the xz- (or yz-) plane.

Solution by William Hoover, Columbus, Ohio.

Let a = the radius of the circle, c = the distance of the chord from the center, 2α = the angle subtended at the center by the arc of the chord; take the middle of the chord for the origin, and the radius at right angles to the chord for the x-axis; the equation to the generating arc is then

$$z^2 + (x + a \cos \alpha)^2 = a^2,$$

or

$$x = \sqrt{a^2 - z^2} - a \cos \alpha.$$

The element of length of the arc is $ds = adz/\sqrt{a^2 - z^2}$, and the required surface

$$S = \int 2\pi x ds = 2\pi \int_0^{a \sin \alpha} (\sqrt{a^2 - z^2} - a \cos \alpha) \frac{adz}{\sqrt{a^2 - z^2}}$$

$$= 2\pi a \left[z - a \cos \alpha \int \frac{dz}{\sqrt{a^2 - z^2}} \right]_0^{a \sin \alpha}$$

$$= 2\pi a^2 \left[\sin \alpha - \cos \alpha \left(\sin^{-1} \frac{z}{a} \right)_0^{a \sin \alpha} \right] = 2\pi a^2 (\sin \alpha - \alpha \cos \alpha).$$

If $\alpha = \pi/2$, $S = 2\pi a^2$, as it should be.

The object of the choice of coördinate axes as assigned in the statement of the problem is not evident.

The volume is

$$V = \pi \int x^2 dz = \pi \int_0^{a \sin \alpha} (\sqrt{a^2 - z^2} - a \cos \alpha)^2 dz$$

$$= \pi \left[a^2(1 + \cos^2 \alpha) z - \tfrac{1}{3} z^3 - 2a \cos \alpha \left(\frac{z}{2} \sqrt{a^2 - z^2} + \frac{a^2}{2} \sin^{-1} \frac{z}{a} \right) \right]_0^{a \sin \alpha}$$

$$= \pi a \{ \tfrac{1}{3}(2a^2 + a^2 \cos^2 \alpha) \sin \alpha - a\alpha \cdot a \cos \alpha \}$$

$$= \pi a \left(\frac{2a^2 + c^2}{3} \sin \alpha - a\alpha \cdot c \right), \qquad c = a \cos \alpha.$$

If $c = 0$, $V = \frac{2}{3}\pi a^3$, as it should be.

431 (Calculus). Proposed by J. W. LASLEY, University of North Carolina.

Explain Bertrand's fallacy:

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dydx = \int_{y=0}^{y=1} \int_{x=0}^{x=1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy,$$

$$\tfrac{1}{4}\pi = -\tfrac{1}{4}\pi, \; 1 = -1.$$

SOLUTION BY HORACE OLSON, Chicago, Illinois.

There is a theorem that the order of integration of a double integral may be reversed *if the integrand is continuous in both variables within the region of integration.* The integrand in this fallacy is discontinuous at ($x = 0$, $y = 0$). Therefore, the theorem does not justify the reversal of the order of integration.

341 (Mechanics). Proposed by PAUL CAPRON, U. S. Naval Academy.

A pole l feet long, with one end on the ground, touches the top of a wall a feet high and slides in a vertical plane perpendicular to the wall. Show that its instantaneous center of rotation is at the intersection of the vertical where it touches the ground with the perpendicular to its axis where it touches the wall, and that the locus of this center is a parabola having the latus rectum a.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let T be the point at the top of the wall and G the point on the ground through which the pole passes at some given instant. Let O be the point on the ground in the same vertical line with T, and let θ be the angle TGO.

The direction of motion of any point is clearly at right angles to the line joining that point to the instantaneous center of rotation. (See Ziwet, *Theoretical Mechanics*, Art. 23; Demartres, *Cours de Géométrie infinitésimale*, Art. 20.) Hence, if the direction of motion of a point be known, the instantaneous center C must lie on the normal at the point to the direction of motion.

Now the direction of motion of that point of the rod which at the instant coincides with T is clearly along the rod. Hence the instantaneous center C is on the normal to the rod at T. Again, the direction of motion of the point G on the ground is along the ground. Hence, C is in the vertical line through G.

To find the locus of C, we choose OG and OT as coördinate axes. Then,

$$x = OG = OT \cot\theta = a \cot\theta \text{ and } y = GC = TG \csc\theta = a \csc^2\theta.$$

Eliminating θ between these two equations, we readily obtain

$$x^2 = a(y-a),$$

which is the equation of a parabola with its vertex at T $(0, a)$, its axis vertical, and latus rectum equal to a. The pole being of limited length l, the locus of C is that part of the parabola for which $0 < x < \sqrt{l^2 - a^2}$.

Also solved by H. R. Howard, William Hoover, J. B. Reynolds, and the Proposer.

342 (Mechanics). Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform rod of length $2a$ is freely hinged at one end, at the other end a string of length b is attached which is fastened at its further end to a point on the surface of a homogeneous sphere of radius c. If the masses of the rod and sphere are equal, find the motion of the system when slightly disturbed from the vertical, and the cubic equation giving the corresponding small oscillations.

Solution by the Proposer.

For symmetry of notation, let there be three bodies of masses m_1, m_2, m_3; of axes of symmetry whose lengths are $2a_1, 2a_2, 2a_3$ with centers of gravity G_1, G_2, G_3 distant a_1, a_2, a_3 from corresponding extremities; radii of gyration about G_1, G_2, G_3 equal to k_1, k_2, k_3; the origin of rectangular coordinates at the fixed extremity of the highest body, (x_1, y_1), (x_2, y_2), (x_3, y_3), the horizontal through the fixed point being the axis of x; and let $\varphi_1, \varphi_2, \varphi_3$ be the angles which the axes of symmetry of the bodies make with the vertical at any time t from the beginning of motion. Then,

$$x_1 = a_1 \sin\varphi_1, \qquad y_1 = a_1\cos\varphi_1, \tag{1}$$

$$x_2 = 2a_1\sin\varphi_1 + a_2\sin\varphi_2, \qquad y_2 = 2a_1\cos\varphi_1 + a_2\cos\varphi_2, \tag{2}$$

$$x_3 = 2a_1\sin\varphi_1 + 2a_2\sin\varphi_2 + a_3\sin\varphi_3, \qquad y_3 = 2a_1\cos\varphi_1 + 2a_2\cos\varphi_2 + a_3\cos\varphi_3. \tag{3}$$

The kinetic potential equation is

$$\begin{aligned} T &= \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + k_1^2\dot{\varphi}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + k_2^2\dot{\varphi}_2^2) + \tfrac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2 + k_3^2\dot{\varphi}_3^2) \\ &= m_1 g a_1\cos\varphi_1 + m_2 g(2a_1\cos\varphi_1 + a_2\cos\varphi_2) \\ &\qquad + m_3 g(2a_1\cos\varphi_1 + 2a_2\cos\varphi_2 + a_3\cos\varphi_3) = V. \end{aligned} \tag{4}$$

From (1), (2), (3),

$$\dot{x}_1 = a_1\cos\varphi_1\cdot\dot{\varphi}_1, \qquad \dot{y}_1 = -a\sin\varphi_1\cdot\dot{\varphi}_1, \tag{5}$$

$$\dot{x}_2 = 2a_1\cos\varphi_1\cdot\dot{\varphi}_1 + a_2\cos\varphi_2\cdot\dot{\varphi}_2, \qquad \dot{y}_2 = -2a_1\sin\varphi_1\cdot\dot{\varphi}_1 - a_2\sin\varphi_2\cdot\dot{\varphi}_2, \tag{6}$$

$$\begin{aligned} &\dot{x}_3 = 2a_1\cos\varphi_1\cdot\dot{\varphi}_1 + 2a_2\cos\varphi_2\cdot\dot{\varphi}_2 + a_3\cos\varphi_3\cdot\dot{\varphi}_3, \\ &\qquad \dot{y}_3 = -2a_1\sin\varphi_1\cdot\dot{\varphi}_1 - 2a_2\sin\varphi_2\cdot\dot{\varphi}_2 - a_3\sin\varphi_3\cdot\dot{\varphi}_3. \end{aligned} \tag{7}$$

These in (4) give

$$\begin{aligned} T &= \tfrac{1}{2}m_1(a_1^2 + k_1^2)\dot{\varphi}_1^2 + \tfrac{1}{2}m_2\{4a_1^2\dot{\varphi}_1^2 + 4a_1a_2\cos(\varphi_1 - \varphi_2)\dot{\varphi}_1\dot{\varphi}_2 + (a_2^2 + k_2^2)\dot{\varphi}_2^2\} \\ &\quad + \tfrac{1}{2}m_3\{4a_1^2\dot{\varphi}_1^2 + 4a_2^2\dot{\varphi}_2^2 + (a_3^2 + k_3^2)\dot{\varphi}_3^2 + 8a_1a_2\cos(\varphi_1 - \varphi_2)\dot{\varphi}_1\dot{\varphi}_2 \\ &\quad + 4a_2a_3\cos(\varphi_2 - \varphi_3)\dot{\varphi}_2\dot{\varphi}_3 + 4a_1a_3\cos(\varphi_1 - \varphi_3)\dot{\varphi}_1\dot{\varphi}_3\} \\ &= m_1 g a_1\cos\varphi_1 + m_2 g\,(2a_1\cos\varphi_1 + a_2\cos\varphi_2) \\ &\qquad + m_3 g(2a_1\cos\varphi_1 + 2a_2\cos\varphi_2 + a_3\cos\varphi_3) = V. \end{aligned} \tag{8}$$

Applying to (8) Lagrange's equations of typical form

$$\frac{d}{dt}\frac{dT}{d\dot{\varphi}_n} - \frac{dT}{d\varphi_n} = \frac{dV}{d\varphi_n}, \qquad n = 1, 2, 3, \tag{9}$$

the φ_1 function is

$$\begin{aligned}\{m_1(a_1^2 + k_1^2) + 4a_1^2m_2 + 4a_1^2m_3\}\ddot{\varphi}_1 + 2a_1a_2m_2 \cos(\varphi_1 - \varphi_2)\ddot{\varphi}_2 + 2a_1a_2m_2 \sin(\varphi_1 - \varphi_2)\dot{\varphi}_2^2 \\ + 4a_1a_2m_3 \cos(\varphi_1 - \varphi_2)\ddot{\varphi}_2 + 4a_1a_2m_3 \sin(\varphi_1 - \varphi_2)\dot{\varphi}_2^2 + 2a_1a_3m_3 \cos(\varphi_1 - \varphi_3)\ddot{\varphi}_3 \\ + 2a_1a_3m_3 \sin(\varphi_1 - \varphi_3)\dot{\varphi}_3^2 = -ga_1(m_1 + 2m_2 + 2m_3)\sin\varphi_1,\end{aligned} \tag{10}$$

and the φ_2, φ_3 functions are

$$\begin{aligned}\{m_2(a_2^2 + k_2^2) + 4a_2^2m_3\}\ddot{\varphi}_2 + \{2a_1a_2m_2 \cos(\varphi_1 - \varphi_2) + 4a_1a_2m_3 \cos(\varphi_1 - \varphi_2)\}\ddot{\varphi}_1 \\ - 2a_1a_2m_2 \sin(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_1^2 - 4a_1a_2m_3 \sin(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_1^2 + 2a_2a_3m_3 \cos(\varphi_2 - \varphi_3)\ddot{\varphi}_3 \\ + 2a_2a_3m_3 \sin(\varphi_2 - \varphi_3)\dot{\varphi}_3^2 = -ga_2(m_2 + 2m_3)\sin\varphi_2,\end{aligned} \tag{11}$$

$$\begin{aligned}m_3\{(a_3^2 + k_3^2)\ddot{\varphi}_3 + 2a_2a_3 \cos(\varphi_2 - \varphi_3)\ddot{\varphi}_2 - 2a_2a_3 \sin(\varphi_2 - \varphi_3)\dot{\varphi}_2^2 + 2a_1a_3 \cos(\varphi_1 - \varphi_3)\ddot{\varphi}_1 \\ + 2a_1a_3 \sin(\varphi_3 - \varphi_1)\cdot\dot{\varphi}_1^2\} = -m_3ga_3 \sin\varphi_3.\end{aligned} \tag{12}$$

Let φ_1, φ_2, φ_3 be so small that one may employ the approximations $\sin\varphi_1 = \varphi_1$, etc.; $\cos\varphi_1 = 1$, etc.; $\dot{\varphi}_1^2 = 0$, etc.; $\dot{\varphi}_1\dot{\varphi}_2 = 0$, etc. Then (10), (11), (12) may be written

$$\begin{aligned}\{m_1(a_1^2 + k_1^2) + 4a_1^2(m_2 + m_3)\}\ddot{\varphi}_1 + 2a_1a_2m_2\ddot{\varphi}_2 + 4a_1a_2m_3\ddot{\varphi}_2 + 2a_1a_3m_3 \cos(\varphi_1 - \varphi_3)\ddot{\varphi}_3 \\ = -ga_1(m_1 + 2m_2 + 2m_3)\varphi_1,\end{aligned} \tag{13}$$

$$\{2a_1a_2m_2 + 4a_1a_2m_3\}\ddot{\varphi}_1 + \{m_2(a_2^2 + k_2^2) + 4a_2^2m_3\}\ddot{\varphi}_2 + 2a_2a_3m_3\ddot{\varphi}_3 = -ga_2(m_1 + 2m_3)\varphi_2, \tag{14}$$

$$m_3\{(a_3^2 + k_3^2)\ddot{\varphi}_3 + 2a_2a_3\ddot{\varphi}_2 + 2a_1a_3\ddot{\varphi}_1\} = -ga_3m_3\varphi_3. \tag{15}$$

These are the three equations of small motion in the general form of the problem.

In the present problem, $m_2 = 0$; $k_1^2 = a_1^2/3$, $k_2 = 0$, $k_3^2 = \frac{2}{5}a_3^2$; and we may put $m_1 = m_3 = m$; also $2a_1 = 2a$, $2a_2 = b$, $2a_3 = 2c$, and (13), (14), (15) are

$$\begin{aligned}\tfrac{16}{3}a\ddot{\varphi}_1 + 3g\varphi_1 + 2b\ddot{\varphi}_2 + 2c\ddot{\varphi}_3 &= 0,\\ 2a\ddot{\varphi}_1 + b\ddot{\varphi}_2 + g\varphi_2 + c\ddot{\varphi}_3 &= 0,\\ 2a\ddot{\varphi}_1 + b\ddot{\varphi}_2 + \tfrac{7}{5}c\ddot{\varphi}_3 + g\varphi_3 &= 0;\end{aligned}$$

or, using another notation of differential equations,

$$\begin{aligned}(\tfrac{16}{3}aD^2 + 3g)\varphi_1 + 2bD^2\varphi_2 + 2cD^2\varphi_3 &= 0,\\ 2aD^2\varphi_1 + (bD^2 + g)\varphi_2 + cD^2\varphi_3 &= 0,\\ 2aD^2\varphi_1 + bD^2\varphi_2 + (\tfrac{7}{5}cD^2 + g)\varphi_3 &= 0.\end{aligned}$$

Eliminate φ_1 and φ_2; then,

$$\{8abcD^6 + g(20ab + 18bc + 52ac)D^4 + g^2(80a + 45b + 63c)D^2 - 45g^3\}\varphi_3 = 0. \tag{16}$$

Let

$$\varphi_3 = L_p \cos(\lambda t + \alpha_p),$$

and after substitution in (22) and simplifying, put $\lambda^2 = \mu$, and we have

$$8abc\mu^3 - g(20ab + 18bc + 52ac)\mu^2 + g^2(80a + 45b + 63c)\mu - 45g^3 = 0,$$

the cubic equation required.

Also solved by J. B. REYNOLDS and W. E. CEDERBERG.

259 (Number Theory). Proposed by E. E. WHITFORD, College of the City of New York.

If p is relatively prime to 10, and if any multiple of p consisting of n digits has its digits permuted cyclically, the number thus formed is also a multiple of p; the number n to be determined by the congruence $10^n \equiv 1 \pmod{p}$. For example, 481, 814, and 148 are each multiples of 37.

SOLUTION BY BENJAMIN F. YANNEY, College of Wooster, Wooster, Ohio.

Let the digits in the initial order be

$$a_n,\ a_{n-1},\ \cdots,\ a_1.$$

Then by hypothesis,

$$a_n10^{n-1} + a_{n-1}10^{n-2} + \cdots + a_210 + a_1 \equiv 0 \pmod{p}.$$

Multiply each member of the congruence by 10, remembering that $10^n \equiv 1 \pmod{p}$, and place the digit a_n in units place. We thus secure one cyclic permutation. Another cyclic permutation is secured by multiplying again by 10; and so on. This completes the proof.

It will be observed that the theorem may be generalized by multiplying all the terms to the left of any specified term by $10^{\kappa n}$, in the case of any cyclic permutation, where κ is any positive integer. Thus, in the example given, 480001, 800000014, and 140000000008 are also each multiples of 37. By successive application of this method, we may obtain different types of cyclic permutations. Thus, 400000080001 is a multiple of 37. We may have other than cyclic permutations, with ciphers, by multiplying any one or more terms of the above congruence by $10^{\kappa n}$, where κ can have a different value for each term multiplied. Thus, 80401 is also a multiple of 37. It is interesting to note in this more general application that no two integers of the original number can ever collide.

Also solved by L. C. MATHEWSON, PHILIP FRANKLIN, W. R. RANSOM, FRANK IRWIN, PAUL CAPRON, HORACE OLSON, and C. C. YEN.

262 (Number Theory). Proposed by C. N. SCHMALL, New York City.

If x, y, z, are 3 integers, consecutive among the integers prime to 3, show that

$$x(x - 2y) - z(z - 2y) = \pm 3.$$

SOLUTION BY EDWARD H. VANCE, Graduate Student, Ithaca, N. Y.

Let $v - 1$ be any number divisible by 3, then any set of three integers consecutive among the numbers prime to 3 may be represented by one of the following sets:

$$v - 2,\ v,\ v + 1;\quad v,\ v + 1,\ v + 3.$$

Substituting $v - 2$, v, $v + 1$ for x, y, z, respectively, in the lefthand side of the given equation we have

$$x(x - 2y) - z(z - 2y) = 3.$$

Substituting v, $v + 1$, $v + 3$ for x, y, z, respectively, we have

$$x(x - 2y) - z(z - 2y) = - 3.$$

Also solved by PAUL CAPRON, N. P. PANDYA, LOUIS O'SHAUGHNESSY, LEWIS CLARK, E. F. CANADAY, GEORGE W. HARTWELL, J. L. RILEY, ALBERT G. RAU, HERBERT N. CARLETON, HORACE OLSON, and V. M. SPUNAR.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

I. ON MAKING MATHEMATICAL RESULTS MORE AVAILABLE FOR ENGINEERS.

By WILLIS WHITED, Harrisburg, Pennsylvania.

Some time ago I received a circular from the Mathematical Association of America regarding the *Annals of Mathematics*. I like very much the idea of a

series of articles setting forth the "state of the art" of the different branches of mathematics in a form that would be intelligible to people who are not specialists in the respective branches.

I am an engineer and know that there are numerous unsolved problems in engineering science which are chiefly mathematical. The engineer studies mathematics primarily for its value as a tool in solving his problems, however fond he may be of the subject for its own sake. Very few engineers find time, in the course of an ordinary lifetime, to acquire a reasonably complete knowledge of all the pure mathematics that they can use to advantage in following up the latest advances in their respective specialties and in doing the research work that devolves upon them. It not infrequently happens that work which appears at the time to be little more than mathematical gymnastics is subsequently developed into something quite useful; but years elapse before the people who need the mathematics learn of its existence. The investigating engineer and the mathematician must keep in closer touch with each other in the future than they have in the past. America must take a larger place in the advancement of science.

The engineering investigator who encounters difficult mathematical problems must have better facilities for acquiring the knowledge he needs of the many powerful methods of mathematical analysis which have been developed within the memory of men now living. Works on advanced mathematics are practically all intended for professional mathematicians. Their contents are almost wholly academic in character and they are beyond the reach of the engineer. Articles in mathematical periodicals are seldom intelligible to any but a very few specialists. This is doubtless unavoidable and perfectly proper, but I would urge that occasional articles be written bringing various branches of the subject down to date, omitting, perhaps, much of the purely academic work and expressing the whole, if possible, in terms that can be understood by the engineer who has kept up his collegiate mathematics.

From what little I know of modern mathematics, I would imagine that progress useful to the engineer has been or soon may be attained in the following branches (among others): differential equations, calculus of finite differences, vector analysis, successive integration, elliptic and hyperelliptic functions, transcendental equations and analytical geometry.

Most of the modern writers on advanced analytical geometry use homogeneous coördinates. This method has some advantages in certain kinds of work, but it is rarely taught to undergraduates in engineering and, moreover, most of the engineer's problems are metrical, so that Cartesian coördinates are better adapted to their solution. Many theorems in projective geometry could be used by the engineer who employs graphical solutions if the theorems were put in such form that he could acquire a knowledge of them in a reasonable time.

Most of the fundamental principles of those branches of science which aspire to become exact can best be expressed in the form of differential equations. Many of these equations have not, thus far, been solved. Approximate solutions are better than none. Hence, I would urge that methods of approximate solu-

tions be so developed as to make them, so far as practicable, accessible to the engineer. In the practical applications of mathematics to engineering and, probably, to other sciences, the solutions of problems are often not exact. Graphical solutions are subject to a very considerable margin of error and arithmetical solutions almost always involve the multiplication or division of decimals in which only a certain number of decimal places are retained. Transcendental functions and radicals are only given approximately in the tables and it may well happen that a solution in a rapidly converging series is just as convenient as an exact solution. If a solution is in the form of a series with general expressions for coefficients, it may be almost as satisfactory as any other kind of a formula. In that case, if a similar problem occurs again, it will only be necessary to substitute the proper values for the constant terms in the coefficients, which can be done by an assistant who is not familiar with differential equations. I therefore hope that mathematicians will publish freely their methods for approximate solutions of differential equations and other problems, preferably in a form that will not compel the busy engineer to search through a multitude of monographs, many of which are in foreign languages and some of which can not be readily obtained, before he can get an adequate idea of the nature of the solution.

Elliptic integrals are met with occasionally and if they merely have to be integrated once approximate methods are available. If successive integration is required, it is apt to be "another story."

It may be that all problems that can be solved by vector analysis can also be solved by the older methods, but this method is often so much simpler that the subject is worthy thorough study.

The engineer often meets with transcendental equations and they usually have to be solved as individual problems. If more general methods, even if only approximate, have been developed, they should be more generally known.

Complex variables are occasionally encountered, chiefly in connection with differential equations. If a practical knowledge of the subject could be imparted without requiring the reader to toil through ponderous tomes in an effort to find an explanation, it would be helpful.

The modern theory of functions is a subject which is very interesting to one who is fond of mathematics for its own sake; but can not some way be found by which the student can get at the pith of the matter in a reasonable time? The subject is chiefly academic, but is very attractive.

II. Relating to New Remainder Terms for Certain Integration Formulæ.

By S. A. Corey, Albia, Iowa.

In the June, 1917, number of the Monthly Professor Daniell notes the fact that at least one of the remainder terms of the integration formulæ which I gave in the June–July, 1912, number of the Monthly is needlessly large. I also observe that the remainder term to my formula 25*s* which he gives is too small, as he has tacitly made the unwarranted assumption that the signs of his S_1 and S_2

must always be alike. To set these matters aright I have computed a new set of remainder terms (based on a form due to Poisson[1]) for all the formulæ originally given. All the work of computation has been carefully checked to eliminate errors as far as possible. We employ the symbol r_{2n},

$$r_{2n} = \pm\,\theta \frac{x^{2n+1}}{(2n)!} \max |f^{(2n+1)}(a + \theta_1 x)|,$$

where $0 \leq \theta \leq 1$, and $0 \leq \theta_1 \leq 1$.

Formula	Remainder Term	Formula	Remainder Term
(1)	$r_4/72$	(2)	$r_4/216$
(3)	$17r_6/24192$	(4)	$11r_6/7560$
(5)	$.000,154r_8$	(6)	$.000,000,586r_8$
(7)	$.000,000,279r_8$	(8)	$.000,057,6r_6$
(9)	$.000,009,56r_8$	(10)	$.000,011,5r_8$
(11)	$.000,029r_8$	(12)	$.000,060,6r_8$
(13)	$.000,010,7r_{10}$	(14)	$.000,015,9r_8$
(15)	$.000,000,584r_{12}$	(16)	$.000,000,252r_{12}$
(17)	$.000,000,16r_{10}$	(18)	$.000,105r_6$
(19)	$.000,29r_6$	(20)	$.000,011r_6$
(21)	$.000,000,143r_{10}$	(22)	$.000,000,000,203r_{16}$
(23)	$.000,010,6r_6$	(24)	$.000,000,002,16r_{12}$
(25)	$r_6/504$	(26)	$r_6/3024$
(27)	$.000,046,9r_8$	(28)	$.000,146r_8$
(29)	$.000,011,7r_{10}$	(30)	$.000,000,006,9r_{10}$
(31)	$.000,001,65r_8$	(32)	$.000,000,335r_{10}$
(33)	$.000,000,419r_{10}$	(34)	$.000,001,16r_{10}$
(35)	$.000,002,76r_{10}$	(36)	$.000,000,537r_{12}$
(37)	$.000,000,425r_{10}$	(38)	$.000,000,020,8r_{14}$
(39)	$.000,000,005,53r_{14}$	(40)	$.000,000,002,56r_{12}$
(41)	$.000,000,118r_8$	(42)	$.000,662r_8$
(43)	$.000,050,9r_8$	(44)	$.000,006,27r_{10}$
(45)	$.000,031,5r_{10}$	(46)	$.000,001,87r_{12}$
(47)	$.000,000,000,15r_{12}$	(48)	$.000,000,097r_{10}$
(49)	$.000,000,055r_{14}$	(50)	$.000,000,24r_{12}$
(51)	$.000,000,021r_{12}$	(52)	$.000,000,000,213r_{16}$
(53)	$.000,372r_{10}$	(54)	$.000,013r_{10}$
(55)	$.000,001,29r_{12}$	(56)	$.000,000,029,8r_{14}$.

[1] See Ford, *Studies on Divergent Series and Summability*, p. 6.

folding.[1] When it is recalled that for over a century the simple method of constructing such polyhedra has been given in text-books, it appears that Wiener's additional contribution consisted almost wholly in constructing, by paper folding, equilateral triangles, squares, and pentagons—a very small portion of Sundara Row's discussion. This fact is not made clear in Klein's remark on page 42 of the English edition of his *Famous Problems in Elementary Geometry,* Boston, Ginn, 1897.

Finally, reference may be given to a note by Fitz Patrick on "La géométrie par le pliage et découpage du papier" in W. W. R. BALL, *Récréations mathématiques et problèmes des temps anciens et modernes,* troisième partie, Paris, Hermann, 1909, pp. 341–360; to *Messenger of Mathematics,* 1905, Vol. 34, pp. 142–3; and to *L'Education Mathématique.*

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Dr. J. B. ROSENBACH has been appointed instructor in mathematics at the University of New Mexico.

Dr. GOLDIE P. HORTON has been appointed instructor in mathematics at the University of Texas.

Dr. G. H. LIGHT, of the University of Colorado, has been promoted to an assistant professorship of mathematics.

Dr. E. W. PONZER, assistant professor of applied mathematics at Leland Stanford University, died on December 20, 1917, as a result of an accidental gunshot wound.

Mr. J. J. TANZOLA, until recently an instructor in mathematics at the U. S. Naval Academy, is now a private with the 305th Machine Gun Battalion, Company C, at Camp Upton, Long Island.

Mr. FREDERICK WOOD, instructor in mathematics at the University of Wisconsin and a charter member of the Association, is now a lieutenant with the 328th Field Artillery at Camp Custer, Michigan.

Dr. J. E. DAVIS, instructor in mathematics at Pennsylvania State College, is enrolled in the 313th Infantry at Camp Meade, Md.

The University of California has been, since May 21, 1917, conducting a school of military aëronautics, in which about five hundred cadets are being trained in an eight weeks' course in flying. Professor B. M. WOODS, of the department of mathematics, a charter member of the Association, is directing the instruction in this school.

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[1] *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente.* Nachtrag. Herausgegeben von W. Dyck. München, Wolf, 1893, pp. 52–54.

Professor A. O. LEUSCHNER, of the University of California, has been delegated to administer instruction in the navigation schools conducted by the U. S. Shipping Board on the Pacific coast.

On account of conditions due to the war, it has been decided by Harvard University that no appointments be made to the Benjamin Peirce instructorships in mathematics during the present year.

Professor J. N. VAN DER VRIES, of the University of Kansas and a member of the Council of the Association, has been chosen as a member of the field division of the U. S. Chamber of Commerce. He has been given a year's leave of absence in order to carry out certain work in connection with the organization of war service committees, which are to take the place of the old national defense council committees and thus to furnish a point of direct contact between specific industries and the government.

Professor HARRIS HANCOCK, of the University of Cincinnati, was elected treasurer of the American Association of University Professors at its Chicago meeting during the holidays.

At the Pittsburgh meeting of the American Association for the Advancement of Science, Assistant Professor G. D. BIRKHOFF, of Harvard University, was elected vice-president of Section A (Mathematics and Astronomy); Professor F. R. MOULTON, of the University of Chicago, was reëlected secretary of the Section.

The initial number of Vol. XL, of the *American Journal of Mathematics*, appeared in January with six mathematical contributions as follows: "Flat-sphere geometry," by J. EISLAND; "Irrational involutions on algebraic curves," by J. V. DE PORTE; "The set of eight self-associated points in space," by J. R. MUSSELMAN; "Associate minimal surfaces," by J. K. WHITTEMORE; "On integral invariants," by F. W. REED; "Fundamental regions for certain finite groups in S_4," by H. F. PRICE.

The *Science Reports* of the Tôhoku Imperial University, November 1917, contains an interesting mathematical discussion by K. TERAZAWA on the "Oscillations of the deep-sea surface caused by a local disturbance." The present paper is a continuation of the writer's former discussion of the problem, *Proc. Roy. Soc.*, London, 1915, and is intended to complete the discussion of the problem of central oscillations with the initial conditions: (1) when an initial displacement of the free surface is given, without initial velocity; (2) when an initial impulse is applied on the surface, without initial surface displacement. The same issue of *Science Reports* also contains a paper by T. KUBOTA on "An application of binary quadratic forms to geometry."

The concluding number of Vol. 12, *Tôhoku Mathematical Journal*, edited by T. HAYASHI, Sendai, Japan, recently appeared containing the following papers: "On the theory of representation of surfaces," by K. OGURA; "A theorem on

limits," by S. Narumi; "On generalized Toeplitz theorems on limits and their applications," by T. Kojima. This twelfth volume of the *Journal* contains 334 pages, consisting, for the most part, of original papers written in the English language.

A continuation of the discussion of "Mathematics in secondary schools," following the investigation conducted by a committee of the Chicago Mathematics Club, appears in *School and Society*, January 12, 1918, by Professor C. N. Moore, of the University of Cincinnati. This paper is devoted chiefly to a criticism of Professor Snedden's discussion of the same subject, in *School and Society*, December 1, 1917. "Mathematics as a test of mental efficiency," by Professor R. E. Moritz of the University of Washington, appears in the January 12 issue of *School and Society*.

"A glimpse at early colonial algebra" is the subject of a most interesting historical paper by Professor D. E. Smith, of Teachers College, Columbia University, in *School and Society*, January 5, 1918. Professor Smith writes concerning the early manuscript copies on mathematics found in the collections of colonial libraries, and in particular he gives a description of a manuscript copy on algebra written by Samuel Langdon under the direction of Isaac Greenwood, Hollis Professor of Mathematics at Harvard University, 1727–1738. At the time of writing the manuscript on algebra, Langdon was fifteen years old; he later became president of Harvard University, serving during the American Revolution.

On February 3, 1918, Dr. Ellery W. Davis, dean of the college of literature, science and arts of the University of Nebraska and professor of mathematics, died of pneumonia after a brief illness. Dr. Davis was a charter member of the Mathematical Association of America. He was present at the recent Chicago meeting of the association and presided at one of the sessions.

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To solve $\frac{\partial^2\varphi}{\partial t^2} = \sqrt{1+(\Delta h)^2}\,\frac{\partial^2\varphi}{\partial x^2}$, put $m^2 = \sqrt{1+(\Delta h)^2}$ and assume $\varphi = \tau(t)\cdot\xi(x)$; so that $\frac{\partial^2\varphi}{\partial t^2} = \frac{d^2\tau}{dt^2}\cdot\xi(x)$, and $\frac{\partial^2\varphi}{\partial x^2} = \tau(t)\,\frac{d^2\xi}{dx^2}$. Substituting these into the original equation, we find that the variables, t and x, can be separated by dividing through by $\tau\cdot\xi$ whereupon we have $\frac{d^2\tau}{dt^2}\div\tau = m^2\frac{d^2\xi}{dx^2}\div\xi$. Since the first of these two equal members cannot vary when t changes nor the second when x changes, both must remain equal to some constant, say $-m^2n^2$. The two resulting equations yield the solutions

$$\xi = K_1\cdot\sin[nx+\beta_1], \qquad \tau = K_2\cdot\sin[mnt+\beta_2]$$

whence $\quad \varphi = K_1K_2\mathrm{Sin}[nx+\beta_1]\,\sin[mnt+\beta_2]$

which we may then reduce to a more useful form:

$$\varphi = \sum_{n=o}^{n=\infty} A_n\sin[n(x\pm mt)+\delta_n].$$

An interesting fallacy results from applying the method of integration by parts, $\int u\cdot dv = uv-\int v\cdot du$, to a case where $u=1/x$ and $dv=dx$: we get

$$\int\frac{dx}{x} = \frac{1}{x}.\int dx - \int x\cdot[-1/x^2]$$

$$= \quad 1 \quad + \int dx/x \qquad \text{whence } 0=1\ !!$$

$$\int\frac{dx}{5+7x^2} = 1/5\int\frac{dx}{1+{}^7/_5x^2} = \frac{1}{5}\sqrt{\overline{{}^5/_7}}\int\frac{\sqrt{{}^7/_5}\,dx}{1+[\sqrt{{}^7/_5}\,x]^2}$$

$$= \frac{1}{35}\ \arctan\ [\sqrt{{}^7/_5}\ x].$$

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RABBI BEN EZRA AND THE HINDU-ARABIC PROBLEM.

By DAVID EUGENE SMITH and JEKUTHIAL GINSBURG.

When Browning put in the mouth of Rabbi ben Ezra the words

"The Future I may face now I have proved the Past,"

he wrote better than he knew, for no scholar of the twelfth century had proved the Past more thoroughly than he, and few could face the Future with greater confidence.[1] Born, very likely at Toledo, between 1093 and 1096,[2] he became known as one of the most learned men of his time, and was the author of numerous books, a traveled scholar, a poet, and a man of great influence. He died in 1167, probably either in Rouen or in Rome.

Of the many lines of research pursued by Rabbi ben Ezra, one of the most interesting to students of the history of mathematics is that relating to the introduction of Hindu astronomy and computation into the Arabian civilization. Of that remarkable activity of the seventh century which resulted in the amalgamation of numerous semi-nomadic tribes into one mighty empire we have abundant knowledge; of the opening of the golden age of Mohammedan civilization and of its development under the caliphs of Bagdad in the eighth and ninth centuries we have well-authenticated records; of the influx of the Greek civilization through the translation of the classics of Alexandria and Athens we have the witness of a large number of manuscripts in the great collections of Europe and

[1] In the text of this article, the name Rabbi ben Ezra will be used because it is familiar to English readers. Properly, the name should be written Abraham ibn Ezra, although it often appears in the Latin translations as Abraham Judæus. In the notes we shall use the form Ibn Ezra.

[2] For a discussion of the matter, see Steinschneider, "Abraham Ibn Esra" in the *Abhandlungen zur Geschichte der Mathematik*, III, 59, Leipzig, 1880.

America; but of the details of the introduction of the Hindu mathematics into the region of Mesopotamia we are still in need of further information. To be sure, we have some recent light upon the subject through the writings of Severus Sebokht, a religious scholar of the seventh century,[1] but the problem is still far from solution.

Among the early sources that throw some light upon the situation, and which are as yet unknown to most European and American scholars, is Rabbi ben Ezra's account of the origin of Arab science, given in the introduction to his translation into Hebrew of a book written by Muhammed ibn Aḥmed el-Bîrûnî (973–1048)[2] on the astronomical tables of Mohammed ibn Mûsâ al Khowarizmi.[3] Although written in Arabic, this work of el-Bîrûnî is known only through two Hebrew manuscripts of ben Ezra's translation, one in the Bodleian Library[4] and the other, with the introduction by Rabbi ben Ezra, in the library at Parma.[5] The Hebrew title of the treatise is Ta'amê Lûchôth al Chowârezmî, and a portion comprising the introduction and the opening paragraphs of the book itself was published by Steinschneider in 1870,[6] with a German translation of the major part of the text.

Inasmuch as only a part of this important work has appeared in any modern language, and that in German, it has seemed desirable to translate all of the fragment from the Hebrew text, this being the part that relates to the introduction of Hindu mathematics among the Arabs. In so doing, we shall make use of Steinschneider's German translation only for purposes of comparison in the notes.[7]

As to the historical accuracy of Rabbi ben Ezra's statements in the introduction to his translation, it has been said that it is doubtful if he ever visited

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[1] See J. Ginsburg in the *Bulletin of the American Mathematical Society*, Vol. 23, page 366; L. C. Karpinski in *Science* for June 12, 1912; E. R. Turner in *Popular Sci. Monthly* for December, 1912.

[2] Except in the case of names which have fairly well-recognized English forms, like al Khowarizmi, the transliteration followed is that of Suter's list in the *Abhandlungen zur Geschichte der Math.*, Bd. XI, Leipzig, 1900.

It was formerly supposed that this author was one el-Mutanâ of whom nothing further is known. Suter, however, corrected the erroneous impression by showing (*Bibliotheca Mathematica* (3), IV Bd., 1903, p. 125) that el-Bîrûnî and el-Mutanâ are the same. Steinschneider (*Orientalistische Literaturzeitung*, 1903, p. 486) explains the mistake as due to some copyist having combined the *beth* and *yod* and also the *resh* and *van* in the name el-Bîrûnî (אלבירוני), thus obtaining el-Mutanâ (אלמתני).

[3] The Mohammed ben Musa, whose algebra, translated into English by Rosen, was published in 1831. As given in Suter's list the name is Muhammed ibn Mûsâ el Chowârezmî (or Chwârezmî), Abû' Abdallâh. See also Suter's introduction to the new edition of al Khowarizmi's tables (Copenhagen, 1914).

[4] Mich. 400; No. 2006 in Neubauer's list of Hebrew MSS. in the Bodleian Library.

[5] Rossi 212. The transcription of this manuscript used by Steinschneider is now in the library of the Jewish Theological Seminary of America, New York.

[6] "Zur Geschichte der Uebersetzungen aus dem Indischen ins Arabische," in the *Zeitschrift der deutschen Morgenländischen Gesellschaft*, Bd. XXIV, p. 325.

[7] Use is also made of the notes by Steinschneider and of the material given by Suter, *l. c.*, and by Wüstenfeld in his *Geschichte der arabischen Aerzte*, Göttingen, 1840. A number of helpful suggestions have also been made by Professor Alexander Marx of the Jewish Theological Seminary of America, New York City.

India,[1] but this would not be at all necessary. He was as careful a writer as any of his time, a scientist of high repute, a student of the history of sciences, and a man less given to the acceptance of mere tradition than was usually the case. It is to be expected from his reputation that he would have consulted the best Arabic sources available, although we have at present only slight knowledge of what those sources were, and although no manuscripts thus far translated throw any light upon the problem.

The following is a translation of the introduction written by Rabbi ben Ezra:

"In the name of the Most Holy and Revered, in whose help I trust, spake Abraham ibn Ezra the Spaniard. In ancient times there was no wisdom and no [true] religion among the sons of Ishmael, the tent dwellers, until the [author of the] Kora[2] came and gave to them from his heart a new religion.

"After him there appeared many sages among them, who wrote many books on their laws; but at last there appeared a great king in Ishmael, called e's Saffah,[3] who heard that there were many sciences in India. And he gave orders to search for a scholar who should know the language of India and that of Arabia, so as to translate for him one of their books of wisdom, although he feared that a calamity might befall him,[4] since profane sciences [were then permitted] in Ishmael in the book of the Koran alone, and whatever of the sciences they received [by tradition] was [believed to be] therein. [He had heard that] in India there was a book, very important in the councils of the kingdom, that was arranged in the form of stories put in the mouths of dumb creatures, the large number of pictures rendering the book very valuable in the eyes of the reader. And the name of the book was Kalilah we-Dimnah,[5] which means the Lion and the Bull, because the first gate[6] of the book refers to them.[7] And the above named king fasted forty days, hoping to see the angel of dreams who should

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[1] Steinschneider adduces a proof that Ibn Ezra did not visit India. See *Zeitschrift der deutschen Morgenländischen Gesellschaft*, Bd. XX, p. 430.

[2] For *koran*.

[3] In the De Rossi codex this name appears as Altsaphak; but evidently the Abbasside el-Saffah is meant, and this fixes the time as the middle or second half of the eighth century. A similar story is told of Nushirvan the Just, who reigned in Persia in the sixth century.

[4] That is, the translator. This calamity might befall one who assumed, by translating such a work, that the Koran was not all sufficient. In the Hebrew a few words are omitted but this seems to be the sense of the passage.

[5] Called by the Hindus the Kurtuk Dumnik.

[6] That is, the first chapter.

[7] Although the first chapter (in some editions the second) of the Kalilah we-Dimnah deals with the bull and the lion, the name of the book is derived from the names of two jackals, Karattaka and Damanaka, which play an important part in the stories of the subsequent chapters. See Theodor Benfey's introduction to G. Bickel's edition of the Syriac *Kalilag und Damnag* (Leipzig, 1876, page 11); Wollaston's English translation under the title *Lights of Canopus* (London, 1904); J. Derenbourg, "Kalîlâh et Dimnâh" in the *Bibliothèque de l'école des hautes études* (Paris, 1881). The first edition appeared in Johannes de Capua's *Directorium humanae vitae*, "Et vocatur liber Kelile et dimne," Strassburg, c. 1488–1493.

In the Talmud, mention is made of Ben Tiglah and Ben Laanah, and the relation of these to the Kalilah we-Dimnah was shown by Wolf, in the *Bibliotheca Hebraea* I, 932, note. See also Steinschneider, in *Jewish Literature*, London, 1857 (p. 279, note 54, *a*). Geiger, in the *Jüd. Zeitschrift*, VII (1869), 139, sets forth his belief that Ben Tiglah and Ben Laanah are the names of two Jewish authors.

allow him to translate the book into Arabic. Then he had a dream in harmony with his thoughts.[1] He thereupon sent for a Jew who lived in his time and who knew the two languages, and he gave him command to translate the book, since he feared that if an Arab should translate it he might die.[2] And when he saw how wonderful the book was, and so it really is, he was overcome by a desire to know more. Then he gave great wealth to the Jew so that he might journey to the city of Arin[3] on the equator, under the signs of the Ram and the Scales, where the day throughout the year is equal to the night, [thinking '] Perhaps he will succeed to bring one of their wise men to the king [']. And the Jew went [there] and indulged in many subterfuges, after which, for a large sum, one of the wise men of Arin agreed to go to the king, and the Jew swore to him that he would not detain him beyond a year and that he would return him to his home. Then this scholar, whose name was KNKH,[4] [was taken to the king] and taught the Arabs the bases of number, that is, the nine numerals.[5]

"From the mouth of the learned man, through the Jew [as] an Ishmaelite interpreter, a scholar by the name of Jacob ibn Sharah[6] translated the book of the tables of the seven planets and the creation[7] of the earth,[8] the degree of rise, the establishing of the houses,[9] and the knowledge of the upper stars and the darkening of the lights.[10] No explanation of these matters was set forth in the book, only operations in the form of rules to be accepted on faith.[11] The average motion of the planets was computed according to Hindu methods, their cycle, called Hazervan, being equal to 432,000 years.[12]

[1] This seems to have been looked upon as a sanction for the translation of profane works, for el Mâmûn waited until he saw Aristotle in a dream before ordering the translation of the Greek philosophers.

[2] That is, an Arab might be punished for the profanity of such a translation while a Jew might escape.

[3] Arin seems to have been an astronomical center of India, like Ujjain. Indeed, Reinaud thinks the two places were the same. Al Khowarizmi's tables were constructed on the basis of this meridian. See Suter's edition of the tables, *l. c.*

[4] In the Hebrew text the vowel points are not given. The name may be Kanka or Kanaka; for, according to Meĝriti (959), Kanka was the inventor of amicable numbers.

The subject is deserving of a more extensive report than is possible in this connection.

[5] Evidently the decimal system, else "bases of number" would have no significance.

[6] Since in the Spanish rabbinical script the Hebrew letters *shin* and *theth* are easily confused, as also the letters *he* and *gaf*, a scribe who was not familiar with the name might easily have written Sharah for Tarik. Hence this Jacob ibn Sharah may have been the same as the famous astronomer and astrologer Ja'qûb ibn Târiq mentioned by el-Bîrûnî as having been living in 777.

[7] Literally, "the whole work."

[8] Was this the Sûrya Siddhânta, the great astronomical work of the Hindus, written in the fourth century?

[9] That is, in the scheme of astrology. The passage is very obscure, and several words omitted at this point in this translation seem to have referred to the ascension and declination of the planets. Steinschneider was also unable to find the exact meaning of the passage.

[10] Eclipses of the sun and moon.

[11] The Hindus gave no proofs of their propositions, as may be seen by examining the works of Āryabhaṭa, Brahmagupta, Mahā īr, and Bhaskara, all of which are now accessible in translation.

[12] In the Sûrya Siddhânta is this passage: "To determine the saura years contained in this aggregate, write down the following numbers, 4, 3, 2, which multiply by 10,000; the product, 4,320,000, is the aggregate or Mahà yuga. . . . Divide the aggregate 4,320,000 by 10 and multiply the quotient by 4 for the satya yug." See *Asiatic Researches*, 1790, II, p. 230.

"After [the death of] Jacob the translator there arose a great scholar in Ishmael who knew the secrets of the wisdom of counting [and of] chronology and [who] reduced the average planetary motion to the era of Ezdeger,[1] the last of the Persian kings, for the Arabs conquered the kingdom of Persia and converted the inhabitants to their own religion. This scholar was Mohammed ibn Mûsâ al Khowârezmî,[2] and all later Arabic scholars do their multiplications, divisions, and extraction of roots as is written in the book of the [Hindu] scholar which they possess in translation.[3] He[4] prepared, in a more convenient form for students, tables which were the equal of the work of KNKH, but he gave no explanation of his statements. After him there arose in Ishmael[5] a scholar called el-Fergânî,[6] who set forth reasons for the words of al Khowarizmi. After el-Fergânî[7] another great scholar translated a very important book about the stars, written by Ptolemy, king of Egypt,[8] a Greek who lived a thousand years ago.[9] This book[10] is perfect, there being nothing higher in the science of the spheres, their secrets, their distances from the earth,[11] and the measure of the upper stars in the sphere of the zodiac. He divided them[12] into six classes, of which the first [class] was called the first glory. He established the number of stars in each class and enumerated all [the stars] in the forty-eight constellations,[13] namely, the constellations of the entire sphere which contains 1,022 stars besides the clouded ones.[14] He gave reasons for all corrections[15] and, in general,

It is interesting to note that the number 432,000 also appears in the Babylonian chronology. According to Berossus, a Babylonian priest (250 B. C.), the antediluvian kings of Babylon reigned for a total period of 432,000 years. See W. H. Roscher, *Die Zahl 40 im Glauben, Brauch und Schrifttum der Semiten*, Leipzig, 1909, p. 97; H. V. Hilprecht, *Mathematical . . . Tablets from . . . Nippur*, Philadelphia, 1906, p. 21.

[1] This begins with the day when Jezdegird III ascended the throne of the Sassanides, June 16, 632, of the Christian calendar of that period. Although the Mohammedan conquest soon imposed the calendar of Islam on most of Persia, the Jezdegirdian calendar survived among the Persians in the southern provinces and in western India.

[2] Mohammed the son of Moses, the Kharezmite, who flourished in Bagdad early in the ninth century and whose *Algebr w'al-Muqabala* gave the name to algebra.

[3] It was probably this work which al Khowârizmî made the basis of the treatise on computation which, as is well known, was translated into Latin in the twelfth century under the title *Liber Algorismi* (Book of al Khowârizmî), from which we have the word *algorism*.

[4] Al Khowârizmî.

[5] In the original, Israel, probably a mistake of the scribe.

[6] Ahmed b. Muh. b. Ketîr el-Fergânî or el-Fergûnî was one of the astronomers and followers of el-Mâmûn.

[7] In the original, "after him."

[8] In the Middle Ages, and even later, Claude Ptolemy (second century A. D.), the great astronomer, was commonly mistaken for one of the kings who reigned under the name of Ptolemy.

[9] This was approximately correct, since Ibn Ezra lived in the twelfth century, while Ptolemy lived in the second century.

[10] The Almagest.

[11] The original may also be translated as motion with respect to the earth.

[12] The stars.

[13] Literally, "figures."

[14] Ptolemy's tables, edited by Peters and Knobel (Washington, 1915), give 48 groups or constellations and 1028 stars. The number 1,022 is not only given by Ibn Ezra but is also given by el-Fergânî in his compendium of the Almagest, as shown in a Hebrew MS. of the latter's compendium now in the Columbia University Library (X 893, T 522). This shows that the number in Ibn Ezra's work was not an error of some scribe. Moreover, in the Manitius edition (1912) of the Almagest the number is given as 1,022.

for everything found in the tables translated by Jacob from the mouth of the Indian scholar.

"All the proofs given by Ptolemy or Talmi in his great book Almagest are perfect, and no man can contradict them, for these are proofs from the science of magnitude or the science of measurements, which is called in the Greek tongue Yeometria and, by the holy sages of Israel, Gematria.[1]

"On this book[2] commentaries were written by many sages in Ishmael, the most distinguished among them, in mathematics and astronomy, being the scholar Muhammed ibn Mutani.[3] He [it was who] compiled for one of his family a very important book about the corrections[4] of the planets and the explanation of the contents of the tables in the book of al Khowarizmi, and briefly mentioned the proofs and illustrations, their principles being taken from the book [called] Almagest. In certain passages, however, his explanations are more complete than those of King Ptolemy, and in these are also places where he sets forth mathematical proofs [which were] invented by himself.[5] In most places he agrees with the theories of el-Fergânî who was mentioned above. For the sake of clearness his book is arranged in the form of questions and answers.

"Said Abraham [ibn Ezra]: Except in a few places there is no difference between the norms[6] of the planets as given in Ptolemy's work and those of the Hindu scholar, and at the proper time I shall mention how this difference arises. I have written a book on the mean motion of the planets, and on the head and tail[7] of the *tali*[8] as observed by the astrolabe, because the positions of the planets

[15] Steinschneider translates the Hebrew word *tikunim* as "norms," but the word also means corrections or interpolations, and this is much more in harmony with the context at this point, although in two later sentences the word "norms" seems more appropriate.

[1] This is not correct, for gematria was an entirely different science, in no way related to geometry. It is probable that the error is that of some copyist of Ibn Ezra's manuscript, although the latter may himself have been at fault. Without the vowel points the word can be read either geometria or gematria. Because of this fact the pronunciation of the word as it appears in the Talmud was quite unsettled until the nineteenth century. Ibn Ezra asserts elsewhere that the Almagest was translated into the Arabic after the time of el-Fergânî, and at any rate after the time of al Khowarizmi. This confirms the view, expressed by Weber (*Naxatra*, I, 321), that the Arabs were familiar with the science of the Hindus before they became acquainted with the works of Ptolemy. (See also Woepcke, *Sur l'introduction de l'Arithmétique indienne en occident*, Rome, 1859, p. 58, and Lassen, *Indische Alterthumskunde*, II, 1139.) Reinaud (*Abulfeda*, pp. XLI–XLIII) states that the Almagest was translated completely into Arabic under el-Mâmûn, but that it was translated into Syriac and Hebrew in the middle of the eighth century, a doubtful statement as to the Hebrews for the reason that they were not generally interested in such matters at that time. See Steinschneider in the *Zeitschrift der Deutsch. Morgenl. Gesellsch.*, XXIV, page 337.

[2] The Almagest.

[3] The Muhammed ibn Ahmed el-Bîrûnî referred to in the note on page 100.

[4] See the note on the Hebrew *tikunim*, *supra*.

[5] Literally, from his heart.

[6] See the note on the word *tikunim*, *supra*. We have taken Steinschneider's translation here, since it seems to make better sense than "corrections." What is apparently meant is "tables."

[7] In the Hindu writings, Rahu and Ketu.

[8] The *tali* was a celestial dragon believed by early writers to be the cause of eclipses and of various other disturbances in the heavens. The Chinese attributed eclipses of the moon to the fact that it was covered by the head or the tail of the dragon, and to prevent it from being devoured they tried to frighten the dragon away by the noise of cymbals and tambourines.

The word *atalû*, used by the Babylonians to mean eclipse, is evidently the source of the

in the tables of al Khowarizmi do not agree with their observed positions by $9\frac{1}{2}$ degrees. It is my opinion that the idea of the Hindu scholars as to mean planetary motion is based upon the representation on the plane, and this is correct according to the science of projection but not according to the science of astronomy.[1]

"The tables in Almagest are useless for the reason that they are evidently corrupted. Moreover, they are not in accord with the paths of the stars.[2] The errors are not due to Ptolemy, however, but to the ancients from whom he derived [his information], and this I shall discuss later when I shall have completed this work. The norms in my book are the same as Ptolemy's and as those used by all Arabian scholars. [The latter] prepared many tables and were more exact in their work than Ptolemy, and I shall hereafter explain the reason. Only the norm of the sun is not[3] the same as Ptolemy's, being less than that by 29 minutes, for he relied upon the observations of ABRKS[4] who lived 208 years before Ptolemy.[5] He could not have relied upon the testimony of Fitin and Afitimon[6] who lived about 1,000 years before him,[7] for they could not have made [as good] an astrolabe [as the one] used by Ptolemy. Hipparchus had stated that the position of the sun at its apogee[8] was in his time at 5 degrees of the Twins,[9] and since Ptolemy found it in nearly the same place he inferred that the position of apogee of the sun, unlike the positions of apogee of the five planets, does not change; but many scholars found that it changes as well as

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Syriac word *atalia* and the Hebrew *tali*, both of which signify dragon and are used in connection with eclipses.

This idea of the power of the *atalia* was ridiculed by Severus Sebokht in the seventh century, and of course Ibn Ezra had no illusions concerning any dragon.

See Morris Jastrow, *Religious Belief in Babylonia and Syria*, page 213; M. F. Nau in the *Journal asiatique*, 1910, ser. X, Vol. 16, page 219.

[1] Literally, "is according to the image of the galgal," etc. Galgal means a circle, and the passage probably means that the Hindus studied planetary motion from the representation of the heavens as circular, on a plane, rather than from a celestial sphere. The passage is more obscure than any other in the book, and Steinschneider was also unable to satisfy himself as to its exact meaning.

[2] Literally, "do not follow the way of the images." The passage is obscure, and may refer to the paths of the planets in the constellations or to the zodiac.

[3] In Ibn Ezra's work here described.

[4] That is, Hipparchus, who lived about 140 B. C. Query: is there any relation of Abrks to the Gnostic term abraxas?

Peters and Knobel, *l. c.*, page 7, agree with Delambre that the catalogue of stars in the Almagest is due to Hipparchus. This was Ibn Ezra's opinion.

[5] This is only approximately correct, since Ptolemy lived in the second century A. D. Probably the 208 is a scribe's error for 280, since Ptolemy himself speaks of Hipparchus as having lived 285 years earlier than he.

[6] That is, Meton and Euctemon, who flourished in the fifth century B. C. Ptolemy states (*Almagest*, III, 1, which is Vol. I, p. 141 in the translation of Manitius) that he had to omit all reference to the observations made by the school of Meton and Euctemon. On the names used by Ibn Ezra, see Isak Israel, Jesod Olam, IV, 7; *Mag. für die Literatur des Auslandes*, 1846, p. 378.

In the fragment of Levi ben Abraham's work on astrology (Cod. Reggio, 13, fol. 56, now Cod. Oxf. 2028) the names appear as Meton and Euctemon.

[7] Probably 600 years would be nearer the truth

[8] Literally, "Place of the high point of the sun."

[9] He found it to be 5° 30′ in the year 140 B. C.

in the case of the planets. Its position now, in the year 1160 of the era of the uncircumcised,[1] is 25 degrees of the Twins.

"And these are the learned men of Ishmael who observed the point of apogee of the sun, not all of whom lived in the same generation:

"The first was the Arabian scholar[2] . . . and Jahjâ ibn Abî Manṣûr[3] and el Merwadi[4] and Ibn al Mokaffa-a,[5] el Kufi,[6] Jacob al Kindi,[7] Thabit ibn Qora,[8] al Hakemi[9] the Hindu, Theon of Alexandria,[10] Ibrahim ez-Zarkali[11] the Spaniard, el-Batani,[12] Ibn Alostay,[13] and Ibn el-Alam.[14]

"And now I shall begin to translate the book of the Ishmael scholar.[15]

"Here beginneth the book of Muhammed ibn al Matani ibn Abdul Karasi[16] ibn Ali Ishmael explaining[17] the tables of al Khowarizmi. You remember what you saw in [the] tables of the planets, [namely,] mistakes and disagreements and [evidences] that their authors did not give any proof for what they told us to do, but they left it[18] to us, and presented them[19] as a matter of tradition without [any] discussion. In the case of a book of this kind the reader may attribute to its

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[1] The date of the work, seven years before Ibn Ezra's death.

[2] The name is omitted in the Hebrew text.

[3] He lived in the time of the caliph el-Mansûr. See the *Zeitschrift für Mathematik*, XII, 31, seq.; Suter's list, page 8.

[4] Steinschneider identifies him as possibly Merwesi or el Merwadsi, the Habas el Hâsib el-Merwazî of Suter's list. He lived at Bagdad in the time of el-Mâmûn.

[5] Not in Suter's list.

[6] Probably an error for el-Sûfî, that is, 'Abderrahmân ibn 'Omar, Abû'l-Hosein, el-Sûfî, who died in 986. His work on the fixed stars was translated into French in 1874. Possibly, however, Ibn Ezra means Muhammed ibn Zijâd ibn el-A'râbî of Kûfa, who wrote on astronomy as well as language, and who died about 846.

[7] For the translation of his works into Hebrew, see *Zeitschrift der Morgenl. Gesellsch.*, XVIII, 131, 181. For his work on Hindu arithmetic, see Woepcke, *Mém. sur la propagation*, 159.

[8] Tâbit ibn Qorra ibn Merwân, Abû'l-Ḥasan, el-Ḥarrânî (826–901), one of the foremost Arab astronomers.

[9] Queried by Steinschneider. There were several Arab scholars of this name, but doubtless the caliph Hakem (996–1021) is meant, after whom the Hakem Tables composed by Ibn Junis were named. The words "the Hindu" are manifestly incorrect.

[10] Of course Theon was not one of the "learned men of Ishmael," so that Ibn Ezra uses the term rather loosely.

[11] Ibrâhîm el Zarquâla, or Zarqâlî, a famous Spanish instrument maker of the eleventh century. See *Zeitschrift für Math.*, XII, 34, 36; Steinschneider, *Etudes sur Zarkali*, Rome, 1884.

[12] Muhammed ibn Ǧâbir ibn Sinân, Abû Abdallâh, el-Battânî, a famous astronomer, known in Europe under the name Albatagnius. He died in 929.

[13] The name is doubtful.

[14] Probably the astronomer and astrologer 'Alî ibn el-Ḥosein, Abû'l Qâsim el-'Alawî, known under the name Ibn el-A'lam who died in 985.

[15] Beginning at this point, Steinschneider published only the Hebrew text of this fragment from Bodleian MS., Michael 400, without translation or comment. The rest of the text is still unpublished. See also the *Zeitung d. Morgenl. Gesellsch.*, XXV, p. 421.

[16] The transliteration is doubtful, Steinschneider gives Al Karuz. On el-Bîrûnî see the note on page 100.

The Codex Mich. 835 begins: "This book was written by Aḥmed . . . elkerim for his brother Muhammed ben Ali ben Ishmael" etc.

[17] Literally, "about the reasons of."

[18] That is, the proving of the operations.

[19] That is, the rules.

author one of two things: either he does not himself understand the explanations, having merely learned the facts from someone, or he is jealous of his great wisdom and does not care to reveal it. We have already seen that scholars of undoubted wisdom did the same, as in the case of Al Khapash[1] in his work on grammar, known as Aloust, the result being that men learned in the science of grammar decided that his book was suited neither to a teacher nor to a student. You recall that you have found the same thing in al-Khowarizmi's tables and you attributed it to the condensed form[2] of the work or to the selfishness of the author.[3] You also remember that you found the work attributed to el-Fergânî very far from perfect and even unsatisfactory for the uses to which you might wish to put it, and you saw that he explained things that are clear and are easy of comprehension while omitting all that was difficult and complicated, and you asked me for explanations so that nothing should be concealed from you. Therefore I explained all you asked for, and this will help your understanding and will satisfy any mathematician and man of science like yourself, and may God be my helper.

"Referring to what you have said about el-Fergânî, [namely,] that you found his work far from perfect,—I have read it and have also found it so; but I have found in el-Fergânî's book many things that showed him to be a wise man;[4] and the thought has occurred to me that el-Fergânî had worked out in his mind the commentary on the book and the proofs [to be furnished], but death overtook him and he was unable to complete it. Now, however, someone has transcribed it and attributed to him [as complete] what still lacks the explanations. It is also possible that he completed his work before he died, but that part of it was lost, or that the book fell into the hands of some ignorant person who ruined its perfection, and so we cannot attribute these faults[5] to a lack of knowledge.[6] It is true that these things will not be understood, however, by a man who is ignorant of mathematics.[7] I have already written a book on selected topics of this work and have arranged it in gates[8] in the form of questions and answers, so that it explains more fully all that you may wish to know, and this is easier to understand and more convenient to study and remember, and I trust in the Creator that I may succeed in satisfying your wish. Question: Why did Muhammed al Khowarizmi say that to compute the Arabian month [it is necessary] to take the whole number of the Arabian year, write it in two different places, multiply one [number] by 354, retaining it; then multiplying the other by 13, dividing [the product] by 30, and then adding the quotient to the other result?"

This completes the translation of the Hebrew text as published by Stein-

[1] The name is doubtful. It is probably Akhfash, who died 830–835.
[2] Literally, shortness or brevity.
[3] That is, his jealousy lest others should share his wisdom.
[4] That is, that he was right.
[5] Literally, "things."
[6] That is, on the part of el-Fergânî.
[7] That is, by such a man as evidently transcribed the work.
[8] *I. e.*, in chapters.

schneider from the Parma codex, and it covers Ibn Ezra's introduction. The text of el-Bîrûnî is in a manuscript in the Bodleian Library, and has never been published.

THE THEORY OF SIMILAR FIGURES.

By ROGER A. JOHNSON, Western Reserve University.

The object of this paper is to place in a new and perhaps a more satisfactory light the elements of the well-known theory of two and of three directly similar figures in a plane. The attempt has been made to keep in the foreground the idea of determining each of such a set of figures by a base-line, that is, by one of a set of homologous line-segments. This notion clarifies the conception, and appreciably simplifies the treatment; further, as shown below, it enables us to avoid certain false proofs which are to be found in the usual discussions of this subject. The use of directed angles[1] is again advantageous, but of course not essential.

The fundamental operations of elementary plane geometry are four in number:

(*a*) *Translation*, or motion of a figure such that every point is moved the same distance in the same direction,

(*b*) *Rotation* about a fixed point,

(*c*) *Reflection* with regard to a fixed line, which is the same in effect as turning the plane over on this line as axis,

(*d*) *Expansion* with regard to a fixed point, whereby the distance of each point of the figure from the fixed point is increased or decreased in a constant ratio (the same word *expansion* is used in all cases, whether the figure is actually enlarged or diminished).

If any figure is subjected to any succession of these operations, the resulting figure is similar to the original one; conversely it is easy to see that if two similar figures lie in a plane, one can be brought to coincide with the other by a combination, for instance, of a translation, a rotation, and an expansion. Our first object shall be to simplify this combination.

As a first lemma, we may note that if we operate on one of two directly similar figures in such a way that two of its points come to coincide with the corresponding points of the other, the figures coincide throughout.

Temporarily we shall use the word *homology* to designate the combination of a rotation and an expansion with regard to the same center.

THEOREM 1. *Given two line segments MN, $M'N'$, there exists a single operation, either a translation or a homology, that carries MN into $M'N'$.*

Case i. If MN and $M'N'$ are equal and parallel, and extend in the same direction, a translation that carries M into M' will carry N into N'.

Case ii. If $MNN'M'$ is a trapezoid, and MM' meets NN' at P, an expansion

[1] "Directed Angles in Elementary Geometry," R. A. Johnson, this MONTHLY, March, 1917, p. 101 ff.

with regard to P, followed by a rotation through 180° when P is between the parallels MN and $M'N'$, carries M and N into M' and N' respectively.

Case iii. If MN and $M'N'$ are segments of the same straight line not coming under case *i*, we may take a segment PQ, so that $M'N'QP$ is a rectangle. Let R be the point located as in case *ii* for MN and PQ, and let C be the foot of the perpendicular from R to the line $MNM'N'$. Then MN is carried into $M'N'$ by an expansion with regard to C, followed if necessary by a rotation of 180° about C.

Case iv. In the general situation, MN and $M'N'$ intersect at a point A. Let circles $MM'A$ and $NN'A$ intersect a second time at C; then triangles MNC and $M'N'C$ are directly similar, and if we rotate $M'N'C$ about C until CM' falls on CM, then expand until M' falls on M, we shall have moved N' to the position N.

In particular, one or two of the given points may fall at A: for instance, let A and M coincide; then for the circle $MM'A$ of the proof just given, we use the circle through M and M' which touches MN at M, and the proof holds as before. If both M and M' are at A, or if the circles are tangent there, then the point C is also there.

THEOREM 2. *If two directly similar figures lie in a plane,*

(*a*) *If they are congruent, with corresponding sides parallel and extending in the same direction, a translation will carry one into coincidence with the other.*

(*b*) *In every other case, there exist a point C, called the center of similitude, a number, the ratio of similitude, and an angle, the angle of similitude; if one figure is rotated about C through this angle, and expanded from C in this ratio, it is thereby made to coincide with the other figure.*

COROLLARY. The center of similitude is self-homologous in the two figures, and is the only self-homologous point; in other words, if we can determine a self-homologous point, we thereby find the center of similitude.

We notice a few interesting applications.

THEOREM 3. *If one vertex of a triangle of constant form is held fast, while a second vertex traces any figure whatever, the third vertex traces a similar figure.*

THEOREM 4. *If one of two directly similar figures is held motionless, while the other rotates about a fixed point, the center of similitude traces a straight line or a circle, according as the given figures are equal or unequal.*

COROLLARY. Obviously two circles may be regarded as similar figures in an infinite number of ways, with any point of the one homologous to any point of the other; the locus of the centers of similitude is a circle, the locus of a point whose distances from the centers of the given circles are proportional to the radii of the latter. This is called the *circle of similitude* of the two circles.

THEOREM 5. *If a figure is moved in a plane without change of form or size, so that two of its points remain constantly on fixed intersecting lines of the plane, the locus of the "instantaneous center of rotation" is a circle whose center is the intersection of the fixed lines.*

We may interpret these results in terms of maps of a plane region superposed,

the same side up, on a table. We see that there is, except in one special case, one point whose images in the two maps coincide; this may be called the *double point* of the two maps. In practise this double point would of course often fall outside the boundaries of actual maps. If one map is held fast, while the other is rotated about a pivot, the double point traces a circle or a straight line, according as the maps are on different scales or the same scale. Our other theorems can as well be interpreted in this way.

We consider next the properties of three maps lying on a table; in other words, the relations among three directly similar figures in a plane. Each two of them determine a double point, or center of similitude. We will designate the figures by I, II, III; then the center of similitude of II and III shall be called C_1, that of III and I, C_2, and that of I and II, C_3. Thus C_1 is self-homologous in II and III; and so on.

It is immediately evident that the situation admits numerous special cases, for even in the simple case of two figures (Theorem 1) there are several cases to consider. We shall confine ourselves to the most general case; similar methods may be applied to any special case, yielding similar results. For instance, we shall exclude case *i* of Theorem 1; thus we shall always have three actual centers of similitude.

Let us then consider three similar figures, determined by homologous segments or base-lines, M_1N_1, M_2N_2, M_3N_3; let these lines be not concurrent nor parallel, but form a triangle $L_1L_2L_3$. If C_1, C_2, C_3 are the centers of similitude, triangles $M_2N_2C_1$ and $M_3N_3C_1$ are similar; and so on. We know that C_1 is the intersection of circles $L_1M_2M_3$ and $L_1N_2N_3$; and so on. Now the circles $L_1M_2M_3$, $L_2M_3M_1$, $L_3M_1M_2$ intersect at a point M, and the circles $L_1N_2N_3$, $L_2N_3N_1$, $L_3N_1N_2$ at a point N.[1]

THEOREM 6. *The points M and N lie on the circle or line $C_1C_2C_3$.*

For

$$\begin{aligned}\measuredangle C_2MC_3 &= \measuredangle C_2M,\ L_2C_2 + \measuredangle L_2C_2,\ L_3C_3 + \measuredangle L_3C_3,\ MC_3\\ &= \measuredangle MC_2L_2 + \measuredangle L_3C_3M + \measuredangle L_2C_2,\ L_3C_3\\ &= \measuredangle L_2C_2,\ L_3C_3.\end{aligned}$$

Similarly

$$\measuredangle C_2NC_3 = \measuredangle L_2C_2,\ L_3C_3.$$

If now L_2C_2 and L_3C_3 are not parallel, but meet at a point X, then the circle through C_3, M and N passes also through C_2, and therefore through C_1. X is also on this circle.

On the other hand, if L_2C_2 and L_3C_3 are parallel, the above equations show that M and N lie on the line C_2C_3, and it comes at once that C_1 is on the same line, and L_1C_1 is parallel to L_2C_2 and L_3C_3.

COROLLARY. We may construct a system wherein the centers of similitude are collinear, or a system wherein they lie on an actual circle.

[1] *Loc. cit.*, p. 103.

For if the triangle $L_1L_2L_3$ and segments M_2N_2 and M_3N_3 are given, C_1 can be found. We may put C_2 anywhere, then C_3, M_1 and N_1 can be located. According as we make C_1L_1 and C_2L_2 parallel or intersecting, $C_1C_2C_3$ will be a line or a finite circle.

In the sequel we shall confine our attention to the case that C_1, C_2, C_3 lie on a circle of finite radius; the reader can easily deduce the corresponding theorems when they are collinear. The circle $C_1C_2C_3$ is called the *circle of similitude* of the three figures, and the triangle $C_1C_2C_3$ is called the *triangle of similitude.*

THEOREM 7. *The lines L_1C_1, L_2C_2, L_3C_3 are concurrent at a point X on the circle of similitude. That is, the triangle of any three homologous lines is in perspective with the triangle of similitude; the center of perspective lies on the circle of similitude.*[1]

For we have just seen that L_2C_2 and L_3C_3 meet at a point X on the circle of similitude; each of these must also meet L_1C_1 on the same circle. But L_3C_3, for instance, meets the circle at C_3 and at only one other point; in general, L_1C_1 and L_2C_2 do not pass through C_3; hence all three of these lines are concurrent at X.

THEOREM 8. *If the lines XL_1, XL_2, XL_3 are divided proportionally at L_1', L_2', L_3'; then $L_2'L_3'$, $L_3'L_1'$, $L_1'L_2'$ are parallel respectively to L_2L_3, L_3L_1, L_1L_2, and are homologous lines in the similar figures. In particular, lines through X and parallel to L_2L_3, L_3L_1, L_1L_2 respectively constitute a set of concurrent homologous lines.*

THEOREM 9. *Conversely, if three homologous lines are concurrent, the point of concurrence lies on the circle of similitude.*

The easiest proof is made by returning to first principles. Let M_1N_1, M_2N_2, M_3N_3 be homologous segments whose lines are concurrent at Z_1. Let an inversion with regard to Z as center carry M_1, N_1, etc., into M_1', N_1', etc. We now have two triangles $M_1'M_2'M_3'$, $N_1'N_2'N_3'$ in perspective at Z; therefore the intersections of corresponding sides are collinear. Let $M_2'M_3'$ meet $N_2'N_3'$ at C_1'; then inverting back to the original figure, the inverse of this point C_1' is the intersection of circles M_2M_3Z and N_2N_3Z, or in other words the center of similitude C_1. But if C_1', C_2', C_3' lie on a line, then C_1, C_2, C_3 lie on a circle through the center of inversion Z, and the proof is complete.

THEOREM 10. *There are three fixed points on the circle of similitude; any set of concurrent homologous lines pass respectively through these points. They are called the invariable points of the system.*

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[1] So far as the writer can ascertain, this well-known theorem has never been proved. The proofs, as given by McClelland and Lachlan, for instance, are based on a false theorem namely: "if two triangles are $A_1A_2A_3$ and PQR; if the lengths of the perpendiculars from P to A_2A_3, A_3A_1, A_1A_2 are p_1, p_2, p_3, and so on; then if A_1P, A_2Q, A_3R are concurrent,

$$\frac{p_2q_3r_1}{p_3r_1q_2} = 1$$

and conversely." The converse, which is used in our theorem, is false: for example, let P and Q lie on the bisectors of the interior angles at A_1 and A_2 respectively, while R is on the exterior bisector of A_3; the equation is satisfied by the perpendiculars, but the triangles are not in perspective. The same flaw vitiates the current proofs of several other standard theorems of this type.

For if XP_2 and XP_3 are homologous lines of II and III, meeting at X on the circle of similitude, and cutting that circle again at T_2 and T_3 respectively, then each of the angles $\measuredangle C_1XP_2$, $\measuredangle C_1XP_3$ is constant and therefore arcs C_1T_2 and C_1T_3 are constant, and T_2, T_3 are fixed points.

THEOREM 11. *The invariable points are homologous; they form a triangle inversely similar to that formed by three homologous lines.*

For (*a*) triangles $C_1T_2T_3$ and $C_2P_2P_3$ are similar, (*b*) $\measuredangle T_1T_3T_2 = \measuredangle T_1XT_2$, etc. Let C_2T_2 meet C_3T_3 at M.

THEOREM 12. *The point C_1', homologous in I to the double point C_1 of II and III, lies on the circle C_2C_3M.*

For

$$\measuredangle C_2C_1'T_1 = \measuredangle C_2C_1T_3 = \measuredangle C_2T_2T_3$$

$$\measuredangle T_1C_1'C_3 = \measuredangle T_2C_1C_3 = \measuredangle T_2T_3C_3$$

whence

$$\measuredangle C_2C_1'C_3 = \measuredangle C_2T_2,\ C_3T_3.$$

THEOREM 13. *Both C_1' and M lie on C_1T_1.*

For

$$\measuredangle C_3T_1C_1' = \measuredangle C_3T_3C_1 = \measuredangle C_3T_1C_1$$

and

$$\measuredangle C_3C_1'M = \measuredangle C_3C_2M = \measuredangle C_3C_2T_2 = \measuredangle C_3T_1T_2 = \measuredangle C_3C_1'T_1.$$

Our results may be summed up as follows:

The invariable triangle and the triangle of similitude are inscribed in the same circle and are in perspective at a point M. The lines C_1T_1, C_2T_2, C_3T_3, which are concurrent at M, meet again the respective circles C_2C_3M, C_3C_1M, C_1C_2M at C_1', C_2', C_3' respectively.

Concerning the properties of triads of homologous points, we merely state the main theorems; proofs by means of directed angles are not difficult.

THEOREM 14. *If P_1, P_2, P_3 are homologous, circles $C_2C_3P_1$, $C_3C_1P_2$, $C_1C_2P_3$ are concurrent at a point X, for which*

$$\measuredangle C_2XC_3 = \measuredangle C_2MC_3 + \measuredangle P_3P_1P_2.$$

COROLLARY. If homologous points P_1, P_2, P_3 move so that $\measuredangle P_3P_1P_2$ remains equal to a given angle α, then each moves on a circle; in particular, P_1 lies on that circle through C_2 and C_3 for which

$$\measuredangle C_2P_1C_3 = \measuredangle C_2MC_3 + \alpha.$$

THEOREM 15. *The loci of collinear homologous points are respectively the circles C_2C_3M, C_3C_1M, C_1C_2M; and the line through the three points passes also through M.*

THEOREM 16. *There is a single triad of homologous points which are the vertices of a triangle similar in a given sense to a given triangle.*

The construction is effected as follows: let $V_1V_2V_3$ be the given triangle. Let X be so located that

$$\measuredangle C_2XC_3 = \measuredangle C_2MC_3 + \measuredangle V_3V_1V_2$$

with similar expressions for $\measuredangle C_3XC_1$ and $\measuredangle C_1XC_2$. There is just one such point X. Now regard circle C_2XC_3 as a part of Fig. 1, and find the homologous circle of II, then P_2, one of the required triad of points, lies on this circle. But P_2 lies also on circle C_3XC_1, and therefore it is uniquely determined.

THEOREM 17. *Similarly, there is a single triad of collinear homologous points, whose distances are in the same ratios as the distances among three given collinear points.*[1]

It is well known that in the special case when three similar figures are constructed on the sides of a triangle $A_1A_2A_3$, in such a way that A_2A_3, A_3A_1, A_1A_2 are homologous segments, then

(*a*) the triangle of similitude is the second Brocard triangle,
(*b*) the invariable triangle is the first Brocard triangle,
(*c*) the circle of similitude is therefore the Brocard circle,
(*d*) the point M is the median point,
(*e*) the circles MC_2C_3, etc., are the Neuberg circles.

NOTE ON A CERTAIN CLASS OF DETERMINANTS.

By W. H. METZLER, Syracuse University.

In this journal for May, 1915, there appeared the translation of a paper by Professor Pascal which considered a class of determinants which have the property of being expressible as the sum of two squares, and it is the main object of this note to point out another proof of this fact.

The determinants in question are of the form:

$$D \equiv \begin{vmatrix} a_{11} & \cdots & a_{1n} & -b_{11} & \cdots & -b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} & -b_{n1} & \cdots & -b_{nn} \\ b_{11} & \cdots & b_{1n} & a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{nn} & a_{n1} & \cdots & a_{nn} \end{vmatrix},$$

which is the determinant of the set of linear homogeneous complex equations for the $2n$ quantities $p_1, \cdots, p_n, q_1, \cdots, q_n$:

$$\begin{aligned} (a_{11} + b_{11}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{1n} + b_{1n}\cdot i)(p_n + q_n\cdot i) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ (a_{n1} + b_{n1}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{nn} + b_{nn}\cdot i)(p_n + q_n\cdot i) &= 0. \end{aligned} \tag{1}$$

[1] Theorems 16 and 17 are doubtless well known, but the author has not chanced to see them explicitly stated. But see McClelland.

The determinant D is obviously unaltered in value[1] by rearranging the last n rows and the last n columns so that they will be in the reverse order, and then multiplying the elements of the first n rows and the last n columns by i. In this form it is seen to be centro-symmetric with all the b's positive and all the a's multiplied by i. It follows, therefore, that it is the product of two factors D' and D'', say, and it has been shown[2] that one of these factors is the sum of two sets of terms and the other is the difference of the same two sets.

Thus

$$D' \equiv \begin{vmatrix} b_{1n} + a_{1n}\cdot i & \cdots & b_{11} + a_{11}\cdot i \\ \cdot & \cdot & \cdot \\ b_{nn} + a_{nn}\cdot i & \cdots & b_{n1} + a_{n1}\cdot i \end{vmatrix}, \quad \text{or} \quad | b_{rs} + a_{rs}\cdot i | ;$$

and

$$D'' \equiv \begin{vmatrix} b_{1n} - a_{1n}\cdot i & \cdots & b_{11} - a_{11}\cdot i \\ \cdot & \cdot & \cdot \\ b_{nn} - a_{nn}\cdot i & \cdots & b_{n1} - a_{n1}\cdot i \end{vmatrix}, \quad \text{or} \quad | b_{rs} - a_{rs}\cdot i | .$$

As D' and D'' are determinants with binomial elements, each may be expanded into a sum of determinants with monomial elements giving

$$D = (X + i\cdot Y)(X - i\cdot Y)$$
$$= X^2 + Y^2.$$

Representing the matrix of D symbolically by

$$\begin{Vmatrix} (A) & (B) \\ (-B) & (A) \end{Vmatrix},$$

Professor Pascal remarks that "it still remains to study determinants whose matrices are of the type

$$\begin{Vmatrix} (A) & (B) & (C) & \cdots \\ (-B) & (A) & (D) & \cdots \\ (-C) & (-D) & (A) & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix}."$$

If our set of equations were of the form:

$$\begin{aligned} (a_{11} + b_{11}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{1n} + b_{1n}\cdot i)(p_n + q_n\cdot i) &= 0, \\ -(a_{12} + b_{12}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{2n} + b_{2n}\cdot i)(p_n + q_n\cdot i) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ -(a_{1n} + b_{1n}\cdot i)(p_1 + q_1\cdot i) - \cdots + (a_{11} + b_{11}\cdot i)(p_n + q_n\cdot i) &= 0, \end{aligned} \tag{2}$$

[1] The sign will be changed if n is odd.

[2] Metzler, On Centro-Symmetric and Skew-Centro-Symmetric Determinants, *Messenger of Mathematics*, New Series, No. 515, March, 1914.

where the diagonal coefficients are all $a_{11} + b_{11}i$, the determinant D would be

$$D = \begin{vmatrix} a_{11} & -b_{11} & a_{12} & -b_{12} & \cdots & a_{1n} & -b_{1n} \\ b_{11} & a_{11} & b_{12} & a_{12} & \cdots & b_{1n} & a_{1n} \\ -a_{12} & b_{12} & a_{11} & -b_{11} & \cdots & a_{2n} & -b_{2n} \\ -b_{12} & -a_{12} & b_{11} & a_{11} & \cdots & b_{2n} & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{1n} & b_{1n} & -a_{2n} & b_{2n} & \cdots & a_{11} & -b_{11} \\ -b_{1n} & -a_{1n} & -b_{2n} & -a_{2n} & \cdots & b_{11} & a_{11} \end{vmatrix};$$

or expressed symbolically

$$D = \begin{Vmatrix} (A) & (B) & (C) & \cdots \\ (-B) & (A) & (D) & \cdots \\ (-C) & (-D) & (A) & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix},$$

the form in question.

By a rearrangement of rows and columns it is readily seen that D may be represented symbolically by

$$\begin{Vmatrix} (A') & (B') \\ (-B') & (A') \end{Vmatrix}.$$

This is what should be expected since the D for equations (2) is a particular case of the D for equations (1).

If the $A, B, C, \cdots$ here are to be of higher order than the second it will be necessary to take equations from algebras of higher order, such as quaternions for the fourth order.

THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America held its fall meeting at St. John's College, Annapolis, Md., Saturday, December 15, 1917. Among the thirty-two persons in attendance were the following members of the Association:

O. S. Adams, U. S. Coast and Geodetic Survey, Washington, D. C.
J. A. Bullard, U. S. Naval Academy, Annapolis, Md.
Paul Capron, U. S. Naval Academy, Annapolis, Md.
G. R. Clements, U. S. Naval Academy, Annapolis, Md.
A. B. Coble, Johns Hopkins University, Baltimore, Md.
A. Cohen, Johns Hopkins University, Baltimore, Md.
J. B. Eppes, U. S. Naval Academy, Annapolis, Md.
J. N. Galloway, U. S. Naval Academy, Annapolis, Md.
H. C. Gossard, U. S. Naval Academy, Annapolis, Md.
Angelo Hall, U. S. Naval Academy, Annapolis, Md.

W. M. Hamilton, Nautical Almanac Office, Washington, D. C.
W. E. Heal, Washington, D. C.
W. W. Johnson, U. S. Naval Academy, Annapolis, Md.
A. E. Landry, Catholic University, Washington, D. C.
Florence P. Lewis, Goucher College, Baltimore, Md.
H. M. Robert, U. S. Naval Academy, Annapolis, Md.
R. E. Root, U. S. Naval Academy, Annapolis, Md.
W. F. Shenton, U. S. Naval Academy, Annapolis, Md.
H. I. Thomsen, Baltimore, Md.
H. R. Tolley, Department of Agriculture, Washington, D. C.
C. E. Van Orstrand, U. S. Geological Survey, Washington, D. C.

The president of the section, Professor Abraham Cohen, occupied the chair at both morning and afternoon sessions, and the following papers were presented:

1. "Report on the 1917 summer meeting of the Association at Cleveland." Dr. G. R. Clements, U. S. Naval Academy.
2. "A method for finding a particular integral for certain linear differential equations." Professor A. B. Coble, Johns Hopkins University.
3. "Report on the construction of certain mathematical tables." Mr. C. E. Van Orstrand, U. S. Geological Survey.
4. "Geometrical stereograms." Dr. W. F. Shenton, U. S. Naval Academy.
5. "Lambert's conformal conic projection." Mr. O. S. Adams, U. S. Coast and Geodetic Survey.
6. "An application of Grassmann's Ausdehnungslehre to the theory of invariants." Mr. W. E. Heal, Washington, D. C.
7. "Pseudo-regular polyhedra." Professor W. W. Johnson, U. S. Naval Academy.

A luncheon at the Hotel Maryland, arranged for those attending these sessions, was participated in by twenty-two persons.

Ralph E. Root,
Secretary.

BOOK NOTICES.

Send communications about books to W. H. Bussey, University of Minnesota.

In 1915 John Wiley and Sons published an *Analytic Geometry* by Professor H. B. Phillips of the Massachusetts Institute of Technology and in 1916 a *Differential Calculus* by the same author. These have been reviewed in this Monthly (see Volume 23, page 17 and Volume 24, page 78). The course in mathematics has now been extended by the publishing of an *Integral Calculus* which may be purchased separately or bound with the *Differential Calculus*. It contains answers to problems, a short table of integrals and a table of natural logarithms.

Every mathematician knows that many a problem is hard or easy according

is not really a case of uncorrelating the correlated. The *Advanced Algebra* contains in revised form and with some additions the material of the chapters on algebra in the original Brenke book, and a similar statement is true of the *Trigonometry.* The books may be had separately or bound together in one volume.

Correlation has been talked about for many years and some books on unified mathematics have been published. But most of the teaching of mathematics in this country is still done by means of the more or less old-fashioned text-books on algebra, trigonometry, etc., as separate subjects. The latest attempt at a correlation of college algebra, trigonometry and analytic geometry is the new *Elementary Mathematical Analysis* by J. W. Young and F. M. Morgan of Dartmouth College. Later it will be reviewed at length in the MONTHLY.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2680. Proposed by C. C. YEN, Tangshan, North China.

The diagonals of a maximum parallelogram inscribed in an ellipse are conjugate diameters of the ellipse. [From Joseph Edward's *Elementary Treatise on Differential Calculus.*]

2681. Proposed by PHILIP FRANKLIN, College of the City of New York.

Given n letters of one kind and $n-1$ letters of another kind, in how many ways can they be arranged so that, moving along the arrangement from one end to the other, the number of letters of the first kind passed over is greater than the number of the second kind at any instant?

2682. Proposed by E. L. REES, University of Kentucky.

Given the diagonal and the angle it makes with a bisector of one of the angles of a parallelogram. Construct the parallelogram so that the rectangle having sides equal to those of the parallelogram may have a given area.

2683. Proposed by J. R. HITT, Student, Mississippi College.

The height of a frustum of a cone is h, the radii of the lower and upper circular bases are a and b, respectively. Deduce a formula for finding the center of gravity of the frustum.

2684. Proposed by B. J. BROWN, Kansas City, Mo.

Find the locus of the center of a conic passing through four fixed points.

2685. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

A particle is describing an ellipse of eccentricity $\sqrt{2/3}$ as a central orbit about a focus when the attracting force suddenly becomes repulsive without changing its magnitude and the particle begins to describe an equilateral hyperbola; find where the change occurred and the angle that the major axis of the new orbit makes with that of the old orbit.

2686. Proposed by EDWIN R. SMITH, State College, Pa.

Given the difference equation

$$u_{x+1} - u_x = \log\left(1 - \frac{x}{sp}\right) - \log\left(1 + \frac{x+1}{sq}\right).$$

where s is a positive integer, p and q are proper fractions such that $p + q = 1$, and x is small when compared with sp and sq. Determine T_x, if $u_x = \log T_x$.

2687. Proposed by N. P. PANDYA, Sojitra, India.

An ellipse intersects a parabola in A and B, and the tangents at A and B to the parabola meet at T. The center C of the ellipse lies within the space enclosed by the parabola and the tangents. Draw a third tangent to the parabola such that C may be the centroid of the triangle formed by the three tangents.

2688. Proposed by FRANK IRWIN, University of California.

With four quantities, a_1, a_2, a_3, a_4, we may, without changing their order, form the following complex fractions:

$$\dfrac{a_1}{\dfrac{a_2}{\dfrac{a_3}{a_4}}} \qquad \dfrac{a_1}{\dfrac{\dfrac{a_2}{a_3}}{a_4}} \qquad \dfrac{\dfrac{a_1}{a_2}}{\dfrac{a_3}{a_4}} \qquad \dfrac{\dfrac{a_1}{\dfrac{a_2}{a_3}}}{a_4} \qquad \dfrac{\dfrac{\dfrac{a_1}{a_2}}{a_3}}{a_4}$$

But these have not all different values; the first and fourth are equal.

Determine how many different rational functions of the quantities a_1, a_2, $\cdots$ a_n may be obtained in this way, and which can be represented in more than one way as a complex fraction of the above kind, and which in only one way.

Restatement of 434 (Calculus). Proposed by E. W. CHITTENDEN, Champaign, Ill.

Evaluate $\int_0^1 f(x)dx$ where

$$f(x) = \sum_{n=1}^{n=\infty} \frac{\operatorname{sgn}(x - x_n)}{2^n}.$$

The function $\operatorname{sgn} x = -1, 0, +1$ according as x is negative, zero or positive. The numbers x_n form the series

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8}, \cdots$$

for which the general formula is $x_n = (2h + 1)/2^k$, where k is the greatest integer such that $2^{k-1} \leqq n$ and $h = n - 2^{k-1}$.

Note. This problem is reprinted because of a misprint in the original statement and also because of a lack of definiteness in the definition of the series of values x_n. An inquiry has also been received in regard to the origin of the notation "sgn x." This notation seems to be due to Kronecker; cf. Werke, Vol. II, p. 500.

SOLUTIONS OF PROBLEMS.

487 (Algebra). Proposed by WILLIAM HOOVER, Columbus, Ohio.

Show in two ways that 0.5623 is not a root of $(1 - m)e^m = 2e^{-1}$, e being the Napierian base. Find the value of m, and test the result in two ways.

SOLUTION BY E. E. WHITFORD, New York City.

Taking logarithms of both sides, we have

$$\log(1 - m) + (m + 1)\log e = \log 2, \quad \text{or} \quad \log(1 - m) = \log 2 - 1 - m.$$

Substituting .5623 for m,

$$\log .4377 = .6931 - 1 - .5623 = 9.1308 - 10.$$

But this is false since $\log .4377 = 9.1733 - 10$. Or, from $(1 - m)e^{m+1} = 2$, on taking $m = .5623$ we have $e^{1.5623} = 4.57$. But this is false, since $4.57 = e^{1.5195}$. One value of m is .5936 approximately; this checks by both the above methods. We may also have $m = -1$.

Also solved by C. E. FLANAGAN and EDWARD H. WORTHINGTON.

488 (Algebra). Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix} = 0,$$

where

$$a_k = \frac{\sin(k\theta + \alpha)}{\sin k\theta},$$

and θ and α have any values that do not make a denominator zero.

SOLUTION BY A. M. HARDING, University of Arkansas.

It is easily shown that the following equations hold for all values of θ:

$$\cot\theta - 2\cot 2\theta + \cot 3\theta = \frac{2\cos 2\theta}{\sin 4\theta + \sin 2\theta},$$

$$\cot\theta\cot 3\theta - \cot^2 2\theta = \frac{2\cos 2\theta\cot 4\theta}{\sin 4\theta + \sin 2\theta};$$

that is

$$\cot\theta\cot 3\theta - \cot^2 2\theta = \cot 4\theta(\cot\theta - 2\cot 2\theta + \cot 3\theta).$$

Now a_k may be written in the form

$$a_k = \cos\alpha + \sin\alpha\cot k\theta.$$

Thence we have

$$a_1 - 2a_2 + a_3 = \sin\alpha(\cot\theta - 2\cot 2\theta + \cot 3\theta),$$

and

$$\begin{aligned} a_1a_3 - a_2^2 &= \sin\alpha\cos\alpha(\cot\theta - 2\cot 2\theta + \cot 3\theta) + \sin^2\alpha(\cot\theta\cot 3\theta - \cot^2 2\theta) \\ &= \sin\alpha(\cot\theta - 2\cot 2\theta + \cot 3\theta)(\cos\alpha + \sin\alpha\cot 4\theta) \\ &= (a_1 - 2a_2 + a_3)a_4. \end{aligned}$$

Hence

$$\frac{a_1a_3 - a_2^2}{a_4} + 2a_2 - a_1 - a_3 = 0;$$

that is,

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix} = 0.$$

Also solved by PAUL CAPRON, OSCAR ADAMS, and EDWARD H. WORTHINGTON.

489 (Algebra). Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove the following:

$$\begin{vmatrix} -x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ -v & -u & y & -x \end{vmatrix}^2 + a\begin{vmatrix} x & -x & -bu & abv \\ y & y & -bv & -bu \\ u & u & x & ay \\ v & -v & y & -x \end{vmatrix}^2$$

$$+ b\begin{vmatrix} x & -ay & -x & abv \\ y & x & y & -bu \\ u & av & u & ay \\ v & -u & -v & -x \end{vmatrix}^2 + ab\begin{vmatrix} x & -ay & -bu & -x \\ y & x & -bv & y \\ u & av & x & u \\ v & -u & y & -v \end{vmatrix}^2 = \begin{vmatrix} x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ v & -u & y & -x \end{vmatrix}^2.$$

SOLUTION BY J. L. RILEY, Stephenville, Texas.

Evaluating the first determinant, we have $[R(R - 2ay^2 - 2bu^2)]^2$, where

$$R = x^2 + ay^2 + bu^2 + abv^2.$$

The second determinant, when expanded, gives $a[-2R(xy + buv)]^2$; the third gives $b[2R(avy - ux)]^2$; the fourth gives 0; and the fifth determinant gives $[-R^2]^2$. But

$$[R(R - 2ay^2 - 2bu^2)]^2 + a[-2R(xy + buv)]^2 + b[2R(avy - ux)]^2 \neq [-R^2]^2$$

unless $(ay^2)(bu^2) = 0$. Hence the relation stated in the problem does not always hold.

520 (Geometry). Proposed by ALBERT A. BENNETT, University of Texas.

On a given tangent to a circle determine a point such that, if a secant be drawn joining this point to the extremity of the diameter which is perpendicular to the given tangent, the segment of this secant exterior to the circle will be equal in length to a given segment.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let us denote the radius of the given circle by r and the length of the given segment by $2d$. Let AB be the diameter perpendicular to the tangent at the point of tangency A. Take a length $AC = d$ along the tangent from A. Join BC. Take D on CB so that $CD = d$. With center at B draw arc DE cutting the circle at E. Produce BE to cut the tangent at P. Then P is the required point.

Proof:

$$BE = BD = \sqrt{d^2 + 4r^2} - d.$$

Since AE is perpendicular to BP it follows that

$$AB^2 = BE \times BP = BE(BE + EP).$$

Hence,

$$4r^2 = (\sqrt{d^2 + 4r^2} - d)(\sqrt{d^2 + 4r^2} - d + EP).$$

From this equation we find

$$EP = 2d.$$

Note: If a point Q be taken on the tangent such that $AQ = AP$, this point Q will also satisfy the conditions of the problem.

Also solved by MAY PHALOR, H. T. AUDE, HERBERT N. CARLETON, OSCAR S. ADAMS, H. C. FEEMSTER, and PAUL CAPRON.

521 (Geometry). Proposed by R. M. MATHEWS, Riverside, Cal.

A variable circle, with center on the line l and passing through a fixed point P, cuts a fixed circle in A and B. Prove that the common chord AB and the perpendicular to l through P intersect in a fixed point.

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois

It is convenient to employ rectangular coördinates. Let l be taken as the axis of x and the point P on the y-axis; then the perpendicular to l through P is the axis of y. Under these conditions, it follows from elementary analytic geometry that the equation of the variable circle is

$$(x - \alpha)^2 + y^2 = r^2,$$

where r is the radius of the variable circle. The equation of the fixed circle may be taken as

$$(x - a)^2 + (y - b)^2 = c^2.$$

Subtracting the first equation from the second, we get

$$2(\alpha - a)x - 2by = c^2 - a^2 + \alpha^2 - r^2 - b^2,$$

which is the equation of a straight line through A and B. To find the point in which this line cuts the axis of y, set x equal to zero and solve the resulting equation for y. Since $r^2 - \alpha^2$ is independent of the position of the variable circle, it follows that the intersection of the chord AB and the perpendicular to l through P is fixed.

It is interesting to note that the above proof still holds when the intersections of the two circles are imaginary points.

Also solved by H. C. Feemster, Oscar S. Adams, Harold R. Schaufler, A. M. Harding, Paul Capron, Margaret F. Willcox, and Roger A. Johnson.

522 (Geometry). Proposed by GEORGE Y. SOSNOW, Newark, N. J.

Prove that the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of the lines joining the middle points of the opposite edges.

Solution by R. M. Mathews, Riverside, California.

Let $ABCD$ be the tetrahedron with X, the mid-point of BD, opposite to Y, the mid-point of AC. In the triangle BYD, we have

$$2XY^2 = BY^2 + DY^2 - 2BX^2,$$

from the theorem: The sum of the squares on two sides of a triangle is equal to twice the square on half the third side plus twice the square on the median to that side.

By the same theorem, we find $2BY^2$ in the triangle ABC and $2DY^2$ in the triangle ADC. Then,

$$4XY^2 = AB^2 + BC^2 + CD^2 + DA^2 - AC^2 - BD^2.$$

Similarly with U, the middle point of AB, and V, of DC:

$$4UV^2 = AC^2 + CB^2 + BD^2 + DA^2 - AB^2 - DC^2;$$

and with T, the middle point of AD, and W, of BC:

$$4TW^2 = AC^2 + CD^2 + DB^2 + BA^2 - AD^2 - BC^2.$$

By addition, we have

$$4(XY^2 + UV^2 + TW^2) = AB^2 + BC^2 + CD^2 + AD^2 + AC^2 + BD^2.$$

Remark: Considering any one of the three equations above we have proved the theorem: Four times the square on the median joining two opposite edges of a tetrahedron is equal to the sum of the squares on the other edges minus the sum of the squares on the edges which it joins.

Also solved by Horace Olson, Oscar S. Adams, J. L. Riley, and H. C. Gossard.

345 (Mechanics). Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

Two particles A and B are together in a smooth circular tube. A attracts B with a force whose acceleration is ω^2 and moves along the tube with uniform angular velocity 2ω, B being initially at rest; prove that the angle φ subtended by AB at the center after a time t is given by the equation

$$\log \tan \frac{\pi + \varphi}{4} = \omega t.$$

Solution by William Hoover, Columbus, Ohio.

Let B' be the initial positions of A and B, and put $BB' = s$, and $AB = x$.

Resolving forces tangentially,

$$\frac{d^2s}{dt^2} = \omega^2 x \cos ABC, \tag{1}$$

BC being the tangent through B.

Now if O be the center of the circle,

$$\angle AOB = \frac{2r\omega t - s}{r}, \qquad \angle ABC = \frac{2r\omega t - s}{2r},$$

$$x \cos ABC = 2r \sin \tfrac{1}{2}AOB \cos ABC = 2r \sin \frac{2r\omega t - s}{2r} \cos \frac{2r\omega t - s}{2r} = r \sin \frac{2r\omega t - s}{r}$$

$$= r \sin (2\omega t - \theta), \ \theta \text{ being the angle } BOB' \text{ and } s = r\theta.$$

Substituting these in (1),

$$\frac{d^2(2\omega t - \theta)}{dt^2} = -\omega^2 \sin(2\omega t - \theta), \tag{2}$$

or, since $2\omega t - \theta = \varphi$,

$$\frac{d^2\varphi}{dt^2} = -\omega^2 \sin\varphi. \tag{3}$$

Multiplying by $2(d\varphi/dt)$ and integrating,

$$\frac{d\varphi^2}{dt^2} = 2\omega^2 \cos\varphi + C. \tag{4}$$

When $\varphi = 0$, $d\varphi/dt = 0$; hence, $C = 2\omega^2$, and (4) is

$$\frac{d\varphi^2}{dt^2} = 2\omega^2(1 + \cos\varphi) = 4\omega^2 \cos^2 \tfrac{1}{2}\varphi. \tag{5}$$

This gives

$$\omega dt = \frac{d\frac{1}{2}\varphi}{\cos\frac{1}{2}\varphi}; \tag{6}$$

integrating and using the fact that $\varphi = 0$ when $t = 0$, we have

$$\log\tan\frac{\pi + \varphi}{4} = \omega t. \tag{7}$$

Also solved by G. Paaswell and J. B. Reynolds.

346 (Mechanics). Proposed by WILLIAM HOOVER, Columbus, Ohio.

Half the length of one of the equal parts of a uniform heavy string resting in equilibrium over a smooth horizontal indefinitely thin peg is cut off; determine the instantaneous change in the pressure on the peg.

Solution by the Proposer.

Let m = the mass of a unit of length of the string, $2a$ = the whole length of the string, and so a, $a/2$, the parts at the instant of cutting, $3a/2$ the length in motion after any time t from the beginning of motion, x = the longer part at the same instant, and T = the corresponding tension in the string; then the equations of motion are, noticing that the momenta of the moving masses are each variable,

$$\frac{d}{dt}\left(mx\frac{dx}{dt}\right) = m\left(\frac{dx^2}{dt^2} + x\frac{d^2x}{dt^2}\right) = mgx - T, \tag{1}$$

$$\frac{d}{dt}\left\{m(\tfrac{3}{2}a - x)\frac{d}{dt}(\tfrac{3}{2}a - x)\right\} = \frac{d}{dt}\left\{-m\tfrac{3}{2}a\frac{dx}{dt} + mx\frac{dx}{dt}\right\} = mg(\tfrac{3}{2}a - x) - T. \tag{2}$$

Subtracting (2) from (1), we have

$$\frac{d^2x}{dt^2} = \frac{2g}{3a}\left(2x - \frac{3a}{2}\right). \tag{3}$$

Multiplying by $2(dx/dt)$ and integrating, noticing that when $x = a$, $(dx/dt) = 0$, we obtain

$$\frac{dx^2}{dt^2} = \frac{g}{3a}\left\{\left(2x - \frac{3a}{2}\right)^2 - \frac{a^2}{4}\right\}. \tag{4}$$

Substituting the values of dx^2/dt^2 and d^2x/dt^2 in (1), and putting $T = T_0$ when $x = a$, we find $T_0 = \frac{2}{3}mga$.

If P = the required initial pressure, $P = 2T_0 = \frac{4}{3}mga = 2mga - \frac{1}{3}\cdot 2mga$, so that the pressure before cutting is diminished by one third.

Also solved by Horace Olson.

264 (Number Theory). Proposed by C. F. GUMMER, Kingston, Canada.

Find a general formula for three squares in arithmetical progression. Is it possible for the common difference to be a perfect square?

SOLUTION BY ARTEMAS MARTIN, LL.D., Washington, D. C.

Let x^2, y^2, z^2 be three square numbers in arithmetical progression; then we must have

$$y^2 - x^2 = z^2 - y^2, \qquad \text{or} \qquad x^2 + z^2 = 2y^2. \tag{1}$$

Assume $z = v + w$, $x = v - w$, and (1) becomes after dividing by 2,

$$v^2 + w^2 = y^2. \tag{2}$$

Take now $v = p^2 - q^2$, $w = 2pq$, and (2) is satisfied. Retracing, we find

$$z = p^2 - q^2 + 2pq, \qquad x = p^2 - q^2 - 2pq, \qquad y = p^2 + q^2.$$

Hence, the required squares are

$$x^2 = (p^2 - q^2 - 2pq)^2, \qquad y^2 = (p^2 + q^2)^2, \qquad z^2 = (p^2 - q^2 + 2pq)^2.$$

Taking $p = 2$, $q = 1$, the numbers are 1, 25, 49; taking $p = 3$, $q = 2$, the numbers are 49, 169, 289; taking $p = 4$, $q = 1$, the numbers are 49, 289, 529; taking $p = 4$, $q = 3$, the numbers are 289, 625, 961; and so on, indefinitely.

The common difference of three square numbers in arithmetical progression can not be a square number. See Barlow's "Theory of Numbers," p. 257. An equivalent theorem is also given in Carmichael's *Diophantine Analysis*, p. 14.

There can not be *four* square numbers in arithmetical progression. Barlow, same page. Therefore there can not be *five*, nor any greater number than three, of squares in arithmetical progression.

Also solved by J. L. RILEY, V. M. SPUNAR, and the PROPOSER.

265 (Number Theory). Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^4 - ax^2 + bx + c = 0$ are rational, prove that $4(a + yz) - 3(y + z)^2$ is a perfect square, y and z being any two roots of the equation.

SOLUTION BY N. P. PANDYA, Sojitra, India.

Since y and z are roots of the given equation, $x^2 - x(y + z) + yz$ is a factor of the left-hand side of the equation.

Since the term in x^3 is wanting, the remaining roots are given by

$$x^2 + x(y + z) + \frac{c}{yz} = 0. \tag{1}$$

The product of (1) with $x^2 - x(y + z) + yz = 0$ gives

$$a = -\frac{c}{yz} - yz + (y + z)^2.$$

Hence,

$$4(a + yz) = -\frac{4c}{yz} + 4(y + z)^2,$$

or

$$4(a + yz) - 3(y + z)^2 = (y + z)^2 - \frac{4c}{yz} = \text{a square},$$

since the roots of (1) are rational and its discriminant is therefore a square.

267 (Number Theory). Proposed by C. C. YEN, Tangshan, North China.

A number theory function $\phi(n)$ is defined for every positive integer n, and for every such number n it satisfies the relation $\phi(d_1) + \phi(d_2) + \phi(d_3) + \cdots + \phi(d_r) = n$, where $d_1, d_2, \cdots, d_r$ are the divisors of n. From this property alone show that

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where p p $p_3 \cdots p_k$ are the different prime factors of n.

SOLUTION BY OLIVE C. HAZLETT, Bryn Mawr College.

The theorem is clearly true for all primes. Accordingly, assume the theorem is true for all divisors of $n = \prod_{i=1}^{k} p_i^{e_i}$ which are less than n. Now for n the defining equation becomes

$$\phi(n) + A_n + p_1^{e_1}\left(1 - \frac{1}{p_1}\right)B_n = n,$$

where A_n is the sum of the ϕ-functions formed for the divisors of $p_1^{e_1-1}\, p_2^{e_2}\cdots p_k^{e_k}$ and B_n is a similar sum formed for all distinct factors of any of the numbers $p_2^{e_2-1}\, p_3^{e_3}\cdots p_k^{e_k}$, $p_2^{e_2}\, p_3^{e_3-1}\, p_4^{e_4}\cdots p_k^{e_6}$, $\cdots$. It is easy to find an expression for A_n, but it is sufficient for our purposes to note that A_n is a polynomial in $p_2, \cdots, p_k$ of degree at most $\sum_{i=2}^{k} e_i - 1$. Therefore $p_1^{e_1}\left(1 - \frac{1}{p_1}\right)$ is a factor of $\phi(n)$. Since this proof is perfectly general, it holds for every expression of the form $p_i^{e_i}\left(1 - \frac{1}{p_i}\right)$ $(i = 1, \cdots, p)$, and thus $\prod_{i=1}^{k} p_i^{e_i}\left(1 - \frac{1}{p_i}\right)$ is a factor of $\phi(n)$. Comparing the coefficients of $\prod_{i=1}^{k} p_i^{e_i}$ our formula is proved.

Also solved by H. C. FEEMSTER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO FINDING DERIVATIVES OF TRIGONOMETRICAL FUNCTIONS.

By T. H. HILDEBRANDT, University of Michigan.

In most textbooks on the elementary calculus the derivatives of the trigonometric functions are based on the derivative of the sine function, which, in turn, is derived from the definition of derivative. The proofs dealing with the value of this derivative seem to have something indirect about them. All goes well until the point is reached where the expression

$$\lim_{\Delta x \doteq 0} \frac{\sin (x + \Delta x) - \sin x}{\Delta x}$$

is to be evaluated, and then one of two methods is used. Either $\sin (x + \Delta x)$ is expanded by the formula for the sine of the sum of two angles and the formula for $1 - \cos x$ in terms of half angles is used, or the formula for the difference of two sines is used. Both of these latter formulæ have long since escaped the memory of the average sophomore student—if they ever had lodging there—and he practically accepts this part of the derivation on faith.

While it must be admitted that the most natural beginning for a chapter on the derivatives of trigonometrical functions is a paragraph devoted to finding the derivative of the sine, this advantage is more than counterbalanced by the simplicity with which it is possible to obtain the derivative of the tangent function directly from the definition of derivative—a fact which seems almost to have

escaped the attention of writers of textbooks on calculus. For this purpose it is possible to proceed in either of two ways, both of which are elegant and altogether natural and direct. If we take $\tan x = \sin x/\cos x$, then

$$\tan(x+\Delta x) - \tan x = \frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\Delta x)\cos x - \cos(x+\Delta x)\sin x}{\cos(x+\Delta x)\cos x}$$

$$= \frac{\sin \Delta x}{\cos(x+\Delta x)\cos x}.$$

If we divide now by Δx and take the limit as Δx approaches zero, then $\lim_{\alpha \doteq 0} (\sin \alpha)/\alpha = 1$ is applicable for evaluating the derivative. Or, proceeding directly, we have

$$\tan(x+\Delta x) - \tan x = \frac{\tan x + \tan \Delta x}{1 - \tan x \tan \Delta x} - \tan x$$

$$= \frac{(1+\tan^2 x)\tan \Delta x}{1 - \tan x \tan \Delta x}.$$

Then by using the fact that $\lim_{\alpha \doteq 0} (\tan \alpha)/\alpha = 1$, we again get at once the value of the derivative of the tangent. Either of these methods yields this derivative without any troublesome trigonometrical transformations.

On the basis of the derivative of the tangent, the remaining derivatives are easily obtained in the order $\sec x$, $\cos x$, and $\sin x$: the secant by taking derivatives in the relation

$$\tan^2 x + 1 = \sec^2 x,$$

the cosine through the fact that it is the reciprocal of the secant, and the sine from one of the relations

$$\sin x = \cos(\pi/2 - x), \qquad \sin x = \cos x \tan x.$$

We get, then, in this way the derivatives which are used most, and we employ only trigonometrical relations which are familiar to the average sophomore student.

II. Relating to the Proof that a Rigid Body Moving about a Fixed Point Is at Each Instant Rotating about an Axis through that Point.

By E. L. Rees, University of Kentucky.

The following proof of this theorem differs from other proofs b; methods in that it is based upon an interesting geometric interpretation of the equations involved.

Let O be the fixed point, and P and Q any two points of the body distinct from O, and let $OP = p$ and $OQ = q$. Using the Gibbs notation we have as the assumptions of rigidity $p^2 =$ constant, $q^2 =$ const., $p \cdot q =$ const., or differentiating with respect to t,

$$p \cdot \dot{p} = 0, \qquad q \cdot \dot{q} = 0, \qquad p \cdot \dot{q} + \dot{p} \cdot q = 0, \tag{1}$$

where $\dot{p}$ and $\dot{q}$ are the velocities of P and Q respectively.

Consider the equations $\dot{p} = w \times p$ and $\dot{q} = w \times q$. Regarding w as the running coordinate, each of these equations represents a line parallel to the position vector and perpendicular to the velocity of the point. If $p \times q = 0$, *i. e.*, if O, P, and Q are collinear, the lines coincide. For, substituting $p = kq$ in $\dot{p} = w \times q$ we get $k\dot{q} = w \times (kq)$, or $\dot{q} = w \times q$. Hence the equations are equivalent.

The condition that these lines shall intersect is $(p^{-1} \times \dot{p} - q^{-1} \times \dot{q}) \times p \cdot q = 0$, the terms in the parenthesis being the perpendiculars (vectors) from O to the lines. Expanding we get $\dot{p} \cdot q - p \cdot \dot{p}p^{-1} \cdot q + p \cdot \dot{q} - p \cdot q^{-1}\dot{q} \cdot q = 0$ which is satisfied by virtue of equations (1). Thus every pair of lines associated as above with a pair of points of the body will intersect. All of the lines will thus pass through the same point (since not all are coplanar) forming a bundle the vertex of which is the tip of the vector w which satisfies all of the equations. Since w is the same for all points at a given instant, we have thus proved that the body rotates instantaneously about an axis through O with the angular velocity w.

To determine w it is necessary to have two non-coincident lines given, *i. e.*, we must have the velocities of two points (not collinear with O) given to determine the velocities of all the other points. Since w satisfies $\dot{p} = w \times p$ and $\dot{q} = w \times q$ it follows that $w = \dot{p} \times \dot{q}/\dot{p} \cdot q = - \dot{p} \times \dot{q}/\dot{q} \cdot p$.

If $\dot{p} = 0$ and $\dot{q} \neq 0$ one of the lines lies along p and the point P is on w, and therefore on the axis of rotation. If $\dot{p} = 0$ and $\dot{q} = 0$ there is no motion since the lines intersect at the origin and $w = 0$.

If the velocities of the two points are coplanar with the vectors of those points, it will be seen easily that the vertex of the bundle will be a point on the normal to the plane through O, and that the lines for all points of this plane will form a pencil in a parallel plane. Since w is perpendicular to every line of this pencil, only one line is needed for its determination. Thus if $\dot{p} = w \times p$, we have $w = p^{-1} \times \dot{p}$. Hence in the case of motion of a rigid body (one point fixed) in a plane the velocity of one point will determine those of all the other points.

If now the fixed point be given a velocity we see that the velocities of all of the points of the body will be determined in space by the velocities of three non-collinear points, and in the plane by the velocities of two non-coincident points.

III. Relating to the Graph of $Y = f(X)$ for Complex Variables.

By E. L. Rees, University of Kentucky.

The following is to suggest a slight modification of the geometric representation of $Y = f(X)$, (X and Y both complex) given by Professor Frumveller in the November, 1917, number of the Monthly.

Let $X = x + iy$ and $Y = u + iv$. Letting u be represented by perpendiculars to the plane of X we get a surface which pictures the variation of the real part of Y. On this surface draw the contours for $v = v_1$, v_2, etc., the consecutive v's differing by a constant. These contours enable us to visualize the variation of v, so that we have pictured the variation of both u and v and, therefore, of Y, all on one surface.

Let the surface be cut by planes parallel to the X-plane, the planes of all of the consecutive pairs being the same distance apart. Project these curves of intersection and the contours on the X-plane. The curves into which these curves project will be the images of the grating of lines parallel to the u and v axes in the Y-plane. We thus establish a simple relation between this scheme and the usual scheme of representation of functional correspondence in complex variable theory.

For the linear function the surface is a plane and the contours are lines in this plane perpendicular to its xy-trace. If the coefficients are real the plane is parallel to the y-axis.

For the quadratic function the surface is a hyperbolic paraboloid and the contours are the intersections of this surface and hyperbolic cylinders.

For $Y = \log X$ the locus is a funnel-shaped surface generated by revolving $u = \log x$ about the u-axis and the contours are the meridians of this surface.

The projections of these contours and the intersections of the surfaces by the parallel planes on the X-plane give the familiar curves associated with these functions.

BUREAU OF INFORMATION.

James Byrnie Shaw, Chairman, University of Illinois, Urbana.

The Bureau of Information was established to furnish information to any one about any mathematical subject or question. So far as possible the committee will answer questions of the following character:

1. Definitions of terms.
2. Brief explanations of recent developments in mathematics.
3. Inquiries as to theorems.
4. Inquiries as to articles upon special topics.
5. Bibliographical references.
6. Historical questions.
7. Inquiries for book reviews.

8. Indication of libraries which will lend books.
9. Information as to graduate courses, fellowships, etc.
10. Inquiries as to societies, prizes, etc.

Whenever an inquiry seems to be of sufficient general interest to warrant publication, an article relative to it will appear in the MONTHLY.

The following question is of general interest and the reply is printed below: "It is well-known that the equilateral triangle, the square and the hexagon, are the only regular figures that will perfectly fill a plane surface. What are the closed forms (polyhedra) that will completely fill space?"

REPLY BY ARNOLD EMCH, University of Illinois.

The cellular subdivision of space is one of the important problems of analysis situs and has been the subject of a great number of investigations. It comprises the theory of crystalline forms and structure, the regular division of the surface of a sphere, etc.

In this connection two representative treatises may be mentioned: *Mathematical crystallography*[1] by H. Hilton and *Vielecke und Vielflache. Theorie und Geschichte*[2] by Dr. M. Brückner, where certain aspects of the problems in question are thoroughly treated. In Brückner's book also numerous historical and other references may be found.

In the plane there are three cases of regular polygons by which a certain portion of the plane may be packed uniformly, namely, the triangle, the square, and the hexagon. In space of three dimensions there are equal cubes, belonging to the class of regular polyhedrons, by which a certain portion of space may be uniformly filled.

If we do not limit ourselves to regular polyhedrons then there are innumerable possibilities of subdividing a certain portion of space by a set of equal polyhedrons of semi-regular and irregular types. A few characteristic cases shall be given.

In the first place space may be subdivided in an infinite number of ways by rhombohedrons or parallelepipedons, whose faces are parallelograms which, in sets of two, are equal.

The subdivision itself is obtained as follows: Take three distinct concurrent, non-coplanar axes which are supposed to be indefinitely extended. Denote the axes by X, Y, Z, and the origin by O. From O measure off on these axes (on each axis with a definite unit-length) equal distances, so that on each axis the scale of real integers is represented. Through the division points on the X-axis draw planes parallel to the YZ-plane; through those of the Y-axis planes parallel to the ZX-plane; through those of the Z-axis planes parallel to the XY-plane. These three sets of planes divide space into a uniform set of equal parallelepipedons, whose vertices form a space-lattice. There is an infinite number of subdivisions possible where the faces of the parallelepipedon are equal rhombs. In this particular case the parallelepipedon is called a rhombohedron. Of the

[1] The Clarendon Press, Oxford, 1903.
[2] B. G. Teubner, Leipzig.

class of rhombohedrons one is of particular importance as will appear from the construction of the rhombus-dodecahedron, Figure 1, which is another example of a semi-regular polyhedron by whose repetition a certain portion of space may be uniformly packed.

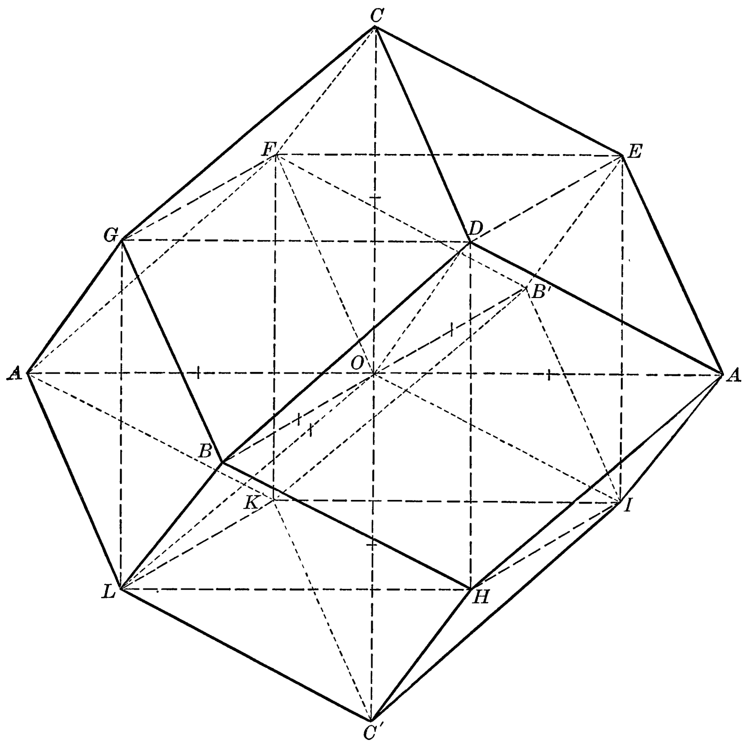

FIG. 1.

Draw three equal, mutually perpendicular axes AA', BB', CC', intersecting at O, and so that $OA = OA'$, $OB = OB'$, $OC = OC'$. Thus $AA'BB'CC'$ are the vertices of an octahedron. Now draw the perpendicular bisecting planes of the segments OA, OA', OB, OB', OC, OC'; they bound a cube $DEFGHIKL$. Join the vertices of the octahedron to those of the cube, as indicated in the figure. The joins thus obtained form the edges of the rhombus-dodecahedron, whose faces are equal rhombs with angles θ and ϕ, for which $\tan \theta/2 = \sqrt{2}$, or $\theta/2 = 54° \ 42' +$, and $\phi = 180 - \theta$. The planes of two adjacent rhombs include an angle of 120°. The rhombus-dodecahedron can be divided into the four equal rhombohedrons $ADOIHBLC'$, $DCFOBGA'L'$, $ADOIDCEB'$, $OIB'FLC'KA'$.

If we make an isometric projection of the figure, *i. e.*, if we assume in the figure OA, OB, OC at angular intervals of 120°, then OD which lies in one of the principal diagonals of the cube will merely appear as a point O, Figure 2. Likewise with the edges AI, HC', BL, GA', CF, EB', which are equal and parallel

to DO. The whole rhombus-dodecahedron appears as a regular hexagon with its radial lines, as represented by the internal hexagon around O in Figure 2. Without entering into further details, this shows that the rhombus-dodecahedron may be slipped into the space of a regular hexagonal prism. We can now add six other equal rhombus-dodecahedrons 1, 2, 3, 4, 5, 6, as shown in the figure, which fit in

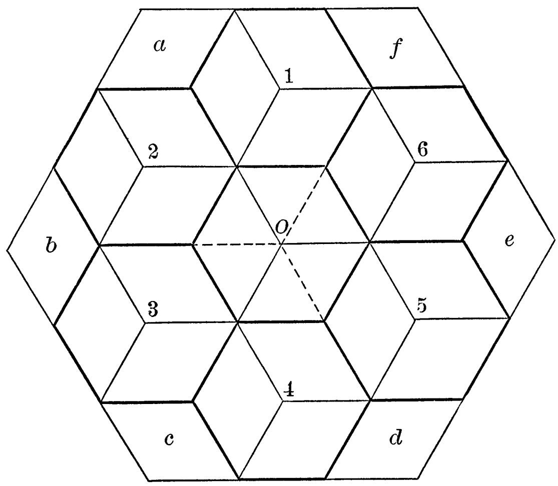

FIG. 2.

closely around O. Other dodecahedrons of the same kind may be fitted in horizontally and vertically, so as to pack uniformly a certain portion of space, which may be made as large as we please. If as shown in Figure 2, we fit in rhombohedrons a, b, c, d, e, f of the kind of which the dodecahedron is composed, *a regular hexagonal prismatic space may be packed uniformly by rhombus-dodecahedrons and rhombohedrons, or by rhombohedrons of the special form alone.*

As a last example of cellular division of space the well-known case of the honey-bees' cell may be mentioned. Geometrically the cell is obtained as follows: Take three faces of a cube adjacent at one of its vertices. Through the six "open" edges of the three faces pass planes parallel to the principal diagonal of the cube passing through the common vertex of the three faces. These six planes together with the three square faces of the cube enclose essentially the form of the bees' cell.

Further references on this extensive subject may be found in the treatises mentioned above.

UNDERGRADUATE MATHEMATICS CLUBS

Edited by R. C. Archibald, Brown University, Providence, R. I.

CLUB ACTIVITIES.

The Mathematics Club of the University of Maine, Orono, Me.

Scientific organizations at the University of Maine have gradually tended to become more special in their nature. The Maine State College, and later University of Maine, Scientific Association was probably organized in the seventies and flourished until some time in the nineties. In a sense The Mathematics and Physics Club was a successor to this organization and lasted for several years before disintegration by reason of the fact that "many of the men formerly interested in it were engineers, and several engineering clubs and societies were formed which occupied the attention of such men. The activities of this club were in part transferred for a time to voluntary meetings of the members of the mathematics department with students in that department, and of the physics faculty with major students in physics." But in February, 1916, it was felt that there was sufficient interest to maintain a separate organization and the new club was founded.

Its object is "to promote interest in the study of mathematics" and its members "consist of all persons connected with the departments of mathematics and astronomy and of any other persons sufficiently interested in mathematics to make application for a membership and who shall be elected by written ballot by a majority of the members present at any regular meeting." Average attendance this year about 8 of the 18 members.

Officers, 1917–18: President, Edith E. De Beck '18; vice-president, Samuel Guptill '20; secretary, Mildred Bisbee '20. The program committee consists of Professor M. O. Tripp, chairman, and the officers.

The following meetings have been held in 1917–18:

November 7: Business, including election of officers to fill vacancies caused by war conditions.

November 14: "Duplication of the Cube" by Samuel Guptill '20.

December 5: "Comparison of the Different Methods Used in the Duplication of the Cube" by Maynard Jordan and Quentin Stauffer, instructors in mathematics.

December 19: "History of Trigonometry" by Edith Deering '21.

January 10: "Mathematics for Engineering Students" by Professor Tripp; "History and Teaching of Mathematics in the United States" by Mildred Bisbee '20.

The Mathematics Club of Northwestern University, Evanston, Ill.

This club was organized in January, 1916, with a membership of 30. Membership was restricted to: (1) Members of the faculty, and graduate students,

in the departments of mathematics, physics, engineering and astronomy; (2) Seniors who are taking their major work in mathematics; (3) Students who have creditably completed 18 semester hours of work in the department of mathematics.

At a recent meeting, an amendment to the constitution was passed, extending membership to Juniors who are taking one or more courses as advanced as "C" in the department of mathematics. Representative courses of "C" grade are: higher algebra, solid analytic geometry, and advanced calculus.

Last year the Mathematics Club contained about 40 active members. At present (February 1) the active membership is about 20; but it is expected that the extension of membership privileges to Juniors will considerably increase the number.

Under the membership requirements the officers of the club are necessarily seniors or graduate students. Officers, 1917–18: President, Theodore Doll '17; vice-president, Franklin Mohr '18; secretary, Mae Campbell '18; treasurer, Helen Maloney '18; faculty adviser, Professor Robert E. Wilson. The program committee is composed of the faculty adviser, president, vice-president, and two members chosen by these three. The other two members are: Anastasia Zachariou '18 and Elizabeth Sheldon '18.

Meetings of the club are now held once in three weeks. Last year they were held once in two weeks. The lectures are about evenly divided between students and faculty. The programs for 1916 were as follows:

February 16: "The Construction of a Conic, given five Elements two of which are Imaginary" by Professor Thomas F. Holgate.

March 1: "What is Mathematics?" by Theodore Doll '17.

March 16: "Computing Machines" by Henry A. Babcock Gr., Fellow in physics.

March 29: "The History of π" by Minna Schick '16.

April 12: "The Method of Least Squares" by Harvey A. Anderson Gr.

April 26: "The Algebra of Al-Khowarizmi" by Ruby Lane '16; "The Hindu Arabic Numerals" by Harriet Knudsen '16.

May 10: "Descartes's Rule of Signs" by Professor David R. Curtiss.

May 24: "The Poincaré World" by Elizabeth Soderholm Gr.

October 12: "The Determination of Stellar Parallax" by Professor Philip Fox, director of Dearborn Observatory.

October 26: "Poincaré's Science and Hypothesis" by Dorothy Andrews '17.

November 9: "Arithmetical Stunts" by Leonard Janes '17.

November 23: "Euler" by Mabel Dinsmore '17; "Laplace" by Laura Hill '17.

December 7: "Railroad Curves" by Professor William H. Burger.

The following programs were given in 1917 and in the early part of 1918:

January 11: "Moritz's Memorabilia Mathematica" by Dorothy Waugh '17, Ruth Austin '17 and Robert Smyth Gr.

January 25: "Mathematical Journals" by Professor Curtiss.

February 15: "Galileo" by Henry Crew, professor of physics.

March 1: "Paper Folding" by Olga Podlesak '17.

March 15: "Circular Diagrams of Stress and Strain, and the Ellipse of Elasticity" by Hugh P. Bersie '17.

March 29: "Triangles and Squares" by Frank E. Wood, instructor in mathematics.

April 12: "Newton as Astronomer" by Luella Sayer Gr.; "Newton the Mathematician" by Helen Forbes '17; "Newton-The Leibnitz Controversy" by Ella Schneider '17; "Newton the Physicist" by William H. Grau, assistant in physics.

April 26: "History of Trigonometry" by Mary K. Voorhees '17; "Fourth Dimensional Geometry" by Frederick L. Kerr, Jr.

May 10: "Line Coordinates" by Professor Chester H. Yeaton.

May 24: "Some Generalizations of Rolle's Theorem" by Minna Schick, Jr.

November 15: "The real Roots of a Cubic Equation" by Professor Wilson.

December 5: "The Trigonometric Solution of the Cubic" by Anastasia Zachariou '18.

January 10: "The Problem of Two Bodies" by Franklin Mohr '18.

January 24: "The Mathematics of Field Artillery" by Professor E. J. Moulton.

February 14: "The Use of Complex Numbers in Electrical Theory" by William C. Bauer, professor of electrical engineering.

A service flag for the Mathematics Club would contain six stars. For all the information given above concerning the Club the editor is greatly indebted to President Doll who expects to be called to the colors before June.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, Eugene, Oregon.

This club was organized in October, 1916, "to promote a broad interest in the whole field of mathematics, especially in phases of it that are not dealt with in the regular courses; also to touch upon interesting and useful applications, the human element and the recreational." All students in the department of mathematics and allied departments are eligible for membership. There are 30 members; average attendence about 20.

Officers, 1917–18: President, Olga Soderstrom '18; vice-president, Ruth Westfall '18; secretary, Helen Wells '18; treasurer, Kenneth Moores '18. The program for each meeting is prepared by the officers assisted by the instructors in the department.

The program for the year 1916–17 follows:

October 6: "Early Reminiscences of the University of Oregon" by Edward H. McAlister, professor of mechanics and astronomy; "The History of Johns Hopkins University" by Roy M. Winger, professor of mathematics.

November 1: "The Fallacies of Arithmetic" by Lucile Watson '17; "The Construction, Theory, and Purpose of the Magic Square" by Loren Roberts '17.

December 6: "The Construction of Polygons by the Use of Ruler and Compasses" by Kenneth Moores '18; "The Problem of Squaring the Circle" by Ruth A. Westfall '18.

March 7: "Reminiscences of the Universities of Wisconsin, Chicago, and Yale" by Edgar E. De Cou, head of the department of mathematics; "Mathematical Recreations, Puzzles and Catch-problems" by Olga Soderstrom '18.

May 23: "The History of Correspondence Schools and the Progress of our University in this Regard" by Chester Kronenberg, instructor in mathematics; "The Application of Mathematics to Wave Surfaces" by Frederick E. Melzer '17.

The programs for the first two meetings of 1917–18 are as follows:

December 13: "The Problem of the Duplication of the Cube" by Elizabeth Carson '18; "Mathematics and the War" by Professor De Cou.

January 16: "The Problem of the Trisection of an Angle" by Cornelia Heess '18; "The Newton-Leibnitz Controversy over Calculus" by Edward E. Bentley '20.

THE MATHEMATICAL AND ASTRONOMICAL CLUB OF SWARTHMORE COLLEGE, Swarthmore, Pa.

This club was founded, in March, 1907, in order "to give students and faculty an opportunity to report upon subjects not usually treated in the classroom. It affords also an opportunity to bring to the students of mathematics from those outside the college two or more addresses which are given each year before the club; but the chief object is to develop in students the responsibility for certain exercises." Anyone desiring to become a member may do so. There are 42 members this year and the average attendance is about 30. No regular club dues are levied, but each year an assessment is made for current expenses; it never exceeds a dollar and is usually less than half that amount.

The following programs were given in the early part of 1917–18:

October 2: "Properties of Sound" by Harvey C. Hayes, professor of physics.

October 16: "Popular Phases of the Fourth Dimension" by Ethelwyn Bower '18.

November 6: "Determination of Latitude and Longitude by Means of Celestial Objects" by John Trimmer '18; "The Use of the Sextant" by Harry Yardley '19; "The Log Book of a Ship" by Ewing Carson '18.

November 20: "Opportunities open to Students of Mathematics" by John A. Miller, head of the department of mathematics and astronomy; "Plain Sailing," by Frank Fetter '20; "Mercator's Projection" by Robert Blau '18.

December 4: "Algol and its Use in the Determination of Light Wave-Lengths" by Dorothy Johnson '18; "History of the Parallax Problem" by Caroline Smedley Gr.; "Use of the Telescope and Galvanometer" by Hazel Brown Gr.; "Researches of Sproul Observatory" by Professors Miller, John H. Pitman (mathematics and astronomy), and Hayes.

December 18: "The Depression Range Finder and its Use in the Present War" by John Trimmer '18.

THE MATHEMATICS CLUB OF VASSAR COLLEGE, Poughkeepsie, N. Y.

This club was organized, in January, 1916, "to bring together students interested in mathematics, and to give them an opportunity to present papers themselves and to hear and discuss those presented by their fellow students. All students who have completed two semestral courses of elective[1] mathematics, and members of the faculty in the mathematics department are eligible for active membership. Students who have taken or are taking one course in elective mathematics, and members of the faculty in other than the mathematics department are eligible for associate membership." The club has 37 active members practically all of whom, and others, are present at each of the monthly meetings.

Officers, 1917–18: President, Beatrice Boyden '18; vice-president, Rachel G. Franklin '19; secretary-treasurer, Louise E. C. Stuerm '19; faculty member of executive committee, Professor Elizabeth B. Cowley; member at large on executive committee, Ruth De Land '18.

The following are programs for 1917–18:

October 30: "The Parallel Curves of the Conics" by Dr. Mary E. Wells.

November 13: "The Mathematics of Warfare" by J. Malcolm Bird, associate editor of *The Scientific American*.[2]

December 13: "The Slide Rule" by Helen Thompson '18; "Magic Squares" by Louise E. C. Stuerm '18.

January: "Planimeters" by Marjorie Wheatley '18; this paper was illustrated by planimeters made by the speaker's father, who is an engineer.

TOPICS FOR CLUB PROGRAMS.

6. WOMEN AS MATHEMATICIANS AND ASTRONOMERS.

The first woman mathematician regarding whom we have positive knowledge is the celebrated Hypatia,[3] recognized head of the Neoplatonic school, who was barbarously murdered at the hands of an Alexandrian mob in the early part of the fifth century. She was the daughter of Theon, also a mathematician and philosopher, who edited Euclid's *Elements*.[4] "Her great eloquence and rare modesty and beauty combined with her remarkable intellectual gifts attracted to her class-room a large number of pupils." She is said to have been the author of commentaries on works by Diophantus, Apollonius of Perga, and Ptolemy.

Maria Gaetana Agnesi, an Italian mathematician, linguist and philosopher of the eighteenth century was one of the twenty-three children of a rich and

[1] Vassar College requires a year of mathematics for freshmen.

[2] Mr. Bird's lecture was open to all members of the College and was attended by about 500 people, including practically all of the students of mathematics, and others. In *The Mathematics Teacher*, September, 1917, vol. 10, pp. 35–51, appeared an article on "The Mathematics of Warfare" by Mr. Byrd.

[3] See *Encyclopædia Britannica*, eleventh edition, 1910, article "Hypatia" with bibliography. Kingsley's romance will be recalled.

[4] *Cf.* HEATH, *The Thirteen Books of Euclid's Elements*, vol. 1, 1908, Chapter 5.

cultured gentleman of Milan who had a taste for science. "When only nine years old she had such command of Latin as to be able to publish an elaborate address in that language, maintaining that the pursuit of liberal studies was not improper for her sex. By the thirteenth year she had acquired Greek, Hebrew, French, Spanish, German and other languages. Two years later her father began to assemble in his house at stated intervals a circle of the most learned men in Bologna,[1] before whom she read and maintained a series of theses on the most abstruse philosophical questions" (*Encyclopædia Britannica*). Her father caused records of these meetings to be published when she was twenty years old. Then followed ten years of arduous labor, entered upon for the benefit of one of her brothers who had a taste for mathematics, and resulting in the publication of *Le Instituzioni Analitiche* (two quarto volumes, Milano, 1748). It is here (Vol. 1, pp. 380–81) that we find a discussion of the curve commonly known now as the "witch of Agnesi." [2]—French and English translations of this work were published in 1775 and 1801[3] respectively. In 1750 she was appointed professor of higher mathematics and natural philosophy[4] at the University of Bologna but never lectured. For the rest of her long life—she lived to be 81 years of age—she was occupied in charitable work and religious meditations. A younger sister was a well-known pianist and composer of three operas.

Among French representatives of women as mathematicians Sophie Germain (1776–1831) is the most notable. She was a pupil of Lagrange and a coworker with Biot, Legendre, Poisson and Lagrange. In 1816 the Institute of France awarded a prize for her memoir on the mathematical theory of the vibrations of elastic surfaces and its comparison with experience. This and other memoirs have been reviewed by Todhunter.[5] Her fame brought her into contact with such men as Ampère, Cauchy and Gauss.[6] Terquem has written an interesting biographical sketch.[7]

Russia produced Sonja Kovalevski (1850–1891) who studied mathematics for two years at the university of Heidelberg (under H. v. Helmholtz, Kirchhoff, Königsberger and du Bois-Reymond) and then proceeded to Berlin, where she

[1] Milan? Compare note 4.

[2] See F. G. Teixeira, *Traité des courbes spéciales remarquables planes et gauches*, Coïmbre, 1908-9, Vol. 1, pp. 108–115, 385–6; Vol. 2, pp. 405–6.

[3] The review of this edition in the *Edinburgh Review* for January, 1804, is interesting.

[4] There is no basis in fact for the statements of the *Encyclopædia Britannica* to the effect that (1) Agnesi's father was professor in the University of Bologna or (2) her appointment as professor had anything to do with his illness. *Cf.* the best biography, L. Anzoletti, *Maria Gaetana Agnesi*, Milano, 1900, pp. 273ff. There is a sketch in *The Catholic Cyclopedia*, New York, Appleton, Vol. 1, 1907.

[5] *History of the Theory of Elasticity and of the Strength of Materials*, Vol. 1, Cambridge, 1886, pp. 147–160; see also Chasles, *Rapport sur les progrès de la géométrie*, Paris, 1870, pp. 35–36; and H. Stupuy, *Œuvres philosophiques de Sophie Germain, suivies de pensées et de lettres inédites et précédées d'une étude sur sa vie et ses Œuvres*, Paris, Quantin, 1879; another edition, 1896.

[6] *Cf.* Boncompagni, *Lettera inedita di Carolo Fredrico Gauss a Sofia Germain*, Firenze, 1871; reviewed by C. Henry in *Bulletin des sciences mathématiques* (Darboux), Paris, tome 14, 1879, pp. 401–406.

[7] *Bulletin de bibliographie, d'histoire et de biographie mathématiques*, tome 6 (1860), pp. 9–12 (*Nouvelles Annales de mathématiques*, tome 19).

prevailed upon Weierstrass to give her private lessons, since the doors of the university were closed to women. He soon found that she had "the gift of intuitive genius to a degree he had seldom found among men, even his older and more developed students." At the end of three years she presented to the university of Göttingen three theses worked out under the direction of Weierstrass through whose strong recommendation she was awarded the degree of doctor of philosophy without examination, and in absentia. Not long after this (1874) she returned to Russia. In 1883 she was appointed lecturer in mathematics at the university of Stockholm, and in 1884 full professor. This post she held till her death. Five of her memoirs were published in *Acta Mathematica* and the most notable is that of 1889, "Sur le problème de la rotation d'un corps solide autour d'un point fixe," which, in 1888, won the prix Bordin (doubled in value because of the remarkable merit of the memoir) of the Institute of France. Sonja Kovalevski "died just as she had attained the height of her fame and had won recognition even in her own country by election to membership of the St. Petersburg Academy of Science." [1]

Of scientists in England a few notes may be given concerning the astronomer Caroline Herschel, who was born at Hanover, in 1750 and started her career in England, with her brother William, as teacher of music in Bath. When her brother accepted the office of astronomer to George III, she became his constant assistant in his observations and also executed laborious calculations which were connected with them. She discovered comets and remarkable nebulæ and completed the reduction to January, 1800, of 2,500 nebulæ discovered by her brother. The Royal Astronomical Society awarded her its gold medal and elected her an honorary member. She died at the age of 98, having spent about fifty years of her chief scientific activity in England.[2]

Perhaps the best single work containing biographies of scores of women mathematicians and astronomers is the second edition of A. REBIÈRE, *Les femmes dans la science*.[3] The bibliographies are full and valuable. Among portraits, it contains those of Agnesi, Sophie Germain, Caroline Herschel, Kovalevski, Agnes Mary Clarke, the English author of various works on the history of astronomy, Marquise du Châtelet[4] (pupil of Clairaut, Maupertius and Jean Bernoulli, and translator of Newton's *Principia*), Martha Mitchell, the astron-

[1] For sketches of Kovalevski's life reference may be given to A. C. LEFFLER, *Sonya Kovalevsky*, biography and autobiography, translated into English, London (1) Walter Scott, 1895, (2) T. Fisher Unwin, 1895; to the article in the eleventh edition of the *Encyclopædia Britannica*, 1911; to G. MITTAG-LEFFLER, "Sophie Kovalevsky, notice biographique," *Acta Mathematica*, Vol. 16 (portrait frontispiece), pp. 385–392, 1893; and to E. W. CARTER, "Sophie Kovalevsky," *Fortnightly Review*, Vol. 63, 1895, pp. 767–783. For other references see *Bibliotheca Mathematica*, 3. Reihe, Band 2, 1901, p. 337.

[2] For detailed biographical material see Mrs. John Herschel's *Memoir and Correspondence of Caroline Herschel*, 2d edition, London, Murray, 1879; *Dictionary of National Biography*, London, Vol. 26, 1891; and the Introduction to Volume I of *The Scientific Papers of Sir William Herschel*, London, Royal Society and Royal Astronomical Society, 1912.

[3] Paris, Novy, 1897, 361 pp.

[4] For other portraits see Carlyle's Works, centenary edition, New York, Scribner, 1908, Vol. 14, p. 191; and G. C. WILLIAMSON, *History of Portrait Miniatures*, London, Bell, 1904, Vol. 2, plate 87.

omer of Vassar, Charlotte A. Scott, the mathematician at Bryn Mawr, Christine Ladd-Franklin of New York, and Dorothy Klumpe, a native of San Francisco, who is the only American who ever received the state doctorate in mathematical sciences in France.[1]

A more recent work by J. A. Zahm was published under the pseudonym H. J. Mozans with the title *Woman in Science with an introductory Chapter on Woman's long Struggle for Things of the Mind.*[2] While the book presents material of value for our topic it contains also many unreliable statements.

The highly interesting article by Gino Loria on "Les femmes mathématiciennes"[3] does not enter so much into details of the lives of the different women as into the consideration of the value of their contributions to science[4] in comparison with those made by many men. On the basis of available facts he finds himself obliged "with regret to make reservations in regard to those whom nature seems to have called to other destinies." Much the same conclusion is reached by the doctor, P. J. Möbius (grandson of A. J. Moebius) in his valuable *Ueber die Anlage zur Mathematik.*[5]

7. The Binary Scale of Notation, a Russian Peasant Method of Multiplication, the Game of Nim and Cardan's Rings.

The binary system is built upon a scale of 2 instead of 10, thus using only two figures, 1 and 0. The succession of integers represented in the decimal notation by

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cdots,$$

would appear in the binary system as follows:

$$1 \quad 10 \quad 11 \quad 100 \quad 101 \quad 110 \quad 111 \quad 1000 \quad 1001 \quad 1010 \quad \cdots.$$

By this means all addition is reduced to counting, all multiplication to simple additions, and division to simple subtractions. One inconvenience of the binary system is the large number of figures which may be required for a relatively small number. The binary representation requires for large numbers approximately 10/3 of the number of figures in the decimal representation. Legendre used binary numeration for calculating high powers and gave the following note[6] on a "very short method for expressing a somewhat large number in binary scale. Let the number be 11,183,445; . . . divide by 64, then the remainder is 21 and the quotient 174,741, which, divided by 64, gives the remainder 21 and

[1] For a time, at least, Miss Klumpe was a member of the staff of the Paris Observatory.

[2] New York, Appleton, 1913. Chapter 3, pp. 136–166: "Women in Mathematics," chapter 4, pp. 167–196: "Women in Astronomy."

[3] *Revue scientifique,* Paris, Dumoulin, 1903, 4e sèrie, tome 20, pp. 386–392. See also G. Loria, "Donne matematiche," *Memoria letta il* 28 *dicembre,* 1901, *Atti e memorie della R. Accademia Virgiliana,* Mantova, 1903, pp. 75–98.

[4] In Note 1 of his Work ("Si la femme est capable de science," pages 287–319) Rebière considers this question. He quotes those holding opposing views, such as Molière, Prudhom, Schopenhauer, and Lalande, Stuart Mill, Anatole France.

[5] Leipzig, Barth, 1900. "Beiträge zur Kenntniss des Mathematischen Talentes. C. Ueber die mathematischen Weiber," pp. 77–86; portraits of Germain, Herschel and Kovalevski.

[6] A. M. Legendre, *Essai sur la thèorie des nombres,* Paris, 1798, p. 229; 2e édition, 1808, p. 209.

the quotient 2730; finally 2730 divided by 64 gives the remainder 42 and the quotient 42; but 21 is expressed in binary numbers by 10101 and 42 by 101010. Then the proposed number is expressed by 101010 101010 010101 010101."

Fractions, decimals, subtraction, division, etc., in the binary system have been discussed by E. Collignon.[1] Reference may be given also to articles by C. Berdellé,[2] to the sketch by É. Lucas in *Récréations mathématiques*,[3] to an extract from the first lesson of Laplace at the *École Normale*, "le 1er pluviôse an III" [1795],[4] and to a note by E. Miot.[5]

Leibniz is often referred to as the founder of binary arithmetic. Whether this be true or not it is certain that the subject interested him considerably and that his publications[6] influenced, directly or indirectly, the production of a series of works such as those by T. F. De Lagny,[7] J. B. Wiedeburg,[8] F. S. Brunetti,[9] and G. F. Brander.[10] As early as 1698 Leibniz set forth the foundations of his binary arithmetic in a letter to Schulenburg and the matter occupied his attention for twenty years. His first published paper on the subject was in 1703. As its title indicates Leibniz believed that this arithmetic solved the mystery of a Chinese symbol Je-Kim (book of mutations) attributed to Fo-hy, the most ancient legislator in China. The symbol is composed of 64 figures each formed of six horizontal lines (some whole, others broken in the middle) one above the other.[11] Interpreting each whole line as corresponding to 1 and each broken line to 0, Leibniz translated the figures as corresponding to 0, 1, 2, $\cdots$, 63. But such an interpretation has been shown to be worthless.[12] In honor of Leibniz's discoveries in connection with binary arithmetic a medal was struck by Rudolph August,

[1] "Note sur l'arithmétique binaire," *Journal de mathématiques élémentaires* (De Longchamps), 1897, tome 21, pp. 101–106, 126–131, 148–151, 171–174.

[2] *Association française pour l'avancement des sciences:* (*a*) "La numération binaire et la numération octovale," 1887; (*b*) "Arithmétique de la gamme," 1897; (*c*) "Les curiosités du calcul," 1898. Also *L'Intermédiaire des mathématiciens*, 1904, tome 11, pp. 269–271.

[3] 2e édition, Paris, Gauthier-Villars, 1891, pp. 143–160.

[4] *L'Education mathématique*, 7e année, 1904, pp. 59–60.

[5] *L'Intermédiaire des mathématiciens*, 1911, tome 18, p. 190.

[6] (*a*) "Explication de l'arithmétique binaire, qui se sert des seuls caractères 0 et 1: avec des remarques sur son utilité et sur ce qu'elle donne le sens des anciennes figures Chinoises de Fohy," *Histoire de l'académie des sciences de Paris*, 1703, pp. 58–63; *Mémoires*, pp. 85–89. Also, *Opera omnia*, Genevæ, vol. 3, 1768, pp. 390–394. (*b*) "De inventione arithmeticæ binariæ a G. G. Leibnitio. Excerpta ex vitâ Leibnitii a D. Jancourt scripta," *Opera omnia*, vol. 3, pp. 346–348. (*c*) "G. G. Leibnitii epistolæ duæ ad J. C. Schulenburgium, De Arithmeticâ Dyadicâ . . .," *Opera omnia*, vol. 3, pp. 349–354.

[7] *L'arithmétique nouvelle*, Rochefort, 1703. (Hutton states that De Lagny proposed a new system of logarithms, on the plan of binary arithmetic, which he found shorter and more easy and natural than the common ones.)

[8] *Dissertatio mathematica de praestantia arithmeticæ binariæ prædecimali* . . ., Jenae, 1718.

[9] *Arithmetica binomica e diadica, in cui tutte le operazioni si fanno colle sole figure Uno, e Zero* (Riflessione al Giornale de'Letterati di Roma an 1746, page 884). Roma, 1746; another edition, 1754.

[10] *Arithmetica binaria sive dyadica, das ist, Die Kunst nur mit zwey Zahlen in allen vorkommenden Fällen sicher und leicht zu rechnen*, Augsburg, 1769; another edition, 1775.

[11] *Cf.* P. Carus, "Chinese Philosophy," *The Monist*, Chicago, 1896, vol. 6, especially pages 191 ff.

[12] *Annales du musée guimet*, tome 8 (1885) and 23 (1893).

duke of Braunschweig. A copy of this medal forms the frontispiece to Wiedeburg's dissertation and is also given on p. 1002 of *Collectanea de Breslau.*

The problem of calculating π in the binary system, proposed several times by Leibniz (*Opera omnia,* tome 3, pp. 521, 547···), was solved by Jacob Bernoulli.[1] For log π in binary system see *Journal de Mathématiques élémentaires* (Bourget), Paris, tome 2, 1878, p. 229.

Peano has indicated applications of the binary system in "La numerazione binaria applicata alla stenografia," *Atti della R. Accademia delle Scienze di Torino,* 1898; (*b*) *Formulaire de Mathématiques,* 1901, pp. 77, 154, 177.

The source of the many references to a Russian peasant method of multiplication seems to have been the publication in 1896 by G. De Longchamps of a request from a Mr. Plackowo (of Tokarewka in the government of Tamboff, Russia) for an explanation of the method. An example will suffice to illustrate both the method and the connection with the binary system of numeration. Suppose, for example, the problem is to multiply 35 by 42. Divide one of the numbers (say 35) by 2, giving 17 (remainder neglected) and multiply 42 by 2; continuing the process we get the following series of corresponding numbers:

35	17	8	4	2	1
42	84	168	336	672	1344.

Add the numbers of the second row corresponding to odd numbers in the first row and we obtain the required product 1470. In *Mathematical Gazette,* 1917, Vol. 9, p. 9, G. H. Bryan has a note on "'Russian Peasant' Multiplication in Roman Numerals." References may be given also to: J. Bowen, "The Russian Peasant Method of Multiplication," *The Mathematics Teacher,* 1912, Vol. 5, pp. 4–8; E. Czuber, "Ueber ein Multiplikationsverfahren," *Zeitschrift für das Realschulwesen,* Dezember, 1915.

The application of the binary scale of notation to "Nim, a Game with a complete Mathematical Theory" was explained by C. L. Bouton in *Annals of Mathematics,* 1901, 2d series, Vol. 3, pp. 35–39. To him is due also the name which was chosen in preference to Fan-tan, which had been used, because it is not the same as the Chinese game of that name. "The game is played by two players. Upon the table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no piles shall be equal at the beginning. A play is made as follows: The player selects one of the piles, and from it takes as many counters as he chooses; one, two, . . ., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins." See also E. H. Moore, "A Generalization of the Game called Nim," *Annals of Mathematics,* 1910, 2d series, Vol. 11, pp. 93–4.

The toy usually known as "Chinese Rings"[2] seems to have been first de-

[1] *Cf.* Leibniz, *Math. Schriften* (Gehrhardt), vol. 3, p. 97.

[2] In French: baguenaudier.

scribed by CARDAN on page 294 of his *De Subtilitate*, Nuremberg, 1550. It is also discussed by John Wallis in the Latin edition of his *Algebra*, 1693 (*Opera*, Vol. 2, pp. 472–478). The toy "consists of a number of rings hung upon a bar in such a manner that the ring at one end (say A) can be taken off or put on the bar at pleasure; but any other ring can be taken off or put on only when the one next to it towards A is on, and all the rest towards A are off the bar. The order of the rings cannot be changed." A pretty full discussion of the number of operations required to remove the rings from the bar (a problem which Cardan and Wallis failed to solve) has been given in W. W. R. BALL, *Mathematical Recreations and Essays*, 5th ed., London, Macmillan, 1911, pp. 230–234. Practically all that is in Ball's sketch, and considerable more, is given in E. LUCAS, *Récréations Mathématiques*, Paris, Gauthier-Villars, tome 1, 2e éd., 1891, pp. 161–186. The discussion of the number of operations by means of the binary scale is due to [L. GROS], *Théorie du Baguenodier*, par un clerc de notaire lyonnais, Lyon, 1872.

NOTES AND NEWS.

Professor E. W. BROWN, of Yale University, was elected a member of the Council of the American Philosophical Society at the January meeting.

The Paris Academy of Science during the year 1917 awarded seventy-seven prizes for especial achievements in science, eight of which were granted to mathematicians.

The Royal Society held its regular meeting on December 6, 1917, under the presidency of Sir J. J. THOMPSON. The only mathematical papers presented were "The series of Legendre," by W. H. YOUNG, and "The Zeros of Bessel functions," by G. N. WATSON. The London Mathematical Society held its monthly meeting on the same date, with vice-president H. HILTON presiding. The following papers were presented: "A new method of describing three-bar curves," by R. L. HIPPISLEY; "Proof of the primality of $N = (10^{19} - 1)/9$," by O. HOPPE; "New Tauberian theorems," by HARDY and LITTLEWOOD; "The curves which lie on the quartic surface in space of four dimensions," by C. V. H. RAO; "The connection between Legendre series and Fourier series," and "Series of Bessel functions," by W. H. YOUNG.

The third volume of the *Proceedings* of the National Academy of Sciences of the United States of America has been completed. The membership of the Academy now numbers one hundred fifty-eight, distributed somewhat irregularly over the United States, and representing the fields of mathematical, physical and biological sciences. Twelve members of the Academy are now, or have been, professors of mathematics. The table of contents of papers presented shows thirteen titles belonging to the field of mathematics, only two of which were credited to members of the Academy. The roster also shows twenty-six foreign associate members, four of whom are noted mathematicians, two belonging to the central, and two to the allied powers.

The Department of Superintendence of the National Educational Association met in Atlantic City on February 26–March 2, 1918. The program was rich in variety and extent, especially in respect to all phases of activity which in any way pertained to the world war. One feature of the program may well be commended to the mathematical fraternity, namely, the two sessions devoted to the teaching of English under the auspices of the National Council of Teachers of English. It has been suggested that our Association could do good service by providing speakers for the program of educational meetings both state and national, and it would seem that this Department of Superintendence offers a most favorable opportunity for propaganda of this kind.

The series of articles on "Valid aims and purpose for the study of mathematics in secondary schools" which is now running in *School Science and Mathematics* is commended to the attention of teachers of mathematics of all grades. These articles are the outcome of the work of a committee of the Mathematics Club of Chicago. Reprints of the whole series may be obtained from the chairman of the committee, Mr. ALFRED DAVIS, of the Francis W. Parker School, 330 Webster Avenue, Chicago.

Two articles bearing on the attacks upon mathematics are printed in *School Science and Mathematics* for January and February, 1918. They are by Professor J. W. A. YOUNG, of the University of Chicago, and relate to the psychological investigations in the disciplinary value of studies. The first article gives a summary of the work which has been done by psychologists along this line with a bibliography and definite references; the second gives some of the author's own theory on the subject, especially as related to mathematics. The psychological discussion of the value of studies is also presented in an interesting address by Professor E. C. MOORE before the Association of Mathematics Teachers of New England, and published in full in *School and Society,* October 27, 1917.

Ten men connected with the department of mathematics at the University of Illinois have resigned to enter the service of the government since the United States entered the war. The following five resignations have not been noted in the MONTHLY: Dr. L. M. KELLS, instructor in mathematics, has entered the Officers' Reserve Training Camp at Battle Creek, Michigan; Dr. J. R. MUSSELMAN, instructor in mathematics, has begun statistical work for the government at Washington; Assistant H. D. FRARY has been appointed director of the wood-testing plant for aëroplanes, at the University of Wisconsin; Assistant W. E. EDINGTON has accepted a position in the research division of the Signal Service and is located at Leavenworth, Kansas; and Assistant A. W. LARSEN has resigned to accept an instructorship in mathematics at the University of Kansas. Two of these vacancies have been filled by the appointment of Dr. J. E. MCATEE, of William Jewell College, as instructor in mathematics, and the appointment of Mr. L. L. STEIMLEY, former instructor at the University of Kansas, as assistant in mathematics.

If departments of mathematics of our colleges and universities, offering summer courses in mathematics, will report to the editor of Notes and News an outline of these courses at once, we shall undertake to insert the announcements in the May issue of the MONTHLY. The outline should be prepared in the form used in the MONTHLY for April, 1917. Last year a fairly complete synopsis of summer work offered at the various institutions was secured by sending personal letters to each institution; no such letters will be sent this year, but it is hoped that all departments offering summer work will send promptly the synopsis for insertion in the MONTHLY.

The University of California will this year conduct two summer sessions, June 24 to August 3, one at the University of Berkeley, the other in Los Angeles. The department of mathematics will offer at Berkeley courses in elementary mathematics, advanced algebra, calculus, differential equations, theory of numbers, theory of functions, and integral equations. The instructors will be Professor D. N. LEHMER, Instructor BERNSTEIN and two assistants, and Dr. G. C. EVANS, of Rice Institute. At Los Angeles Professor C. A. NOBLE, and Professor V. SNYDER, of Cornell University, will offer courses in elementary algebra, plane analytic geometry, calculus, and higher geometry.

The April meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, April 12, 13. Besides the usual research papers on both days, there will be a Symposium on Summable Series on Friday afternoon. Professor R. D. CARMICHAEL, of the University of Illinois, will present a paper on "General aspects of the theory of summable series," and Professor C. N. MOORE, of the University of Cincinnati, will treat "Applications of the theory of summability to developments in orthogonal functions."

At the coming summer meeting of the ASSOCIATION a session is to be arranged for papers to be presented by members. The Program Committee requests that members who are ready to discuss questions of interest in connection with collegiate instruction of mathematics send abstracts of their proposed papers to the chairman, Professor R. C. ARCHIBALD, Brown University, as soon as possible, and not later than May 15. Such papers should, in general, be of such length as may be delivered in fifteen to twenty minutes. It is hoped there will be a large response to this appeal, in the interests of a live and varied program.

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To solve $\frac{\partial^2\varphi}{\partial t^2} = \sqrt{1+(\Delta h)^2}\,\frac{\partial^2\varphi}{\partial x^2}$, put $m^2 = \sqrt{1+(\Delta h)^2}$ and assume $\varphi = \tau(t)\cdot\xi(x)$; so that $\frac{\partial^2\varphi}{\partial t^2} = \frac{d^2\tau}{dt^2}\cdot\xi(x)$, and $\frac{\partial^2\varphi}{\partial x^2} = \tau(t)\,\frac{d^2\xi}{dx^2}$. Substituting these into the original equation, we find that the variables, t and x, can be separated by dividing through by $\tau\cdot\xi$ whereupon we have $\frac{d^2\tau}{dt^2}\div\tau = m^2\frac{d^2\xi}{dx^2}\div\xi$. Since the first of these two equal members cannot vary when t changes nor the second when x changes, both must remain equal to some constant, say $-m^2n^2$. The two resulting equations yield the solutions

$$\xi = K_1\cdot\sin[nx+\beta_1], \qquad \tau = K_2\cdot\sin[mnt+\beta_2]$$

whence $$\varphi = K_1K_2\mathrm{Sin}[nx+\beta_1]\,\sin[mnt+\beta_2]$$

which we may then reduce to a more useful form :

$$\varphi = \sum_{n=0}^{n=\infty} A_n\ \sin[n(x\pm mt)+\delta_n].$$

An interesting fallacy results from applying the method of integration by parts, $\int u\cdot dv = uv-\int v\cdot du$, to a case where $u=1/x$ and $dv=dx$: we get

$$\int\frac{dx}{x} = \frac{1}{x}\cdot\int dx - \int x\cdot[-1/x^2]$$

$$= \quad 1 \quad + \int dx/x \qquad \text{whence } 0=1 \ !!$$

$$\int\frac{dx}{5+7x^2} = 1/5\int\frac{dx}{1+{}^7/_5x^2} = \frac{1}{5}\sqrt{{}^5/_7}\int\frac{\sqrt{{}^7/_5}\,dx}{1+[\sqrt{{}^7/_5}\,x]^2}$$

$$= \frac{1}{35}\ \arctan\ [\sqrt{{}^7/_5}\ \ x].$$

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DESCRIPTIVE GEOMETRY AND ITS MERITS AS A COLLEGIATE AS WELL AS AN ENGINEERING SUBJECT.[1]

By WILLIAM H. ROEVER.

1. The Evolution of Descriptive Geometry. To the needs of the architect on the one hand and to those of the artist on the other, can be attributed the origin and much of the development of that branch of mathematics which is now called descriptive geometry. With the development of the builder's art, there must also have developed the process of making plans and elevations of buildings. In fact, the Roman architect Vitruvius used the terms "ichnographia" and "orthographia" for plan and elevation, respectively. In the attempts to solve the problems encountered in the construction of arches and other architectural forms, the art of stereotomy (stone and wood cutting) was gradually developed in the Middle Ages. To solve these problems the objects under consideration were represented by their plans and elevations, the usefulness of which was thus extended from that of mere representation to that also of the solution of stereometric problems. A remarkable step was taken in 1738 when Frézier separated the geometric constructions of this art from their technical applications and used them to solve problems of space. But it was Gaspard Monge (1746–1818) who, by enriching and systematizing this new constructive geometry of space, elevated it to the dignity of a pure science and gave to it the name descriptive geometry. Monge also solved with it the ordinary problems of geodesy and topography.

The needs of the artist, as has already been implied, were responsible for the early development of perspective. Exhumed mural paintings from ancient

[1] An address delivered at the third annual meeting of the Mathematical Association of America, held at the University of Chicago, December 27–28, 1917.

Rome revealed a knowledge of the vanishing point and of other notions concerning perspective. Though lost in the Middle Ages, when oblique parallel projection took its place, perspective was again discovered at the time of the Renaissance simultaneously in Italy, the Netherlands, and Germany. In the mathematical sense it was treated in the seventeenth and eighteenth centuries by men like Desargues, Taylor, Zanotti, and Lambert. In the nineteenth century its considerations brought forth the modern projective geometry at the hands of Poncelet, Möbius, Steiner, von Staudt, and others.[1]

Projective geometry made possible a deeper inquiry into the characteristics of projection than did the descriptive geometry of Monge, and hence the former science soon exerted a marked influence upon the latter. Some writers, especially in Germany, interwove projective with descriptive geometry. Besides contributing to the development of descriptive geometry through the modern projective geometry which it created, perspective may also be regarded as responsible for the development of parallel projection and axonometry. It might even be said that the oblique parallel projection of the Middle Ages was a pseudo-perspective used because of the lack of knowledge of true perspective. In the perspective coördinates of Desargues may already be gleaned notions of modern axonometry, while Lambert in his "Free Perspective" (1759) considered parallel projection as a limiting or special case. The nineteenth century witnessed many further contributions to oblique projection and axonometry at the hands of men like Möllinger, Weisbach, and Pohlke. To perspective also may be attributed the origin of photogrammetry, which has for its object the problem inverse to that of perspective, that is, the determination of the plan and elevation of an object from perspectives (*i. e.*, central projections) of that object.[2]

Of material aid in the representation of space objects by drawings in a plane are the representations of the shadows which they cast upon themselves and upon other objects, as well as the shading, *i. e.*, the reproduction of the intensity of illumination, or "brightness," of the different parts of the surface of the body represented. Thus the theories of shadows and illumination became a branch of descriptive geometry. For this and other reasons a better knowledge of the theory of surfaces was necessary. The nineteenth century witnessed in France the development of the theory of surfaces and their curvature, the theory of shadows, and, at the hands of Monge and his pupils, the theory of illumination. The latter theory was also developed in Germany and Italy.

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[1] A type of perspective for a special purpose is the stereographic projection, which was invented by Hipparchus about 140 B. C. for representing a spherical surface upon a plane, and is used to make celestial and terrestrial maps.

[2] A generalization of ordinary perspective is exhibited in what is called relief perspective, in which the half of space lying on one side of the picture plane is represented in the space between the picture plane and a plane parallel thereto. It is a remarkable fact that in 1447 when ordinary perspective was just beginning to be understood, there should have been produced such a perfect example of relief sculpture as that by Lorenzo Ghiberti in the doors of the main portal of the baptistry at Florence. This work, which Michael Angelo declared to be worthy of the Gates of Paradise, appears in general to have been constructed in accordance with the rules of relief perspective, which were not formulated until many years later.

2. The Position of Descriptive Geometry within the Realm of Mathematics. Concerning the position of descriptive geometry within the realm of mathematics, nothing that can be said will have more weight than the views of Professor Gino Loria of the University of Genoa.[1] According to this eminent geometer, mathematicians are not agreed as to the position of descriptive geometry within the entire domain of mathematical disciplines. Some (among which he does not count himself) incorporate it, following Fiedler's example, with projective geometry. Others, without denying its inner relation with the theories of modern projective geometry, regard it as amongst the applied disciplines. The latter view can, no doubt, be approved if one thinks of it only as a means to the proper construction of drawings in architecture, technology, crystallography, etc. But one is strongly inclined to abandon this view when he considers that the conceptions on which descriptive geometry rests and the methods which it continually applies are, on the whole, only such as are taught by the old Euclidean geometry and the modern projective geometry, and that throughout its entire domain the procedure is so very exact that it is comparable with analysis and algebra. In fact, descriptive geometry offers the most beautiful and instructive examples of a rigorous and complete development of several useful methods for the unambiguous representation of the whole of space upon a plane. We regard, therefore—inasmuch as we include in applied descriptive geometry the theory of illumination and shadows—the science with which we will now occupy ourselves as a province of the great realm of pure mathematics, though a region on the border thereof over which we may be transferred into the field of applied mathematics.

3. The Purposes of Descriptive Geometry. From the preceding account of the evolution of descriptive geometry it appears that the principal objects which have been sought, and in the course of time attained, are:

1. *The representation of the objects of space by means of figures which lie in a plane.*

2. *The solution of the problems of space by means of constructions which can be executed in the plane, i. e., the transformation into plane constructions* (like joining two points by a line, cutting a line by a line, and drawing a circle of given center and radius)[2] *of the graphically impossible space operations* (like passing a plane through a point and a line, cutting a line by a plane, cutting a plane by a plane, and describing a sphere of given center and radius).[3]

The criterion of what shall be meant by a good plane representative of a

[1] See preface to Vol. I, Loria, G., *Vorlesungen über Darstellende Geometrie* (1907), B. G. Teubner, Leipzig.

[2] These plane operations are sometimes called the postulates of construction.

[3] These objects may be extended so as to include: (*a*) The representation of a surface upon a plane, and the corresponding transformation of operations on the surface to those of the plane. The stereographic projection of Hipparchus, which has already been mentioned, is an example of such a representation. However, the general problem of this type properly belongs to mathematical geography; (*b*) The representation of all, or part, of space within a portion thereof, and the corresponding transformation of operations. As an example of this, relief perspective has already been mentioned.

space object has, more or less unconsciously, been taken to be *that the plane representative, when properly placed, shall produce upon the retinal surface of the eye the same image as that produced by the object itself.*[1] That it is possible to satisfy this criterion, at least approximately, results from the following physiological facts:[2]

1. All of the rays of light which emanate from a point P (Fig. 1) and pass through the pupil into the eye undergo several refractions and then unite again in a point P_n of the retinal surface of the eye. The particular ray PP_n passes through the eye practically unbroken. This ray may be taken as a substitute for the slim bundle of rays which come from P and pass through the pupil into the eye. Its direction is that in which the eye perceives the point P to lie. It is called the sight-line of P.

2. The sight-lines from the various points $P, Q, R, \cdots$ of a body Σ all pass through a fixed point K of the eye. This point is called the optical center of the eye (see Fig. 2). Hence these sight-lines belong to a bundle of lines of which the common point is K. The retinal image Σ_n, which is composed of the points $P_n, Q_n, R_n, \cdots$, is thus seen to be the geometrical intersection of the retinal surface of the eye with those lines of this bundle which come from the points of Σ.

From these two facts there results the fundamental geometrical relation which every drawing Σ' of a space object Σ must satisfy in order that it should produce upon the retinal surface of the eye the same image as that produced by the object itself. From fact 1 it follows that any light-emitting point P' which lies on the sight-line of P has the same retinal image as P. In order therefore to set up a plane picture Σ' which shall produce the same retinal image as that which is produced by Σ, it is sufficient to replace the points $P \cdots$ of Σ by the points $P' \cdots$

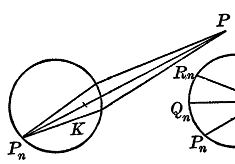


FIG. 1.

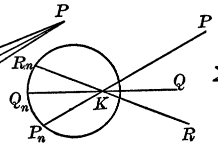


FIG. 2.

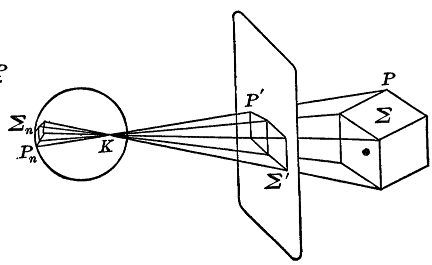


FIG. 3.

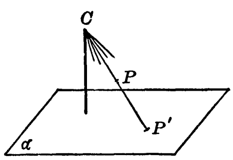
in which the sight-lines of the points $P \cdots$ pierce the picture plane (see Fig. 3). Since by fact 2 all of the sight-lines which come from the points of the space object Σ belong to a bundle of lines of common point K (optical center of the eye),

[1] In relief perspective the criterion of good representation is that a straight line of the space to be represented shall go over into a straight line of the representing space.

[2] The remainder of this paragraph, including statements 1 and 2, has been taken, with some modifications, from A. Schoenflies, *Einführung in die Hauptgesetze der Zeichnerischen Darstellungsmethoden* (1908), B. G. Teubner, Leipzig.

it follows that the representative Σ' in the picture plane is simply the intersection of this plane with those lines of the bundle K which come from the points of Σ. In other words, *a plane representative (i. e., a picture) of a space object which will, when properly placed, produce the same retinal image as the object itself produces, must be a central projection of that object.*[1] Even when such a picture is not properly placed with respect to the eye, or when such a picture is replaced by one which is obtained by parallel instead of by converging rays (whether these rays be perpendicular or oblique to the picture plane), the eye, because of its power of accommodation, recognizes therein the object represented. Thus we are led to the conclusion that *a central or parallel (orthographic or oblique) projection of a space object is a good plane representative of that object.* These types of projection form the basis of nearly all of the methods of representation which are used in descriptive geometry.

In order to attain the second object of descriptive geometry, *i. e.*, to solve by constructions in a plane the geometric problems of space, it is essential that there shall exist an unambiguous correspondence between the elements of space (points, lines, planes, etc.) and the plane representatives of these elements. By this it is meant that it must not only be possible to pass without ambiguity from a space element to its plane representative, but that it must also be possible to pass back again without ambiguity from the plane representative to the space element. It is evident (see Figs. 4, 5, 6) that a single projection does not satisfy these



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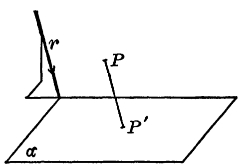

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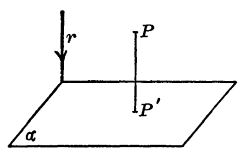


FIG. 6.

requirements. For, while to each point P of space there corresponds but one point P' of the picture plane α, it is not possible to pass back again without ambiguity from P' to P, because any point whatever of the projecting ray of P could be taken as corresponding to P'. In fact, the very criterion which has been given for the good plane representation of space objects involves the ambiguity that a general point P' (Fig. 3) of the plane representative may be the picture of any point whatever of its sight-line KP'. Now, in so far as the solution of space problems is concerned, a method of representation which has the property of *unambiguous correspondence* will suffice, even if it does not satisfy the *criterion of good representation* which has been given. On the other hand we have already seen that the different types of projection satisfy the criterion of good

[1] By a *central projection*, from a point C, of the points P of space upon a plane α, is meant those points P' in which the lines CP pierce the plane α (see Fig. 4). If the projecting lines instead of emanating from a point C are parallel to a straight line r, the projection is said to be *parallel* instead of central. If the line r is *not* perpendicular to α, the parallel projection is called *oblique* (see Fig. 5); while if r is perpendicular to α, the projection is called *orthographic* (see Fig. 6).

representation. But more than one projection is needed in a method of representation which shall have the property of unambiguous correspondence.[1] That which is used in addition to one projection may be another projection or it may be something else.

4. The Methods of Descriptive Geometry Exemplified. In the *Mongean Method of Double Orthographic Projection* a point P is projected orthographically upon each of two perpendicular planes. These planes, which are generally supposed to be horizontal and vertical and are therefore called respectively the *horizontal plane* π_1 and the *vertical plane* π_2, intersect in a straight line which is called the *ground line* g (see Fig. 7). The orthographic projections of P upon π_1 and π_2 are denoted by $'P$ and P'' respectively. In order to avoid the incon-

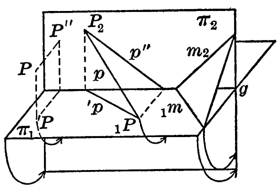
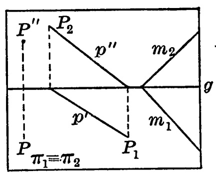


FIG. 7.



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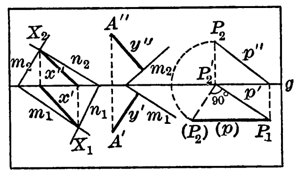

FIG. 9.

venience of two planes, the plane π_1 is rotated around the ground line into coincidence with π_2, so that $'P$ then assumes in π_2 the position P'. The plane π_2, with which the revolved position of π_1 now coincides, is called the *drawing plane* (see Fig. 8). Thus we see that to the point P of space there correspond in the drawing plane the two points P' and P'' which lie on the same perpendicular to the ground line, and, conversely, to such a pair of points in the drawing plane there corresponds but one point in space. Hence this method of representation possesses the property of unambiguous correspondence. Similarly, to a line p of space there corresponds a pair of lines p' and p'' in the drawing plane, and, conversely, to any pair of lines in the drawing plane there corresponds (in general) but one line in space. Instead of being determined by its projections p' and p'', a line p may also be given by its traces P_1 and P_2, that is, by the points in which it pierces π_1 and π_2, respectively. A plane of space is represented by its traces and hence we can say that to a plane μ of space there corresponds in the drawing plane a pair of lines m_1 and m_2 which intersect on the ground line, and, conversely, to such a pair of lines of the drawing plane there corresponds but one plane of space.[2]

From what has been said it is evident that (1) the condition that a point be on a line is that the projections of the point lie on the corresponding projections of the line; (2) the condition that a line lie in a plane is that the traces of the

[1] One is here reminded of the fact that binocular vision makes possible the perception of three-dimensional space.

[2] While $'P$ and P'' are the projections of the point P, the pair of points (P', P'') is the representative of P in the drawing plane. However, it is customary to speak of P', P'' as the projections of P. Similar remarks apply to the projection of a line and to the traces of a line and of a plane.

line lie in the corresponding traces of the plane. It can also easily be shown that (3) the condition that a line and a plane be perpendicular to each other is that the projections of the line be perpendicular to the corresponding traces of the plane.

The last condition makes possible the solution of perpendicularity problems. To solve the metrical problems use is made of a process called *rebatting*. This consists in revolving a plane figure around the horizontal or vertical trace of its plane until it coincides with the picture plane. In Fig. 9 are given the solutions by the Mongean method of each of the three problems:

Problem I. To find the line of intersection $x \equiv (x', x'')$ of the two planes $\mu \equiv [m_1, m_2]$, $\nu \equiv [n_1, n_2]$.

Problem II. To find the line $y \equiv (y', y'')$ which passes through the point $A \equiv (A', A'')$ and is perpendicular to the plane $\mu \equiv [m_1, m_2]$.

Problem III. To find the length of that portion of the line $p \equiv (p', p'')$ which is contained between its traces P_1 and P_2.

The solution of Problem I depends upon condition 2 and the fact that line x lies in both of the given planes. The solution of Problem II depends upon conditions 1 and 3. To solve Problem III the right triangle $P_2P_2{}'\,{}_1P$ (Fig. 7) is revolved around p' into coincidence with π_1. Hence in Fig. 9 the segment $\overline{P_2{}'(P_2)}$ is drawn through $P_2{}'$ perpendicular to p' and equal to $\overline{P_2{}'P_2}$. Thus we have solved by constructions in a plane three problems of space. Of these, I is a problem of geometry of position, II is a perpendicularity problem, and III is a metrical problem.[1]

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[1] The problems of space which involve the point, line, and plane have been divided into three groups (see Loria, *Vorlesungen*, etc., already cited): (A) Problems of Geometry of Position, (B) Perpendicularity Problems, (C) Metrical Problems. In each of these groups there is an indefinitely great number of problems, but they may all be solved when a few problems, called the fundamental problems of the group, can be solved.

The fundamental problems of group A may be taken to be:

I. To find the line connecting two points.
II. To find the line of intersection of two planes.
III. To find the plane through a point and a line.
IV. To find the point common to a plane and a line.

In I one of the points may lie at infinity, and in III either the point or the line may lie at infinity For these special positions the problems become respectively: Ia. Through a point pass a line parallel to another line; IIIa. Through a line pass a plane parallel to another line; IIIb. Through a point pass a plane parallel to another plane. The problem: "Through a point A to pass a line. x intersecting the lines b and c" may be solved in either of the following ways: (1) Find the intersection x of the plane of A and b by the plane of A and c. (2) Find the point X where b pierces the plane of A and c, and then x is the line connecting A and X. Similarly every problem of geometry of position can be solved by a few applications of one or more of the fundamental problems.

The fundamental problems of group B may be taken to be:

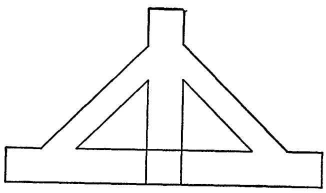
I. Through a point pass a line perpendicular to a plane.
II. Through a point pass a plane perpendicular to a line.
III. Through a point pass a line perpendicular to and touching a line.
IV. Through a line pass a plane perpendicular to a plane.
V. Find the common perpendicular to two non-intersecting lines.

The fundamental problems of group C may be taken to be:

I. To determine the length of the segment connecting two points.
II. To determine the magnitude of the angle formed by two lines.
III. To determine the magnitude of the angle formed by two planes.

This brief account of the Mongean method is sufficient to show its value as a means of solving graphically (with the instruments of the geometer) the problems of space. As to its value in the production of pictures, let it first be observed that most of the objects of architecture and technology contain, among their bounding surfaces, planes which are mutually perpendicular and which intersect in edges of the object. It is on these planes, or on planes parallel to these, that the objects are generally projected (orthographically) in order to obtain their Mongean representatives (*i. e.*, their pictures). Thus certain planes (namely those perpendicular to both of the planes of projection) are shown merely by lines, and therefore the Mongean pictures of such objects fail to convey a satisfactory notion of the space forms of such objects. This point is well illustrated by Figs. 10 and 11, which are the plan and elevation, respectively, and together form the Mongean representative of a bracket-shaped object. On the other hand, Fig. 12 is a projection (also orthographic) of the same object on a plane which is not parallel to one of the mutually perpendicular planes of the object. Evidently Fig. 12 conveys to the mind a much better notion of the

FIG. 10.



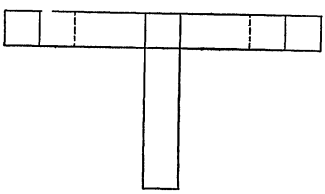


FIG. 11.

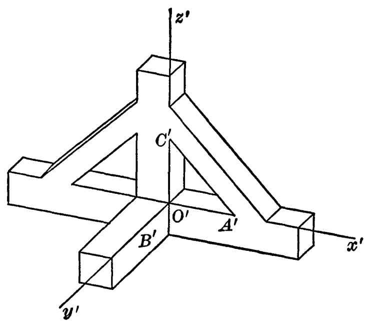


FIG. 12.

space form of the object which it represents than do Figs. 10 and 11, from which, however, the dimensions of the object may easily be taken. It is thus evident why an architect will give the plans and elevations of a building to his contractor whereas he will show to his client a picture of the type of Fig. 12.

Notwithstanding the fact that Fig. 12 is not a true perspective (*i. e.*, a central projection) it produces a retinal image which does not differ much from that produced by the object itself. Such pictures, as well as true perspectives and oblique projections, can be constructed from their plans and elevations by well-known processes of the Mongean method. However, a quicker and more satisfactory method for their production is contained in the so-called *Axonometric Method*, about which something will now be said. Inasmuch as the axonometric

method furnishes but a single picture, one naturally asks whether this method—which is capable of producing such good pictures—can also be used to solve graphically the problems of space. The answer to this question is that this method can so be used. In order to see this let us think of the various points of an object to be represented as referred to a system of three mutually perpendicular axes Ox, Oy, Oz (which, if the body is of the type already mentioned, may be three of its edges). Let a general point be denoted by P, its projections (orthographic) on the planes yOz, zOx, xOy by P', P'', P''', respectively, and its projections (orthographic) on the axes Ox, Oy, Oz by P_x, P_y, P_z, respectively. Then these seven points, together with the origin O, determine the so-called projecting parallelopiped $PP''P_xP'''P'P_zOP_y$ of the point P. In projecting the object by a system of parallel rays on a general plane π (plane of the paper), the axes project into the lines O^*x^*, O^*y^*, O^*z^*, respectively, and the points P, P', P'', P''', P_x, P_y, P_z project into the points P^*, P'^*, P''^*, P'''^*, P_x^*, P_y^*, P_z^*, respectively, so that the lines $P^*P'^*$, $P''^*P_z^*$, $P'''^*P_y^*$ are parallel to O^*x^*, $P^*P''^*$, $P'''^*P_x^*$, $P'^*P_z^*$ are parallel to O^*y^*, and $P^*P'''^*$, $P''^*P_x^*$, $P'^*P_y^*$ are parallel to O^*z^* (see Fig. 13). It is evident that from any one of the six pairs of points

(1) (P^*, P'^*), (P^*, P''^*), (P^*, P'''^*), (P''^*, P'''^*), (P'''^*, P'^*), (P'^*, P''^*)

the complete projection $P^*P''^*P_x^*P'''^*P'^*P_z^*O^*P_y^*$ of the projecting parallelo-

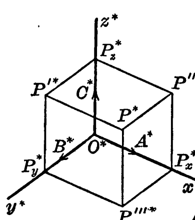


FIG. 13.

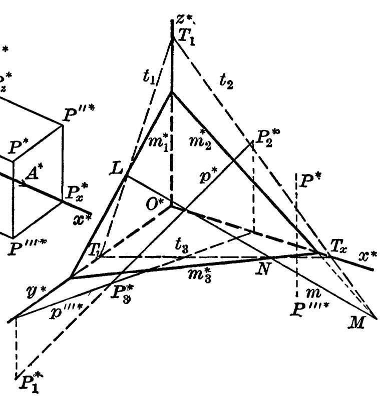


FIG. 14.

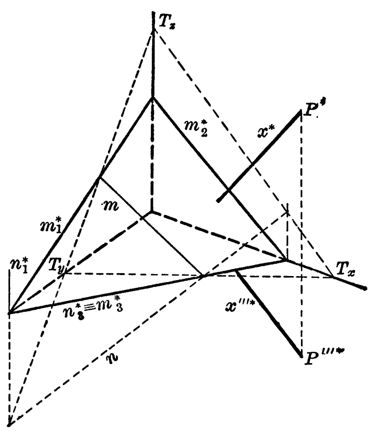


FIG. 15.

piped of the point P can be obtained. Let us suppose now that the segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ (Fig. 13) are the projections on the picture plane of unit segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ of the axes Ox, Oy, Oz, respectively. Then if the coördinates x, y, z, of a point P with respect to the axes Ox, Oy, Oz, are known, the projection P^* of this point upon the picture plane can easily be found by taking $\overline{O^*P_x^*} = x \cdot \overline{O^*A^*}$, $\overline{P_x^*P'''^*} = y \cdot \overline{O^*B^*}$, $\overline{P'''^*P^*} = z \cdot \overline{O^*C^*}$. However, the position of P^* alone would not be sufficient for the reversal of this process. But if any one of the six pairs of points (1) were given, the train of segments

depends upon the angles ξ, η, ζ (which may be chosen subject merely to the condition that they shall be obtuse). The construction shown in Fig. 16 enables one to choose the angles ξ, η, ζ and the segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ in their proper relations. In this construction Δ and Γ are circles of common center O^*, while $O^*(A)$ and $O^*(B)$ are perpendicular radii lying below the horizontal line which passes through O^*. Through the points where $O^*(A)$ cuts Δ and Γ, horizontal and vertical lines are drawn, respectively, meeting in A^*. In a similar manner B^* is determined. A vertical tangent t is drawn to Δ meeting Γ in (C) above O^*, and then through (C) a horizontal line is drawn cutting the vertical line through O^* in C^*. The three segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ thus found are of proper lengths and directions.

If the projection is oblique instead of orthographic, the relation between the angles ξ, η, ζ and the segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ depends not only upon the inclinations of the axes to the picture plane, but also upon the direction of the projecting rays. But *no matter how one may choose the angles* ξ, η, ζ *and the lengths* l, m, n *of these segments, the three directed segments* (*i. e.*, vectors) $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ *of the picture plane may always be regarded as the parallel oblique projection of three mutually perpendicular concurrent space vectors of common length.* This is a statement of *Pohlke's Theorem,* which was discovered in 1853.

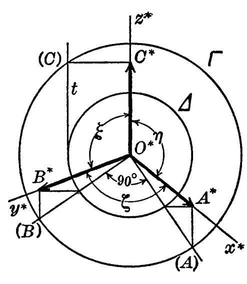


FIG. 16.

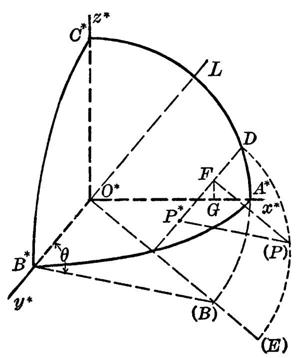


FIG. 17.

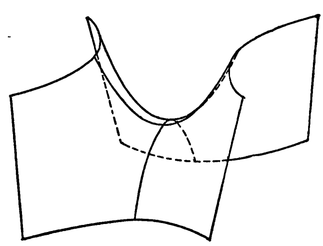


FIG. 18.

Long before the mathematical facts concerning parallel projections were known, however, it was used—often as a substitute for perspective, the rules of which were either not known or too complicated for the draughtsman. Forms of it were used in the Middle Ages before perspective was understood. The particular type of parallel projection is determined by the direction of the projecting rays and the inclination of the picture plane to the mutually perpendicular axes Ox, Oy, Oz. In the choice of the particular assumption, one is guided by practical considerations. The two particular assumptions which are commonly made are the following: *First,* the picture plane is parallel to the plane of two of the axes; *second,* the picture plane passes through the vertical axis, but is inclined to the other two. It is here assumed that the projecting rays are inclined to

the picture plane; the case where they are perpendicular, having already been considered to some extent. The first type is essentially that used in the science of fortifications, and is called cavalier perspective (also military or bird perspective). The second type, and also more general types, are much used in the representation of crystals. Of the two types, the second is preferable because it coincides approximately with the middle part of perspective at a great distance, whereas in cavalier perspective there is a considerable distortion. Cavalier perspective is also used in mathematical text-books to represent space relations. Thus Figs. 4, 5, 6 and 7 of this paper are such representations. In Fig. 17 the first octant of a sphere is so represented. In the same figure is given a solution of the

Problem. The projection P^* of a point P of the sphere is given; required the coördinates of P.

Solution: A projecting ray through the point B (where the positive y-axis pierces the sphere) is projected orthographically upon the picture plane into the line O^*y^*. The plane which passes through this projecting ray BB^* and is perpendicular to the picture plane cuts from the first octant of the sphere a quadrant of a great circle. By revolving this plane around its trace y^*O^*L into the picture plane the quadrant assumes the position $LA^*(B)$ and the ray BB^* assumes the position $(B)B^*$. The angle $\theta = \measuredangle\, O^*B^*(B)$ is the inclination of the projecting rays to the picture plane. Through P a plane parallel to the plane just revolved is passed. This cuts from the sphere a quadrant of a small circle, which on being turned into the picture plane (as was the large quadrant) assumes the position $D(P)(E)$. From this the revolved position (P) of P can be found by drawing $P^*(P)$ parallel to $B^*(B)$. Then the coördinate y is the perpendicular $\overline{F(P)}$ from (P) to P^*D, z is the perpendicular $\overline{FG}$ from F to O^*x^*, and x is eqaul to $\overline{O^*G}$. The meeting of DF and $O^*(B)$ on the ellipse is accidental.

In Fig. 18 a hyperbolic paraboloid is represented in cavalier perspective.

It has already been stated that parallel projection (both orthographic and oblique) was used as a substitute for perspective. It furnishes pictures which, while not producing (as does perspective) retinal images identical with those of the object, are nevertheless representations which convey to the mind adequate notions of the objects pictured. It has the advantage over perspective that its pictures may be "scaled off" and thus used as working drawings, just as the Mongean representatives. It also furnishes a means of solving graphically the problems of space.

Now, just as parallel projection was modified, or rather augmented, so as to make of it a method which can be used for the graphical solution of space problems, so central projection, or perspective, can also be thus modified. In this method, which is called that of *Free Perspective,* a line of space is represented by two points in the picture plane, namely, by its trace and its vanishing point, a plane of space is represented by its trace and its vanishing line, and a point is represented by its projection and the representative of a line or a plane on which it lies. Thus there exists an unambiguous correspondence between

space and the plane, making possible the solution of space problems by this method.[1]

To solve problems concerning fortifications, military engineers use a method which is employed in representing the natural surface of the ground in topographical and hydrographical maps. It is called the *Method of Contour Lines.* In this method a point is represented by its orthographic projection on a datum plane and the number which represents its distance from this plane, a line is represented by its projection and a scale on this projection (the points of the scale being the points into which are projected the intersections of the line by equally spaced planes parallel to the datum plane), and a plane is represented by one of its lines of greatest slope. Thus is made possible the solution of space problems by this method.

It is hoped that the preceding account of the several methods of descriptive geometry will have given the reader a notion of the nature of the subject. Many things have not been mentioned, but those interested are referred to the books already mentioned.

5. Didactic Considerations. Having discussed the objects and nature of descriptive geometry, let us now consider what place it should occupy in the educational curricula. Developed from technical needs, this science was, and essentially still is, peculiar to the technical schools. It was taught and developed there almost exclusively. Germany and Austria followed France (where Monge published his lectures simultaneously with the opening of the École Polytechnique in Paris in 1795) into the field early in the nineteenth century after the founding of their technical schools. Italy did not follow until later, and, beyond the work by Brook Taylor on perspective, England took no part in the development of the subject. In Italy the most noteworthy work was done not in a technical school, but in the University of Padua by a professor of science, namely, by Bellavitis about 1851.

In this country also the teaching of descriptive geometry is confined almost exclusively to the technical schools. But the amount of time which is here devoted to the subject is so limited and the course so restricted in its scope, that the average student at the completion of his course knows hardly more than the elements of the Mongean method and very little, if anything, about other parts of the subject. Furthermore, many teachers have only a very limited knowledge of the subject. There seems to be no high requirement for a teacher of descriptive geometry as there is, for instance, for teachers of general mathematics. This being the case there is absolutely no hope of disseminating a profound and extensive knowledge of this subject.

There is a tendency in our technical schools to diminish, rather than to increase, the amount of time devoted to descriptive geometry. Therefore, if the subject is worth keeping alive—to say nothing of a hope for doing research

[1] Those interested in the methods of free perspective and contour lines are advised to read Loria's *Vorlesungen über Darstellende Geometrie,* from which much of what has been said in this paper about the Mongean and axonometric methods has been taken.

work in it, for which there are still possibilities—it seems to me that it is the duty of our collegiate departments to offer courses in the subject and to make the scope of these courses rather extensive. Another possibility might be to raise the grade of our technical schools to that of the European technical schools, and then possibly the professors of descriptive geometry might take the interest in the subject which exists abroad. The latter alternative, *i. e.*, raising the standard of our technical schools, seems unlikely. To the former the colleges might say: "Why teach a technical subject in a college?" In reply to this, I would refer to the remarks of Professor Loria already given, namely, that descriptive geometry lies within the realm of pure mathematics but on the border thereof, so that it is an easy step to the applied. But there are other justifications for its introduction into a college curriculum. Gauss recommended it as "a nutriment-giving element to animate the real geometric spirit" [1] and Christian Wiener says, "It possesses the capacity above all to develop the power of space visualization and thus to lay the foundation for the study of higher geometry." [2]

I have found that students who study analytic geometry after, or simultaneously with, descriptive geometry find the former subject more interesting and less difficult than those who have not studied the latter subject. In the study of projective geometry also, a knowledge of descriptive geometry would be of great help in making the drawings of the space relations which must necessarily be made in that subject. On the other hand, some knowledge of projective geometry is essential in the higher branches of descriptive geometry. According to Professor Loria these two subjects should be taught independently, as has been done at the University of Genoa. I consider a course in descriptive geometry in the college quite as important as one in analytic geometry and projective geometry.

Another reason for the introduction of descriptive geometry into the colleges is the almost complete ignorance concerning the subject which exists among mathematicians. This lack is generally accompanied by an inability to visualize space forms. This inability is a great handicap not only in some branches of geometry, but also in mechanics and physics, where very frequently a mental image of a situation is of as much value as refined analysis. Moreover, a general course in mathematics without graphical solutions of plane and space problems with the instruments of the geometer is almost as great an incongruity as general courses in physics and chemistry without laboratory work.

Descriptive geometry is now required for an honors degree at the University of Cambridge, England.[3] There are many advocates of its introduction into the universities of Continental Europe, and some universities actually do give courses

[1] *Encyklopädie der Mathematischen Wissenschaften*, III AB 6, page 560. This article by E. Papperitz was consulted in writing this paper.

[2] *Chr. Wiener, Lehrbuch der Darstellenden Geometrie*, Vol. I, p. 61. From this work were taken most of the historical notes of this paper.

[3] Smith, D. E. Report of Sub-Commission A of the International Commission on the Teaching of Mathematics: Intuition and experiment in mathematical teaching in secondary schools.

in it. Some might hold that its introduction into a university could not be justified unless it offered problems worthy of a Doctor's thesis. I believe that it does offer such problems, and as one such, mention the theory of illumination. But even if descriptive geometry did not offer such problems, its introduction into the college, and into the university, is justifiable because of the many reasons already given.

It is a fact that the authors of our textbooks on mathematics are deplorably ignorant concerning the making of drawings to represent space objects. Klein evidently had this in mind when he said, "Is it not as worthy an object of mathematics to be able to draw correctly as to be able correctly to calculate?" It seems only reasonable to demand that the drawings in mathematical textbooks should be made by some one who understands the underlying principles, and it would be very desirable for the author himself to be able to make his drawings.

As to the qualifications of a teacher of descriptive geometry, it would seem that he should have besides an extensive knowledge of the subject, covering the various branches above outlined, a knowledge also of plane and solid analytic geometry and projective geometry, besides the ability to make good drawings in ink as well as in pencil. The time is at hand when it is desirable to require of prospective teachers the completion of such a course. A person with such qualifications could be attached to either a department of mathematics or a department of drawing. If to the former, he would certainly be required to know more mathematics than the minimum just laid down, and if to the latter, he would probably be required to be proficient in other branches of drawing.

While this address was being prepared for publication, death claimed that one person who by many was regarded as the best teacher of descriptive geometry which this country ever had. I had the good fortune to be his pupil, and it was he who inspired me with that love for the subject which it is my earnest desire to transmit to others. I refer to the late Dr. E. A. Engler, at one time professor of mathematics at Washington University.

GEOMETRY FOR JUNIORS AND SENIORS.[1]

By E. B. STOUFFER, University of Kansas.

An examination of the list of papers presented to the American Mathematical Society during the last five years shows that on the average not more than one paper in six is concerned essentially with geometry. If the papers presented to the Society are assumed to picture approximately the research activity of America, it may well be asked whether geometry is receiving the attention which it deserves. Is a ratio of one to five or six a healthy sign for mathematics as a whole in this country?

It is true, there is no fundamental distinction between geometry and analysis—

[1] Read at the second summer meeting of the Association, Cleveland, Ohio, September 7, 1917.

the laws which govern the process of reasoning in the two are the same. Yet frequently a theorem which requires a long and careful argument for its proof by the methods of analysis is intuitively evident when viewed geometrically. To see a proposition thus standing out by itself, stripped of its attendant proofs, is very much of an aid in grasping its full significance. Try, for instance, to explain to a class of sophomores the fundamental theorem of the integral calculus without showing that the integral may be represented as the area under a curve. Yet the proof is purely analytical.

The development of many a branch of mathematics may be aided by the development of geometry along some particular line. It cannot be denied, for instance, that the geometry of space of more than three dimensions has been a distinct asset to analysis where more than three variables are involved. The terms "points," "curves," "surfaces," etc., the notion of other elements than points as generating elements of geometrical figures, the propositions analogous to those in ordinary space, these may well suggest analytical theorems, otherwise unexpected, and may also greatly simplify their statement and proof. Thus, the problem of the equipartition of energy in a gas seems far removed from geometry, yet the notion of a sphere of n dimensions is used in the proof of some of the results already obtained towards its solution.

If, therefore, geometry has not kept pace in the scientific advance of America, it is pertinent to ask the cause and the cure; if it has maintained its proper position, it is still desirable to seek methods of insuring that, in the future, it does not retard the scientific world but rather leads the way.

The tendency to give geometry a secondary place is evident in many American institutions, both in the graduate and in the undergraduate courses. A man who received his master's degree in mathematics a year ago at one of our most important graduate institutions had had no geometry in advance of the ordinary plane and solid analytic geometry. Moreover, neither the student nor the institution is so very exceptional. As for the undergraduate student majoring in mathematics, he is required to take the traditional course in plane analytic geometry, and may be advised to take a brief course in solid analytic geometry, but there the list of courses in geometry open to him very frequently ends. He has no chance to gain the inspiration and delight which accompany a knowledge of the theorems bearing the names of Desargues, Pascal, Brianchon, no opportunity to recognize the geometry already studied as a special case of something far more general and more beautiful.

There can be no question but that many additional keen-minded students might become interested in mathematics if geometry were presented to them more attractively in their undergraduate courses. Sir G. H. Darwin, in the opening address of the Fifth International Congress, expressed the belief that analysis and geometry offer different attractions to mathematical minds and then added, "I suspect that the mathematician will drift naturally to one branch or another of our science according to the texture of his mind and the nature of the mechanism by which he works."

A fundamental cause then of our slow growth geometrically is our undergraduate course of study. The remedy, naturally, lies in the introduction of additional geometry of the proper kind, in suitable amount and in the best way. The questions before us are: What is the proper kind of geometry? What is a suitable amount? What is the best way of presenting it?

We assume the prerequisites of the ordinary freshman and sophomore mathematics: algebra, trigonometry, plane analytic geometry, differential and integral calculus. The student may have had additional courses—for example, one in synthetic geometry, in his freshman year. If so, it is his good fortune, but such a course should not replace one taken after the student's mind has been developed and broadened by the analytic geometry and the calculus.

There are two particular classes of students for whom the course should be planned: first, those who expect to teach mathematics in the secondary schools; and second, those who expect to continue their study of mathematics in a graduate school. There may be other students in the course—for instance, those who take the course for its so-called cultural value—but their number will probably be too small to require special attention. Besides, a course fulfilling the needs of the first two groups will undoubtedly be satisfactory to the stray student.

By no means all American institutions offer a course in geometry for juniors and seniors, but where there is such a course it is as a rule largely projective geometry. It might well be asked, however, whether there is not some other branch of geometry which would fit the needs of the class of students mentioned above better than the projective. In this paper attention will be called to some general principles which are believed to be fundamental in the selection and arrangement of the material for the course, regardless of whether it is projective geometry or not. If some other branch satisfies these requirements best, it should be given. However, in the discussion of these principles almost exclusive use has been made of the notions of projective geometry because, in the author's belief, no other branch of geometry can so well fulfill the conditions.

I. *The course should couple together the earlier courses in geometry, and should also form a connecting link between them and future courses if such are taken.*

There has been too much of a tendency to separate mathematics, especially for the beginner, into watertight compartments, each compartment containing the material for one course. The student frequently regards it as more important that he recognize the course in which a particular theorem has been presented than that he understand the theorem itself and its relation to other results. Most of us, I presume, have met the student who, when asked to solve a quadratic or simplify a complex fraction in trigonometry, explains that that is algebra and expresses surprise that such questions should be asked in a trigonometry class. That the unfortunate effects of this method of presenting the elementary courses have been recognized is evidenced by the number of texts which have recently appeared combining in various ways the algebra, trigonometry, analytic geometry and calculus. This question was even considered worthy of formal discussion at the summer meeting of the Association a year ago.

It is equally important that the student recognize that the traditional divisions of *geometry* are for the purpose of emphasizing some unique method of attack or some particular kind of results. The high school work in geometry is entirely synthetic and is largely a course in memory work and logical reasoning; the early college courses, on the other hand, are as a rule concerned with teaching the "analytic method," that is, with showing the combined use of coördinates and algebra. It is true that the student recognizes some of the terms, like point, line, circle, as common, and even a few identical theorems may be proved in both sets of courses; but the methods employed seem to have nothing in common.

This evil is especially serious for one who goes out to teach mathematics in the high school without further training in geometry. He may have understood fully his college course in analytic geometry but he can only teach his Euclid in the way he first learned it, with no inspiration for himself and little for his students. Klein calls this jump from the synthetic geometry of the high school to the analytic geometry and back again a "double discontinuity" and adds that the real purpose of a university course for the teacher of mathematics is "that you may be able in a large measure to draw inspiration for your teaching from the great body of knowledge that has been presented to you."

II. *The course should be both synthetic and analytic, with the emphasis on the synthetic.*

After Poncelet and his followers, in the early part of the nineteenth century, had shown the possibilities of synthetic work in projective geometry, the wave of enthusiasm for it threatened for a time to drive the analytic school out of existence. Although the brilliant work of von Staudt in freeing synthetic geometry from analytic and metric relations was a triumph for the synthetic school, the later work of Plücker and Cayley showed that each method had its distinct advantages and that it was a decided loss to cultivate one at the expense of the other.

The entire elimination of the analytic work from the first course in projective geometry is especially unfortunate for the student who has just finished a course in analytic geometry. For, it is only by the introduction of some analytic work that the two courses can be connected in a vital way. This fact is well expressed by Segre in a strong appeal for combined analytic and synthetic methods in geometry: "The method of coördinates serves to pass from one to the other and unites them intimately, or rather so welds them together that we may say that every advance in the one means an advance in the other."[1]

It is not desirable to introduce a large proportion of analytic work, as the student is already much better trained in its methods than in synthetic methods. Results which can be obtained equally well or better by the synthetic method should be presented in that way. However, even a relatively small amount of analytic work in the projective geometry will strengthen and enlarge both the elementary analytic geometry and the projective geometry. For instance,

[1] Segre, *Rivista di Matematica,* Vol. I (1891), pp. 42–66; translation by J. W. Young, *Bulletin of the American Mathematical Society,* Series II, Vol. 1, pp. 442–468.

the student who has had only the usual course in analytic geometry feels no freedom in the use of coördinate systems. The idea that other quantities than distances—for example, the ratio of distances or the value of an angle—may be used as coördinates, the notion that lines and planes can be determined by coördinates in place of equations, the concept of homogeneous coördinates and the resultant elimination of the infinitely distant point as a special case, all these are new viewpoints which broaden previous concepts remarkably.

On the other hand, the projective geometry gains decidedly from the elementary analytic geometry. The latter assumes implicitly that there is a one-to-one correspondence between the points on a line and the numbers of the real number system. The real basis for the assumption is admittedly too difficult for inclusion in an undergraduate course, but the same may be said of the alternative assumption, the postulate of continuity in some form or other. The introduction of analytic methods into the projective geometry makes possible the explicit assumption of an abscissa system of coördinates for the points of a range, an assumption already familiar to the student and in fact quite natural and easy of comprehension. From this assumption follows directly the projective coördinate system and the proof of the fundamental theorem of projective geometry.

Again, in the introduction of imaginaries into geometry, analysis comes to the rescue. If statements are to be made as general as possible—and that is one of the principal features of this course—imaginary elements must be considered. Von Staudt's use of involutions without double points for defining imaginary points on a line, while a fine piece of mathematical work, is too difficult for such a course as we are outlining. By the use of algebra the whole subject of imaginaries is carried back to a field with which the student is already somewhat familiar. The necessity of treating imaginaries in geometry by means of algebraical quantities is well expressed by Russell in his *Foundations of Geometry*, where he says: "All the fruitful uses of imaginaries in geometry are those which begin and end with real quantities, and use imaginaries only for the intermediate steps. Now in all such cases, we have a real spatial interpretation at the beginning and end of our argument, where alone the spatial interpretation is important; in the intermediate links, we are dealing in a purely algebraical manner with purely algebraical quantities and may perform any operations which are algebraically permissible."

The possibility of giving an analytical statement to the projective transformation is another gain from the introduction of analytic methods. The equation of the transformation not only makes the whole transformation appear much more real but affords a solid peg on which to hang its double points. Moreover, the slight acquaintance here with the linear transformation will make the friendship ripen much more rapidly when it is met in some other course in geometry, in function theory, or elsewhere.

The prospective graduate student in mathematics will take this course as a preparation for the more intensive graduate courses. He should therefore

have training in both analytic and synthetic methods. "Often it will be found convenient to alternate between the synthetic method which appears more penetrating, more enlightening, and the analytic which is in many cases more powerful, more general, or more rigorous; and it will frequently happen that the same subject will not be quite clear from all points of view unless treated by both methods. . . . The most important thing is that the geometer should not be the slave of a single method and that he should put himself in a position where he can make use of any instrument to obtain an important result."[1]

III. *The course should begin with undefined terms and unproved propositions, specifically stated.*

"Foundations" is too large a word for what should be included, but there must always be some undefined terms and some unproved propositions. These should be as simple as possible. The course might begin with about the kind of material Enriques uses in the first three sections of his projective geometry. The point, line, and plane are assumed without definition, the nine fundamental forms are then defined, and eight or ten propositions from elementary geometry, so simple as to appear intuitively correct, are introduced without proof. Also the assumptions are made which are necessary for the elimination of the special cases of infinite elements.

The question of foundations is extremely important for the prospective teacher of Euclidean geometry, not so much because he will present that question to his students as because his own clear ideas will enable him to judge what to emphasize and what to omit. It is either an unusually brilliant or an exceedingly careless teacher who has never been troubled with logical difficulties in the treatment of certain topics in algebra and geometry. The emphasis given to logical considerations in the report of the National Committee of Fifteen on Geometry Syllabus is sufficient evidence of the importance of this subject in elementary geometry; but it must be difficult for the teacher to grasp the significance of this report if the idea of making necessary assumptions and deducing results therefrom has not been presented to him in some form or other in college courses.

IV. *The course should show the generalizing methods and principles which have been so characteristic of modern work in geometry.*

This should undoubtedly be the chief aim of the course. The unfolding to the interested student of the geometrical concepts developed during the early part of the nineteenth century will cause him to experience in his own mind much the same transformation as that through which the mathematical world passed during those years. In fact, there is a noticeable parallelism between the development of mathematics through the centuries and its development in the mind of the student under the modern curriculum. There is in both first arithmetic, then algebra and Euclidean geometry, followed in order by Cartesian geometry and the calculus.

Historically the next great advance opened up the field of the so-called modern geometry. The developments of the eighteenth century had been

[1] Segre, *loc. cit.*

almost entirely analytic. The newly discovered calculus and its unlimited applications, with the assistance of the analytic geometry, had crowded out of consideration any other forms of geometry. But by the end of the eighteenth century this field of research was beginning to appear exhausted and there was a revival of interest in the almost forgotten work of Desargues and Pascal. The French mathematician Monge really reopened the subject of projective geometry by the publication in 1800 of his *Descriptive Geometry*. Besides discovering many properties of curves and surfaces useful for classification purposes, he first suggested the use of imaginaries in pure geometry. His work introduces into geometry, as Hankel says, "the hitherto unknown idea of geometric generality and geometric elegance."

In 1822 there appeared a treatise by Poncelet, a pupil of Monge, in which is introduced for the first time the valuable concept of figures in homology. Here are discussed also such notions as the line at infinity, the circular points at infinity, imaginaries, and projection and section. Plücker and Steiner soon followed with treatises, that of Steiner being especially worthy of notice because in it projective pencils are first made use of in the generation of conics. "In the ten years which embrace the publication of these immortal works of Poncelet, Plücker and Steiner, geometry has made more real progress than in the two thousand years which had elapsed since the time of Apollonius. The ideas which had been slowly taking shape since the time of Descartes suddenly crystallized and almost overwhelmed geometry with an abundance of new ideas and principles."[1]

If the mathematical world was so inspired by the richness and beauty of the projective geometry after a period devoted exclusively to analytic geometry and the calculus, is it not to be expected that there will be a favorable reaction in the mind of the college student when, after the two years spent on the usual freshman and sophomore courses, he sees the generality of the projective methods? Duality will appeal to him not merely as a great labor-saving device, but also as a new and unexpected correlation of widely separated theorems. Nothing of the kind has appeared to him in previous courses. Likewise the process of projection and section is both easy to understand and fascinating in its possibilities. Again, the elimination of the consideration of the infinitely distant element as a special case—the horror of the average student—brings undiluted delight. The invariance of the anharmonic ratio under projective transformations, the possibility of generating figures by means of other elements than points, the notion of coordinates attached to lines and planes, the construction of conics by means of two projective pencils, these and many other such general notions cannot fail to astonish and enthuse a student who has any interest at all in geometry.

V. *Metric geometry should be introduced as a special case, but emphasis should be put on the fact that it is metric.*

The introduction of metrical considerations brings the whole subject back

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[1] Pierpont, "History of Mathematics in the Nineteenth Century," *Bulletin of the American Mathematical Society*, Series II, Vol. 11, pp. 136–159.

to actual experience. The possibilities of the applications of projective geometry to physical problems and the presentation of metric geometry as a special case of a more general geometry are immediate results. Moreover, in no other way can the important part the infinitely distant point plays be so easily emphasized.

There might be mentioned several other general principles which should be considered in the selection and arrangement of the course. For instance, the material should be arranged as far as possible in the order in which the student would of himself seek information; there should be an abundance of construction problems; the problems of the course should not only afford drill in the theory but should also offer chance for originality without being so difficult as to produce discouragement.

To leave these somewhat safer generalities, what should be the content and arrangement of topics in order to fulfill best the requirements mentioned? A very brief outline of a plan for a course is here suggested.

After the undefined elements and unproved propositions from elementary geometry have been introduced and the method of eliminating the consideration of the ideal elements as special cases has been explained, the principle of duality may be explained and illustrated. We are then ready for a discussion of projection and section, and of perspectivity, and for the definition of a projectivity as a sequence of perspectivities. Desargues's theorem on triangles should be proved as an illustration of the possibilities of the use of projectivities.

We are now ready for the introduction of coördinate systems for one-dimensional forms. Emphasis should be put on the fact that we seek a system satisfactory for projective geometry, that is, such a system that the coördinate of an element need not be changed when the form is subjected to a projective transformation. The abscissa system, introduced by means of the assumption already mentioned, and the simple ratio system, which follows immediately, both fail to satisfy the requirements of projective geometry. The double-ratio coördinate follows most naturally and is found to be invariant under a projectivity. As already mentioned, it is now easy to prove the fundamental theorem of projective geometry.

The projective transformation may next be shown to be equivalent to the linear transformation. The double points of a projectivity appear at once. Some metric geometry may be introduced and also a number of interesting construction problems. If time permits a discussion of geometric addition and multiplication will be found illuminating.

Harmonic sets with some metric properties should now be introduced. Involutions may follow, being defined most directly perhaps as projectivities of period two. The analytic representation of an involution makes intelligent a discussion of its double elements, even though they are conjugate imaginaries.

We are now ready for the discussion of conics. The field of selection here is so large and varied that the time at the disposal of the class must determine what is given, but certainly there should be included the famous theorems of Pascal, Brianchon and Desargues, projectivities on a conic, the theory of poles and polars, and some metrical properties of conics.

Conics might be either preceded or followed by a discussion of two-dimensional projective coordinates. The duality between points and lines, both as to coördinates and equations, the use of the abridged notation, the expression of the double ratio of four elements in terms of their parameters, the equations of conics, are all worthy of particular notice. Any time remaining might well be given to further discussion of projectivities between two-dimensional forms.

The course as outlined will probably consume the time of a three-hour course throughout the year. If only half that amount of time can be given to the course, as is unfortunately true in many institutions, it will be necessary to omit the work on two-dimensional coordinate systems and to make brief the treatment of some of the other subjects mentioned. In any case, however, the course is such that the prospective teacher will have new inspiration as well as much new material, and that the future graduate student will have an increased interest in geometry and a first-class foundation for its further study.

BOOK REVIEWS AND NOTES.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

BOOK REVIEWS.

An Introduction to Statistical Methods. By HORACE SECRIST, PH.D., Associate Professor of Economics and Statistics, Northwestern University. The Macmillan Company, New York, 1917. xxi + 482 pages. $2.00.

We are told in the preface to this book that "the treatment is non-mathematical for several reasons, chief of which are that the mathematical phases of the subject are treated in other places and that there seems to be an urgent need for a fundamental discussion of the non-mathematical but not less vital processes in statistical investigation and analysis."

With a non-mathematical treatment, we may well question the propriety of reviewing the book in a mathematical journal, but the methods involve certain mathematical concepts such as average deviation, standard deviation, correlation coefficient, and probable error.

It may well be said that it seems to the reviewer that the illustrations drawn from economics and business are a very useful and interesting part of the book. There is given a valuable discussion of the collection of statistical data with special reference to the dangers of interpreting results from figures collected without a thorough knowledge of the conditions and limitations under which the figures are produced. The main sources of economic and business statistics are also well discussed.

In the chapters on "Tabular Presentation," "Diagrammatic Presentation," and "Graphic Presentation," the author indicates how the choice of different devices depends largely on the purpose of the presentation.

In the chapter on "Averages as Types," the uses and abuses of different

averages are discussed. The two chapters on "Index Numbers" show very well not only some of the purposes to which such numbers may be put but the serious consequences of the loose and indiscriminate use of indices.

The part of the book that uses mathematical concepts is well illustrated by concrete applications, but the reviewer offers the following criticisms:

On p. 387 we find the statement: "If, however, signs are disregarded, the aggregate deviations are larger when taken from the arithmetic mean than when taken from any other average, for the reason that this average is affected both by the size of items and the frequencies." This statement is incorrect, as can be shown easily by giving a well-selected illustration in which the mode is near the end of the distribution. That is to say, we can easily give an illustration in which the average deviation from the mode is greater than from the arithmetic mean.

On p. 388 there is the statement that "mathematical consistency seems to demand that the median be used. On the other hand, the *average* deviation requires that the total be averaged, that is, divided by the number of items, and logical consistency seems to demand that they be computed from the mean." It is not clear to the reviewer what this statement means, as it is not indicated what distinction exists between mathematical consistency and logical consistency.

On p. 453 there occurs the statement: "If $r = 0$, no correlation exists, changes in the two phenomena being indifferent." This statement does not seem to be correct. To be sure, when two variates are independent, we have $r = 0$, but the converse is not necessarily true. The statement quoted seems to imply that the converse is true.

As shown by illustrations in this book, the use of the correlation coefficient and the attendant theory may help very much to throw light on the tendency of two sets of economic phenomena to change together, but such applications should be made with a clear understanding of the limitations imposed in the mathematical development of the theory.

H. L. Rietz.

University of Illinois.

Elliptic Integrals. By Harris Hancock. John Wiley and Sons, New York, 1917. 104 pages. $1.25.

Professor Hancock's book is the eighteenth in the series of "Mathematical Monographs" edited by Mansfield Merriman and Robert S. Woodward. It furnishes a useful guide to students who have completed an elementary course in the calculus and wish to extend their study to a consideration of types of integrals not treated in a first course. As is to be expected in so short a discussion, Professor Hancock's book treats only the formal aspects of the subject and chiefly the integrals of the first and second kinds with the standard transformations into the normal forms. The discussion of elliptic functions and their transformations is brief but well balanced, the interest being centered where it

should be, upon the doubly periodic nature of the functions. A great many examples and problems are given. The five-place tables at the end of the book and the many reduction and other formulas of integrals in the text give in compact form the material most often needed for reference.

RAYMOND W. BRINK.

THE UNIVERSITY OF MINNESOTA.

NOTES ON NEW BOOKS.

"Isaac Barrow was the first inventor of the infinitesimal calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673." This is a quotation from the preface to "The geometrical lectures of Isaac Barrow, translated with notes and proofs and a discussion on the advance made therein on the work of his predecessors in the infinitesimal calculus" by J. M. Child, B.A. (Cantab.), B.Sc. (Lond.) The book is No. 3 of the "Open Court Series of Classics of Science and Philosophy." It is published by the Open Court Publishing Company, Chicago and London.

A revised edition of "Plane geometry with problems and applications" by H. E. Slaught and N. J. Lennes has just been published by Allyn and Bacon, Boston and Chicago. Practical applications to everyday life are a feature of the book. Here are two sample ones: (1) "In kicking a goal after a touchdown in the game of football, the ball is brought back into the field at right angles to the line marking the end of the field. The distance between the goal posts being given, and also the point at which the touchdown is made, find by a geometrical construction how far back into the field the ball must be brought in order that the goal posts may subtend the greatest possible angle." (2) "A car wheel is broken, and it is required to determine its diameter by construction when only a fragment of it is given." At the very beginning of the book there are two pages of "reasons why geometry should be studied by those who wish to obtain a generous culture, a broad outlook and a mental development characterized by logical thinking and clear expression." These two pages and the book itself are a good answer to those who are questioning the value of geometry as a subject of study in the high school.

The number of trigonometry textbooks on the market is still increasing. P. Blakiston's Son and Co. of Philadelphia have recently published "Plane trigonometry with tables" by Eugene Henry Barker, head of the department of mathematics of the Polytechnic High School of Los Angeles, California.

The society in England which corresponds to the "Mathematical Association of America" is called the "Mathematical Association." It publishes the *Mathematical Gazette*. One of its past presidents, Professor A. N. Whitehead, has recently published in one volume eight addresses which he has given in recent years on educational, mathematical and scientific topics. They ought to be of

especial interest to readers of the MONTHLY because three of them are presidential addresses to the "Mathematical Association." The title of the book is "The organization of thought." It is published by Williams and Norgate, London.

The D. Van Nostrand Company has recently published a book on "Recreations in mathematics," by H. E. Licks. It has chapters on arithmetic, algebra, geometry, trigonometry, analytic geometry, calculus, astronomy and the calendar, mechanics and physics, and an appendix. It is not so extensive as the well-known "Mathematical recreations and essays" by W. W. R. Ball, but it contains considerable material not to be found in that standard work. The eight pages devoted to the cell of the honey bee will be new to many readers.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2689. Proposed by E. V. HUNTINGTON, Cambridge, Mass.

Show that the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \varphi + \theta)}$$

is $y_1 = (\cos \theta - \rho)/(\cos \theta + \rho)$, where $\rho = \sqrt{\sin^2 \varphi - \sin^2 \theta}$.

This problem was suggested to the proposer by a professor of civil engineering, and has important applications in the theory of conjugate stresses.

Note.—It may facilitate the work to let $\xi = 2x + \varphi + \theta$.

2690. Proposed by E. V. HUNTINGTON, Cambridge, Mass.

Find the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \beta + \theta)}.$$

2691. Proposed by ROGER A. JOHNSON, Hamline University.

Show by purely geometric methods, without the use of the calculus, that the envelope of all circles whose centers are on a fixed circle and which touch a fixed diameter of that circle is a two-arched epicycloid. (Cf. Calculus problem, 423.)

2692. Proposed by J. L. RILEY, Stephenville, Texas.

A cube is cut at random by a plane, what is the chance that the section is a hexagon?

2693. Proposed by W. F. HARLOW, Portland, Oregon.

A cow is tethered with a rope, length l, to a peg on the opposite side of a wall, height h, the peg being at a distance a from the wall. Find the area over which the cow can graze.

2694. Proposed by N. P. PANDYA, Sojitra, India.

Find the locus of the centroid of a triangle, whose vertex lies on a given parabola, whose base of given length is a segment of a given straight line of unlimited length, and one of whose base angles is known.

2695. Proposed by FRANK IRWIN, University of California.

A positive number, which for convenience we will write as a fraction, a/b, is developed into a continued fraction,

$$a_1 \frac{1}{-a_2 -} \frac{1}{a_3 -} \cdots,$$

by the following process:

$$a = a_1 b - r_1, \qquad b = a_2 r_1 - r_2, \qquad r_1 = a_3 r_2 - r_3, \text{ etc.}$$

Here $a_1, a_2, \cdots$ are positive integers, and, if we suppose a, b to have been taken positive, $b > r_1 > r_2 > \cdots > 0$. Show that if a/b be but slightly larger than a positive integer or zero, the numerators (as likewise the denominators) of the successive convergents will for a time be in arithmetical progression, and determine how long this phenomenon will continue.

2696. Proposed by L. E. LUNN, Heron Lake, Minnesota.

An air pipe 18 inches in diameter passes diagonally through a room from one lower corner to the opposite upper corner leaving through elliptical openings in the floor and ceiling, so that the ellipses are tangent to two boundaries of the floor and to the two opposite boundaries of the ceiling. If the room is 10 x 12 x 8, find the remaining cubic capacity of the room.

2697. Proposed by H. S. UHLER, Yale University.

Show how to reduce the left-hand members of the following identities to their respective right members:

$$\sin^2 (x + \tfrac{1}{2}y) - \sin (x + \tfrac{3}{2}y) \sin (x - \tfrac{1}{2}y) = \sin^2 y,$$

$$\sin (x + y) \sin (x + \tfrac{1}{2}y) - \sin x \sin (x + \tfrac{3}{2}y) = \sin \tfrac{1}{2}y \sin y,$$

$$\sin x \sin (x + \tfrac{1}{2}y) - \sin (x - \tfrac{1}{2}y) \sin (x + y) = \sin \tfrac{1}{2}y \sin y.$$

2698. Proposed by WARREN WEAVER, Throop College of Technology, Pasadena, California.

An urn contains N balls numbered from 1 to N. Of these n are drawn out and are arranged linearly according to the numbers on each. A certain ball is observed to be the kth in this line. What is the most probable number written on this ball?

SOLUTIONS OF PROBLEMS.

490 (Algebra). Proposed by HENRY HEATON, Atlantic, Iowa.

Show that $\sin 3° = \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \frac{1}{8}(\sqrt{5 + \sqrt{5}} - \sqrt{15 + 3\sqrt{5}})$.

Solution by S. E. Rasor, The Ohio State University.

Since $3° = 12° - 9°$, $12° = 30° - 18°$, and, for $\theta = 18°$, $2\theta = 90° - 3\theta$, the sine and the cosine of 9°, 12°, 18°, and thus sin 3° may be found as follows:

We have

$$\sin 2\theta = 2 \sin \theta \cos \theta = \sin (90° - 3\theta) = \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta,$$

$$4 \sin^2 \theta + 2 \sin \theta = 1,$$

and

$$\sin 18° = \tfrac{1}{4}(\sqrt{5} - 1), \quad \cos 18° = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}}.$$

Also from the identities, $\sin \frac{1}{2}A \pm \cos \frac{1}{2}A = \pm \sqrt{1 \pm \sin A}$ for $A = 18°$, we have

$$\sin 9° = \tfrac{1}{4}(\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}), \qquad \cos 9° = \tfrac{1}{4}(\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}).$$

Also,

$$\sin 12° = \sin (30° - 18°) = \sin 30° \cos 18° - \cos 30° \sin 18° = \tfrac{1}{8}(\sqrt{10 + 2\sqrt{5}} - \sqrt{15} + \sqrt{3}),$$

$$\cos 12° = \tfrac{1}{8}(\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1).$$

Therefore,

$$\sin 3° = \sin (12° - 9°) = \sin 12° \cos 9° - \cos 12° \sin 9°$$

$$= \left(\frac{\sqrt{10 + 2\sqrt{5}} - \sqrt{15} + \sqrt{3}}{8}\right)\left(\frac{\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}}{4}\right) - \left(\frac{\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1}{8}\right)\left(\frac{\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}}{4}\right)$$

$$= \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{16}\left(\sqrt{10 + 4\sqrt{5}} + \sqrt{10 - 4\sqrt{5}} - \sqrt{30 + 12\sqrt{5}} - \sqrt{30 - 12\sqrt{5}}\right)$$

$$= \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{16}(1 - \sqrt{3})\left(\sqrt{10 + 4\sqrt{5}} + \sqrt{10 - 4\sqrt{5}}\right).$$

By simplifying the expression in the last parenthesis and then collecting under the radical sign, we have, after a simple reduction,

$$\sin 3^\circ = \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{8}\left(\sqrt{5 + \sqrt{5}} - \sqrt{15 + 3\sqrt{5}}\right).$$

Also solved by L. E. Mensenkamp, J. L. Riley, Horace Olson, H. S. Uhler, H. C. Feemster, Florence Rae, R. M. Mathews, H. L. Agard, Gilbert A. Ault, Frank Irwin, C. E. Githens, A. M. Harding, and the Proposer.

491 (Algebra). Proposed by J. W. Lasley, University of North Carolina.

Solve the equations $xy = x^2 - y^2$ and $x^2 + y^2 = x^3 - y^3$ for x and y.

Solution by J. W. Baldwin, Ann Arbor, Michigan.

Solving $xy = x^2 - y^2$ for x in terms of y we have $x = \frac{1}{2}(1 + \sqrt{5})y$ and $x = \frac{1}{2}(1 - \sqrt{5})y$. These values substituted in $x^2 + y^2 = x^3 - y^3$ give, after simplification, $y^2(y - \frac{1}{2}\sqrt{5}) = 0$ (1) and $y^2(y + \frac{1}{2}\sqrt{5}) = 0$ (2). From (1), $y = 0, 0, \frac{1}{2}\sqrt{5}$ and from (2) $y = 0, 0, -\frac{1}{2}\sqrt{5}$. Hence, the corresponding values of x are $0, 0, (5 + \sqrt{5})/4$ and $0, 0, (5 - \sqrt{5})/4$. From (1) or (2) it is seen that two branches of the curve represented by the second equation pass through the origin. It is readily determined that these branches are imaginary and, hence, the origin is a conjugate point.

Also solved by S. E. Rasor, F. H. Holestin, Elgin E. Groseclose, H. N. Carleton, A. M. Harding, J. L. Riley, Polycarp Hansen, E. B. Escott, G. Y. Sosnow, J. Q. McNatt, O. S. Adams, Horace Olson, T. C. Amick, L. E. Lunn, Pauline Sperry and the Proposer.

433 (Calculus). Proposed by Louis O'Shaughnessy, University of Pennsylvania.

Solve the differential equation

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} - \frac{y}{x} = 0.$$

I. Solution by Emil L. Post, New York City.

We have

$$x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = y. \tag{1}$$

Operating on both sides by $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ (see next problem), we have

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left[x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right] = \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}. \tag{2}$$

But for any operation $f(D)$

$$f(D)u\cdot v = uf(D)v + \frac{du}{dx}\frac{f'(D)v}{1!} + \frac{d^2u}{dx^2}\frac{f''(D)v}{2!} + \cdots.$$

Let

$$f(D) = D^{\frac{1}{2}} = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}; \qquad u = x; \qquad v = \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}.$$

Then

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left[x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right] = x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + \frac{1}{2}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = x\frac{dy}{dx} + \tfrac{1}{2}y. \tag{3}$$

Substituting from the original equation and (3) into (2), we have

$$x\frac{dy}{dx} + \tfrac{1}{2}y = \frac{y}{x}. \tag{4}$$

From this we have

$$\log y = -1/x - \tfrac{1}{2}\log x + \log C,$$

or

$$y = Ce^{-(1/x)}x^{-\frac{1}{2}}.$$

II. Solution by the Proposer.

According to Professor Kelland (*Trans. Royal Society of Edinburgh*, Vols. XIV and XVI) the general differential operator may be defined as follows:

$$\frac{d^\mu x^n}{dx^\mu} = (-1)^\mu \frac{\Gamma(-n+\mu)}{\Gamma(-n)} x^{n-\mu},$$

for all values of n and μ. We have $\Gamma(n+1) = n\Gamma(n)$.

Let us assume $y = A_0 + A_1x^{-\frac{1}{2}} + A_2x^{-1} + A_3x^{-\frac{3}{2}} + \cdots$.

Then

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = (-1)^{\frac{1}{2}}\left\{\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}A_1x^{-1} + \frac{\Gamma(\frac{3}{2})}{\Gamma(1)}A_2x^{-\frac{3}{2}} + \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}A_3x^{-2} + \cdots\right\}$$

and

$$\frac{y}{x} = A_0x^{-1} + A_1x^{-\frac{3}{2}} + A_2x^{-2} + A_3x^{-\frac{5}{2}} + \cdots.$$

Hence,

$$i\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}A_1 = A_0 \text{ and } A_1 = \frac{\sqrt{\pi}}{i}A_0 = -i\sqrt{\pi}A_0.$$

For, since $\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi$ when p is a fraction less than one, then $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Also, $\Gamma(1) = 1$.

Furthermore,

$$i\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}A_2 = A_1 = -i\sqrt{\pi}A_0, \text{ or } A_2 = -2A_0,$$

since $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$; also

$$i\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}A_3 = A_2 = -2A_0, \text{ or } A_3 = -\frac{\sqrt{\pi}}{i}A_0 = i\sqrt{\pi}A_0.$$

Finally, $y = A_0(1 - i\sqrt{\pi}x^{-\frac{1}{2}} - 2x^{-1} + i\sqrt{\pi}x^{-\frac{3}{2}} + \cdots)$.

Note.—This problem, which is very similar to one solved by Professor Kelland, was submitted because it was thought that it might prove of interest to the readers of the Monthly. The proposer would be glad to see some discussion of the subject of general differentiation.

The above solution may be considered as a reply to the following question:

360 (Calculus.) Proposed by ELMER SCHUYLER, Brooklyn, New York.

What interpretation must be given to

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} \text{ so that } \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right) = \frac{dy}{dx}?$$

435 (Calculus). Proposed by B. F. FINKEL, Drury College.

Show that

$$\int_0^\infty e^{-x^2-a^2/x^2}dx = \frac{\sqrt{\pi}}{2e^{2a}}$$

by a transformation, rather than by the usual method of differentiating under the sign of integration, as, for example, in Byerly's *Integral Calculus*, page 106–107.

SOLUTION BY OTTO J. RAMLER, Catholic University of America.

We have the known definite integral

$$\int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}.$$

Now let $z = (x - a/x)$. Then

$$dz = (1 + a/x^2)dx,$$

and the integral becomes

$$\int_0^{\infty} e^{-(x-a/x)^2}(1 + a/x^2)dx = \sqrt{\pi},$$

or

$$\int_0^{\infty} e^{-(x-a/x)^2}dx + a\int_0^{\infty} e^{-(x-a/x)^2}\frac{dx}{x^2} = \sqrt{\pi}.$$

In the second integral, let $x = a/y$. Then $dx = -a/y^2dy$, and the second integral becomes

$$\int_0^{\infty} e^{-(a/y-y)^2}dy \equiv \int_0^{\infty} e^{-(y-a/y)^2}dy.$$

Hence,

$$\int_0^{\infty} e^{-(x-a/x)^2}dx + \int_0^{\infty} e^{-(y-a/y)^2}dy = \sqrt{\pi}$$

or

$$2\int_0^{\infty} e^{-(x-a/x)^2}dx = \sqrt{\pi}$$

or

$$2\int_0^{\infty} e^{-x^2-a^2/x^2+2a}dx = 2e^{2a}\int_0^{\infty} e^{-x^2-a^2/x^2}dx = \sqrt{\pi}.$$

Whence,

$$\int_0^{\infty} e^{-(x^2-a^2/x^2)}dx = \frac{\sqrt{\pi}}{2e^{2a}}.$$

Solved similarly by O. S. ADAMS.

339 (Mechanics). Proposed by C. N. SCHMALL, New York City.

A roll of cloth of very small uniform thickness a is coiled up tightly in the form of a circular cylinder of diameter d and is laid horizontally across a perfectly rough incline so that its axis is parallel to the intersection of the plane with the horizontal. It is then allowed to unroll (without slipping) down the plane. Neglecting the motion of its center of gravity in the direction perpendicular to the plane, show that it will unroll entirely in the time

$$T = \frac{\pi}{4}\sqrt{\frac{6d^2}{ag\sin\phi}},$$

where ϕ is the inclination of the plane to the horizontal plane, and g is the acceleration of gravity.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

After t seconds, let the radius of the cylinder be r; let s be the distance its center of gravity has moved parallel to the plane; θ, the angle through which it has turned; $W = mg = 2\rho gr^2$ its weight; and let the reaction against it at its point of application with the plane be composed of R lbs. perpendicular to the plane and S lbs. upward along the plane. The moment of inertia is then $I = \rho r^4$. Since a is very small, assume

$$r = \frac{d}{2} - \frac{a\theta}{2\pi}, \qquad rd\theta = ds.$$

Using the dot and double dot to indicate first and second derivatives with respect to the time, we have the equations for motion of the center of gravity along and perpendicular to the plane and of the cylinder about its geometric axis:

$$\frac{d}{dt}(m\dot{s}) = W\sin\phi - S, \qquad \frac{d}{dt}(m\dot{r}) = R - W\cos\phi, \qquad \text{and} \qquad \frac{d}{dt}(I\dot{\theta}) = rS.$$

The second of these is to be neglected. Eliminating S from the first and last, we have

[A] $$\frac{1}{r}\frac{d}{dt}(I\dot{\theta}) + \frac{d}{dt}(m\dot{s}) - W\sin\phi = 0.$$

During the motion, r varies from $d/2$ to 0, θ from 0 to $\pi d/a$, and s from 0 to $\pi d^2/4a$.

(I) The proposed result can be obtained as follows: Assume

$$(1)\ \frac{d}{dt}(m\dot{s}) = m\frac{d\dot{s}}{dt}; \qquad (2)\ \frac{d}{dt}(I\dot{\theta}) = I\frac{d\dot{\theta}}{dt}; \qquad (3)\ \frac{d\dot{s}}{dt} = r\frac{d\dot{\theta}}{dt}$$

Then [A] becomes

[B] $$r\ddot{r} + \frac{ag\sin\phi}{3\pi} = 0.$$

Integrating, we find, if $k = ag\sin\phi/3\pi$, $\log(d/2r) = p^2$,

$$T = \frac{d}{\sqrt{2k}}\int_0^\infty e^{-p^2}dp = \frac{\pi}{4}\sqrt{\frac{6d^2}{ag\sin\phi}} = \frac{0.3392d}{\sqrt{a\sin\phi}}$$

These assumptions seem unwarranted, however.

(II) If we let $x = 2r/d = 1 - (a\theta/\pi d)$, we have

$$ds = rd\theta = \frac{d}{2}x\cdot d\left[\frac{\pi d}{a}(1-x)\right] = -\frac{\pi d^2}{2a}xdx; \qquad s = \frac{\pi d^2}{4a}(1-x^2);$$

$$\dot{\theta} = -\frac{\pi d}{a}\dot{x}, \qquad \dot{r} = \frac{d}{2}\dot{x}, \qquad \dot{s} = r\dot{\theta} = -\frac{\pi d^2}{2a}x\dot{x}, \qquad m = \rho d^2x^2/2, \qquad I = \rho d^4x^4/16.$$

Thus [A] becomes

[C] $$3x\ddot{x} + 10\dot{x}^2 + 10k = 0, \qquad \text{where} \qquad k = 2ga\sin\phi/5\pi d^2.$$

If $\dot{x}^2 = y$, $3xdy + 20(y+k)dx = 0$. Whence, by integration,

$$x^{20/3}(y+k) = \text{Const.} = k,$$

since at the start

$$x = \frac{2r}{d} = 1, \qquad y = \left(-\frac{a\dot{\theta}}{\pi d}\right)^2 = 0.$$

[D] $$x^{20/3}\dot{x}^2 = k(1 - x^{20/3}) \qquad \text{or} \qquad \frac{-x^{10/3}\dot{x}}{\sqrt{1-x^{20/3}}} = \sqrt{k}.$$

$\dot{x} < 0$ throughout the motion; and t varies from 0 to T as x varies from 1 to 0. Hence,

$$T = \frac{1}{\sqrt{k}}\int_0^1 \frac{x^{10/3}dx}{\sqrt{1-x^{20/3}}}.$$

Since

$$\int_0^1 x^m(1-x^n)^p dx = \frac{1}{n}\cdot\frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{\Gamma\left(p+1+\frac{m+1}{n}\right)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \text{and} \qquad \Gamma\left(\frac{13}{20}\right) = \frac{20}{13}\Gamma\left(\frac{33}{20}\right)$$

we have

$$T = \frac{3\pi d}{26}\sqrt{\frac{10}{ag\sin\phi}}\frac{\Gamma(1.65)}{\Gamma(1.15)} = 0.195d/\sqrt{a\sin\phi} \qquad \text{(units feet and seconds).}$$

(III) Reverting to the original equations, we find

$$-\frac{k^2}{u^3} - \frac{h^2}{v^3}\frac{dv}{du} = 0, \qquad \frac{dv}{du} = \frac{v^3k^2}{u^3h^2}$$

$$S = \frac{1}{r}\frac{d}{dt}(I\dot{\theta}) = -\frac{\pi\rho d^4}{8ax}\frac{d}{dt}(x^4\dot{x}) = -\frac{\pi\rho d^4x^2}{8a}(x\ddot{x} + 4\dot{x}x^2).$$

From [C],

$$S = -\frac{\pi\rho d^4x^2}{12a}(\dot{x}^2 - 5k).$$

From [D],

$$S = \frac{\pi\rho d^4x^2}{12a}k(6 - x^{-20/3}) = \frac{\rho g \sin\phi}{30}d^2x^2(6 - x^{-20/3}).$$

When

$$x^{-20/3} = 6, \qquad x = x_0 = .76433, \qquad \dot{x}^2 = 5k = 2ga\sin\phi/\pi d^2, \qquad \dot{r}^2 = ga\sin\phi/2\pi,$$

$$\dot{\theta}^2 = 2\pi g\sin\phi/a, \qquad \dot{s}^2 = \pi g d^2 \sin\phi x_0^2/2a = .29210\pi g d^2\sin\phi/a.$$

From this point on, the discussion in (II) calls for a negative value of S. Again,

$$R = W\cos\phi + \frac{d}{dt}(m\dot{r}) = \frac{\rho d^2}{2}\left[gx^2\cos\phi + \frac{d^2}{2}\frac{d}{dt}(x^2\cdot\dot{x})\right] = \frac{\rho d^2x}{4}[2gx\cos\phi + (x\ddot{x} + 2\dot{x}^2)d.]$$

From [C] and [D],

$$R = \tfrac{1}{2}\rho d^2gx\left[x\cos\phi - \frac{2a\sin\phi}{5\pi d}(1 + \tfrac{2}{3}x^{-20/3})\right].$$

When $x = x_0$,

$$R = \tfrac{1}{2}\rho d^2gx_0\left[x_0\cos\phi - \frac{2a\sin\phi}{\pi d}\right];$$

and if

$$\tan\phi \not> \frac{\pi d x_0}{2a}, \qquad R \not< 0.$$

In general,

$$R > 0 \text{ as long as } \tan\phi < \frac{5\pi d}{2a}\cdot\frac{x}{1 + \frac{2}{3}x^{-20/3}}.$$

As a is very small, there is no point in considering the condition $R > 0$ as a further restriction. As soon as S becomes zero

$$\left(\text{after } T_0 = \frac{1}{\sqrt{k}}\int_{x_0}^{1}\frac{x^{10/3}dx}{\sqrt{1 - x^{20/3}}}\text{ secs.}\right),$$

the cylinder begins to rotate so rapidly that the uncoiled cloth is no longer laid down and the cylinder ceases to resemble even remotely a rigid body. It would seem, therefore, that without additional data, the problem is not to be solved.

IV. It may be a reasonable assumption that after T_0 secs., S continues to be zero, the roll loosening just enough to effect this condition, and that the cloth is still laid down flat, so that $\pi dm = -2\rho a ds$. [$\rho = \pi lw/2g$, if l is the length of the cylinder in feet, and w its weight in lbs. per cu. ft.]

If this is the case, $m = \frac{1}{2}\rho d^2x_0^2 - (2/\pi)\rho as$, and the first of the equations of motion is

$$\frac{d}{dt}(m\dot{s}) = \left(\tfrac{1}{2}\rho d^2x_0^2 - \frac{2}{\pi}\rho as\right)\ddot{s} - \frac{2}{\pi}\rho a\dot{s}^2 = g\sin\phi\left(\tfrac{1}{2}\rho d^2x_0^2 - \frac{2}{\pi}\rho as\right),$$

or $(s - c)(\ddot{s} - g\sin\phi) + \dot{s}^2 = 0$. Letting $\dot{s}^2 = z$, $(c = \pi d^2x_0^2/4a)$,

$$(s - c)dz + [2z - 2g\sin\phi(s - c)]ds = 0.$$

In III, when $x = x_0$, $t = T_0$,

$$s = \frac{\pi d^2}{4a}(1 - x_0^2) = \frac{\pi d^2}{4a} - c, \qquad \dot{s}^2 = \frac{\pi g d^2 x_0^2\sin\phi}{2a} = 2gc\sin\phi.$$

Taking these final conditions of the phase of the motion according to III as the initial conditions for the present discussion, we have

$$z = \dot{s}^2 = \tfrac{2}{3}g\sin\phi\left[(s - c) + \frac{4c^3}{(s - c)^2}\right].$$

The distance to be traversed is $(\pi d^2/4a) - ((\pi d^2/4a) - c) = c$. Hence, the duration of this second phase is

$$T_1 = \sqrt{\frac{3}{2g\sin\phi}}\int_0^c \frac{(s-c)ds}{\sqrt{4c^3+(s-c)^3}}\sqrt[3]{2}\sqrt{c}\int_a^0 \frac{xdx}{\sqrt{1+x^3}}, \qquad \text{where} \qquad \alpha = -\tfrac{1}{2}\sqrt[3]{2}.$$

The full time of the motion is thus $T = T_0 + T_1$; or

$$T = \sqrt{\frac{5\pi d^2}{2ga\sin\phi}}\int_{x_0}^1 \frac{x^{10/3}dx}{\sqrt{1-x^{20/3}}} + \sqrt{\frac{3\pi d^2 x_0^2}{8ga\sin\phi}}\sqrt[3]{2}\int_a^0 \frac{xdx}{\sqrt{1+x^2}}.$$

It is more convenient to write the first of these integrals as

$$\int_{x_0}^1 = \int_0^1 - \int_0^{x_0},$$

using the value found in (II).

By the Binomial Theorem,

$$\int_0^{x_0} \frac{x^{10/3}dx}{\sqrt{1-x^{20/3}}} = \sum_0^\infty T_k$$

where now

$$T_0 = \tfrac{3}{13}\, x_0^{13/3}, \qquad T_1 = \tfrac{1}{22}\, x_0^{11}, \cdots \frac{T_{k+1}}{T_k} = \frac{20k+13}{20k+33}\,\frac{2k+1}{12(k+1)}x_0, \qquad (x_0 = 6^{-3/20}).$$

$$\Sigma T = \tfrac{3}{13}\,(.31204 + .010244 + .000797 + .000080 + .000009) = .07458,$$

$$\sqrt{\frac{5\pi}{2g}}\,\Sigma T = 0.03685,$$

$$\sqrt[3]{2}\int_{-a}^0 \frac{xdx}{\sqrt{1+x^3}} = \sum_0^\infty T_k \qquad \text{where now} \qquad T_0 = \tfrac{1}{4}, \cdots, \frac{T_{k+1}}{T_k} = \frac{3k+2}{3k+5}\cdot\frac{2k+1}{8(k+1)},$$

$$\Sigma T = .25 + .0125 + .001465 + .000222 + .000036 + .000007 + .000001 = .26423$$

$$\sqrt{\frac{3\pi x_0^2}{8g}}\,\Sigma T = 0.03865,$$

$$T = \frac{d}{\sqrt{a}\sin\phi}(0.1950 - 0.03685 + 0.03865) = \frac{0.1968d}{\sqrt{a}\sin\phi} \text{ (units feet and seconds).}$$

It may be noted that since $\pi dm = -2\rho a ds$ and $m = 2\rho r^2$, $2\pi r dr = -a ds$, so that

$$r^2 = \frac{d^2 x_0^2}{4} - \frac{as}{\pi},$$

and when $s = c$, at the end of the motion,

$$r^2 = \frac{d^2 x_0^2}{4} - \frac{ac}{\pi} = 0.$$

The roll of cloth, intact would roll $\pi d^2/4a$ feet down the plane in

$$\sqrt{\frac{3\pi d^2}{4ga\sin\phi}} = \frac{0.2713d}{\sqrt{a}\sin\phi} \text{ secs.}$$

II. Solution by William Hoover, Columbus, O.

Let $d = 2b$ (1), r = the radius of the roll after any time t from the beginning of motion, the width of the strip being taken as the unit, x = the distance the roll has moved down the plane, and ρ, k, the density and radius of gyration about the axis. Then we have plainly

$$\rho\pi(b^2 - r^2) = \rho a x. \tag{2}$$

This is an instance in which we must employ the general definition of momentum and consider the mass, as well as the velocity, of the moving body as variables.

Let F = the friction; then resolving along the inclined plane, for the linear motion we have

$$\frac{d}{dt}(\rho\cdot\pi r^2\cdot\dot{x}) = \rho\cdot\pi r^2 g\sin\varphi - F \tag{3}$$

and for the angular motion, taking moments about the axis of the roll, θ being the angular motion corresponding to x, and so

$$x = r\theta \cdots \quad (4), \qquad \frac{d}{dt}(\rho \cdot \pi r^2 \cdot k^2 \dot{\theta}) = rF. \tag{5}$$

We have $k^2 = r^2/2$ (6), and from (4), $\dot{\theta} = \dot{x}/r$ (7). Substituting (6) and (7) in (5) and eliminating F from the resulting equation and (3), developing the derivatives and reducing,

$$\ddot{x} + \frac{7}{3} \cdot \frac{\dot{r}}{r} \dot{x} = \frac{2g \sin \phi}{3}. \tag{8}$$

Now, from (2), $\dot{x} = -(2\pi r/a)\dot{r}$ (9), $\ddot{x} = -(2\pi/a)(\dot{r}^2 + r\ddot{r})$ (10), and substituting these in (8) and arranging,

$$\ddot{r} + \frac{10}{3r} \dot{r}^2 = -\frac{ag \sin \varphi}{3\pi r}. \tag{11}$$

Putting $\dot{r}^2 = y$ (12), (11) is thrown into the form $(dy/dr) + Py = Q$ (13), in which $P = 20/3r$ (14), and $Q = -(2ag \sin \varphi/3\pi r)$ (15), the integral (13) being

$$y = e^{-\int P dr} \int e^{\int P dr} Q + Ce^{-\int P dr}. \tag{16}$$

Substituting (14) and (15) in (16) and performing the operations indicated,

$$y = -\frac{2ag \sin \varphi}{3\pi} r^{-20/3} \cdot \frac{3}{20} r^{20/3} + Cr^{-20/3} = -\frac{2ag \sin \varphi}{3\pi} + Cr^{-20/3}. \tag{17}$$

But when $r = b$, $y = 0$, and, hence, $C = agb^{20/3} \sin \varphi/(30\pi)$, and (17) becomes

$$\dot{r}^2 = \frac{dr^2}{dt^2} = \frac{ag \sin \varphi}{30\pi} \left(\frac{b^{20/3} - r^{20/3}}{r^{20/3}} \right), \tag{18}$$

giving for the required time

$$t = \sqrt{\frac{30\pi}{ag \sin \varphi}} \int_0^b \frac{r^{10/3} dr}{\sqrt{b^{20/3} - r^{20/3}}}. \tag{19}$$

The last factor may be put into the form,

$$I = \int_0^b \frac{r^{10/3}}{b^{10/3}} \left(1 + \frac{r^{20/3}}{2b^{20/3}} + \frac{3}{8} \frac{r^{40/3}}{b^{40/3}} + \cdots \right) dr, \tag{20}$$

the series appearing to converge suitably for the degree of approximation for t in (19) by the conditions of the problem, the result though differing from that stated in the problem.

347 (Mechanics). Proposed by E. B. ESCOTT, Kansas City, Mo.

A cord $ABCD$ is suspended from points A and D which are 20 feet apart in horizontal distance. D is 4 feet lower than A. At B and C are suspended weights 100 and 200 lbs. $AB = 8$ feet, $BC = 10$ feet, $CD = 12$ feet. Find angles α, β, γ made by AB, BC, CD, respectively, with the horizontal. Also find the tensions T_1, T_2, T_3 in AB, BC, CD.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

For the equilibrium of $w_1 = 100$, resolving vertically and horizontally,

$$T_1 \sin \alpha - T_2 \sin \beta = w_1 = 100,$$

$$T_1 \cos \alpha = T_2 \cos \beta;$$

similarly for $w_2 = 200$,

$$T_3 \sin \gamma + T_2 \sin \beta = w_2 = 200,$$

$$T_3 \cos \gamma = T_2 \cos \beta.$$

The sums of the horizontal and vertical projections of AB, BC, CD give

$$8 \cos \alpha + 10 \cos \beta + 12 \cos \gamma = 20,$$

$$8 \sin \alpha + 10 \sin \beta - 12 \sin \gamma = 4.$$

These six equations furnish the theoretical solution of the problem.

The PROPOSER furnished a complete solution.

348 (Mechanics). Proposed by ALTON L. MILLER, Ann Arbor, Michigan.

If equilateral triangles be constructed on the sides of any triangle, their centers are the vertices of a new equilateral triangle. Show that the center of gravity of this new equilateral triangle coincides with the center of gravity of the original triangle.

SOLUTION BY EMMA M. GIBSON, Springfield, Mo.

Let ABC be the given triangle and let the coördinates of the vertices A, B, C referred to the rectangular axes ox and oy be (c, o), (o, b), (a, o), respectively. The equation of the line through (a, o) and (o, b) is

$$y = -\frac{b}{a}x + b. \tag{1}$$

The line from D, the third vertex of the equilateral triangle on BC, through $(a/2, b/2)$ and perpendicular to (1) is

$$y = \frac{a}{b}x + \frac{b^2 - a^2}{2b} \tag{2}$$

The line through C making an angle of 60° with (1) is

$$y = \frac{b + a\sqrt{3}}{b\sqrt{3} - a}(x - a). \tag{3}$$

Solving equations (2) and (3), the values of the coördinates of D are found to be $[(a + b\sqrt{3})/2, (b + a\sqrt{3})/2]$.

Similarly the coördinates of F and E are found to be $[(a + c)/2, \sqrt{3}(c - a)/2]$ and $[(c - b\sqrt{3})/2, (b - c\sqrt{3})/2]$, respectively.

Now the centers H, G, I of the three equilateral triangles are

$$\left(\frac{3a + b\sqrt{3}}{6}, \frac{3b + a\sqrt{3}}{6}\right), \quad \left(\frac{a + c}{2}, \frac{\sqrt{3}(c - a)}{6}\right), \quad \left(\frac{3c - b\sqrt{3}}{6}, \frac{3b - c\sqrt{3}}{6}\right),$$

respectively, since $\bar{x} = \frac{1}{3}(x_1 + x_2 + x_3)$, $\bar{y} = \frac{1}{3}(y_1 + y_2 + y_3)$. These points are the vertices of the new triangle and by the formula for the length of a line between two points, the three sides are proved equal. Hence, the new triangle is equilateral.

The coördinates of the center of gravity of the original triangle are $\bar{x} = \frac{1}{3}[a + c]$, $\bar{y} = \frac{1}{3}b$. The coördinates of the center of gravity of the new triangle are

$$\bar{x} = \frac{1}{3}\left[\frac{a + b\sqrt{3}}{2} + \frac{a + c}{2} + \frac{c - b\sqrt{3}}{2}\right] = \tfrac{1}{3}(a + c),$$

$$\bar{y} = \frac{1}{3}\left[\frac{b + a\sqrt{3}}{2} + \frac{\sqrt{3}(c - a)}{2} + \frac{b - c\sqrt{3}}{2}\right] = \tfrac{1}{3}b,$$

which are the same as those obtained for the original triangle.

Also solved by HORACE OLSON and ROGER JOHNSON.

268 (Number Theory). Proposed by FRANK IRWIN, University of California.

Show that in any arithmetical progression, whose first term a_1 and common difference d are positive integers, any required number of consecutive terms may be found, no one of which is a prime number.

SOLUTION BY B. F. YANNEY, College of Wooster, Ohio.

Suppose that the theorem is not true, and that n is the greatest number of consecutive terms in the progression no one of which is a prime.

Consider any $n + 1$ consecutive terms of the progression, as

$$A_1 = a_1 + kd,\ A_2 = a_1 + (k+1)d,\ A_3 = a_1 + (k+2)d,\ \cdots,\ A_{n+1} = a_1 + (k+n)d.$$

Set $M = A_1A_2A_3\cdots A_{n+1}$. Then will $A_1' = A_1(M+1)$, $A_2' = A_1(M+1) + d$, $\cdots$, $A_{n+1}' = A_1(M+1) + nd$ be $n+1$ consecutive terms of the progression, no one of which is a prime. For consider any one of them, as

$$A_{r+1}' = A_1(M+1) + rd = (a_1 + kd)(M+1) + rd = (a_1 + kd + rd)(M+1) - rdM,$$

which is evidently not prime, since M is a multiple of $a_1 + kd + rd$. We are thus led to a contradiction. Hence the denial of the theorem must be withdrawn, and the theorem is true.

Also solved by H. N. CARLETON, ELIJAH SWIFT, HORACE OLSON and LOUIS CLARK.

269 (Number Theory). Proposed by ARTEMAS MARTIN, Washington, D. C.

Find three rectangular parallelepipedons whose edges are rational whole numbers, and whose solid diagonals are equal, and rational whole numbers.

I. SOLUTION BY THE PROPOSER.

Let w, x and y denote the lengths of the edges, and z the solid diagonal, of any one of the three required solids; then we must have

$$z^2 = w^2 + x^2 + y^2.$$

Put $x = np$, $y = nq$, $z = w + nr$; then

$$(w + nr)^2 = w^2 + (np)^2 + (ny)^2 = w^2 + 2nrw + n^2r^2,$$

which gives, after dividing by n,

$$w = \frac{n(p^2 + q^2 - r^2)}{2r}, \qquad \text{and} \qquad z = \frac{n(p^2 + q^2 + r^2)}{2r}.$$

Now take $n = 2r$ and we get the integral values

$$z = p^2 + q^2 + r^2,\quad w = p^2 + q^2 - r^2,\quad x = 2pr,\quad y = 2qr,$$

for one of the solids. The other two solids are obtained by interchanging the values of p, q, r in the expressions for w, x, and y.

Hence

$$\begin{aligned}(p^2 + q^2 + r^2)^2 &= (p^2 + q^2 - r^2)^2 + (2pr)^2 + (2qr)^2 \\ &= (p^2 + r^2 - q^2)^2 + (2pq)^2 + (2qr)^2 = (q^2 + r^2 - p^2)^2 + (2pq)^2 + (2pr)^2.\end{aligned}$$

Take $p = 4$, $q = 2$, $r = 1$; then the solids are

21, 19, 8, 4; 21, 16, 13, 4; 21, 16, 11, 8 (diagonals 21).

Take $p = 4$, $q = 3$, $r = 2$; then they are

29, 21, 16, 12; 29, 24, 12, 11; 29, 24, 16, 3 (diagonals 29).

The values of p, q, r may be chosen at pleasure.

II. SOLUTION BY C. F. GUMMER, Queen's University, Kingston.

We have to find three solutions of the Diophantine equation,

$$r^2 - z^2 = x^2 + y^2, \tag{1}$$

having the value of r in common.

From the theory of the form $x^2 + y^2$, the general solution of (1) is given by

$$r + z = m(a^2 + b^2), \qquad r - z = m(c^2 + d^2),$$

$$x = m(ac + bd), \qquad y = m(ad - bc),$$

where at most one of the integers a, b, c, d is zero, and either m or $a^2 + b^2 + c^2 + d^2$ is even. It follows that

$$r = \tfrac{1}{2}m(a^2 + b^2 + c^2 + d^2), \qquad z = \tfrac{1}{2}m(a^2 + b^2 - c^2 - d^2).$$

By permutation of a, b, c, d we obtain six solutions (generally distinct) with r in common from which three may be selected with no other value in common.

Thus, if $m = 2$, $a = 5$, $b = 3$, $c = 1$, $d = 2$, we get for x, y, z the sets of values

22, 14, 29; 34, 2, 19; 26, 26, 13;

26, 2, 29; 22, 26, 19; 34, 14, 13;

of which the first or last three are entirely distinct, and in each case the sum of the squares is 39^2.

Evidently, by taking for r a value having various divisors expressed in the form $a^2 + b^2 + c^2 + d^2$, we can obtain any desired number of rectangular parallelepipeds with integral sides and a common integral diagonal.

Also solved by Elijah Swift, B. F. Yanney, H. C. Feemster and J. L. Riley.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

DISCUSSIONS.

I. Relating to the Derivation of the Distance Formula.

By H. T. Burgess, University of Wisconsin.

The derivation of the distance formula independent of Hesse's "normal form" as presented by Mr. Mathews on page 476 of the December Monthly appeals to me as an excellent idea. Hesse's form in this connection may well be relegated to Professor Miller's collection of "Obsoletes." [1]

Since the form $y = mx + b$ includes all lines not parallel to the Y-axis and is the only one ever needed in ordinary straight-line problems, I suggest the derivation below as a possible alternative.

Given a point $P = (x_1, y_1)$ and a line $y = mx + b$, to find the distance d from the point to the line.

Write the equation of the parallel through P in the form $y = mx + b'$. When a drawing is made, it is obvious that, numerically,

$$d = (b' - b) \cos \alpha = \frac{b' - b}{\sqrt{1 + m^2}},$$

where $\alpha = \arctan m$.

If one must be conventional, we have

$$d = \frac{b' - (\pm b)}{\pm \sqrt{1 + m^2}} = \frac{y_1 - mx_1 - (\pm b)}{\pm \sqrt{1 + m^2}}$$

to take care of the sign of d.

[1] *Cf.* pp. 453–456 in the December, 1917, Monthly.

II. Relating to the Demonstration of a Geometrical Theorem.

By William E. Heal, Washington, D. C.

Prof. H. E. Slaught has informed me that Mr. W. J. Greenstreet of London has called his attention to the fact that the geometrical theorem (if the bisectors of two angles of a triangle are equal, the triangle is isosceles) demonstrated in the Monthly for September, 1917, page 344, is not necessarily true if the bisectors of the exterior angles be taken into account, and has suggested that it would be of interest to readers of the Monthly to discuss the question fully.

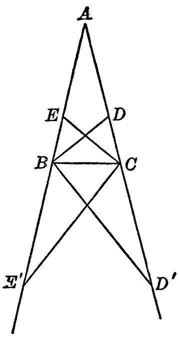

Fig. 1.

I will divide the discussion into four parts.

Case I. When the bisectors of the interior base angles are equal to each other. This is the case already discussed in the Monthly and it is shown that the triangle is isosceles.

Case II. When the bisectors of the external base angles are equal to each other. In Fig. 1 let, for brevity, $AE = a$, $AE' = a'$, $EB = b$, $E'B = b'$, $AD = c$, $AD' = c'$, $DC = d$, $D'C = d'$, $EC = f$, $E'C = f'$, $DB = g$, $D'B = g'$, $BC = h$.

We have

$$AB \times BC = AD' \times D'C - D'B^2,$$

$$AC \times BC = E'B \times E'A - E'C^2.$$

If $E'C = D'B = f' = g'$ we have

$$(a' - b')h = c'd' = f'^2, \qquad (c' - d')h = a'b' - f'^2$$

$$(a' - b')h - c'd' = (c' - d')h - a'b'.$$

Also

$$AB : BC = D'A : D'C, \qquad AC : BC = E'A : E'B,$$

$$(a' - b') : h = c' : d', \qquad (c' - d') : h = a' : b',$$

$$c'h = (a' - b')d', \tag{1}$$

$$a'h = (c' - d')b'. \tag{2}$$

Substituting the values of $a'h$, $c'h$ above and dropping $b'd'$ from both sides,

$$b'c' - b'h - c'd' = a'd' - d'h - a'b', \qquad b'(a' + c' - h) = d'(a' + c' - h).$$

Hence $b' = d'$, since $(a' + c') > [(a + b) + (c + d)] > h$.

From this we have by (1) and (2)

$$\frac{a'}{c'} = \frac{c' - b'}{a' - b'}, \qquad \text{or} \qquad a'^2 - a'b' = c'^2 - b'c', \qquad (a' + c')(a' - c') = b'(a' - c');$$

hence we have $a' = c'$ and $b' = a' + c'$.

The last supposition is absurd and hence we have again the triangle isosceles.

Case III. Suppose

$$BD = CE', \quad AC \times BC = AE' \times BE' - CE'^2, \quad AB \times BC = AD \times CD + BD^2,$$

$$(c + d)h = a'b' - f'^2, \tag{3}$$

$$(a' - b')h = cd + f'^2. \tag{4}$$

Also

$$AB : BC = AD : CD, \qquad AC : BC = E'A : E'B,$$

$$ch = (a' - b')d, \tag{5}$$

$$a'h = (c + d)b', \tag{6}$$

$$(c + d)h + (a' - b')h = a'b' + cd.$$

Introducing the values of $a'h$ and ch into (3) and (4) and reducing, we have

$$b'(c - h - a') = d(c - h - a'). \tag{7}$$

Hence

$$b' = d \text{ if } c \neq (a' + h);$$

$BDCE'$ is a parallelogram and the vertex A of the triangle ABC is at infinity.

If $c = a' + h$ we have from (7) the relation

$$\frac{b'}{d} = \frac{c - h - a'}{c - h - a'} = \frac{0}{0}$$

and we cannot infer $b' = d$.

If the triangle ABC is isosceles we have, dividing (5) by (6),

$$\frac{a' - b'}{c + d} = \frac{b'c}{a'd} = 1; \qquad \text{or} \qquad a' : b' = c : d.$$

The triangles ABD, $AE'C$ are similar and BD is parallel to $E'C$ and they cannot be equal. Therefore the triangle ABC is not isosceles.

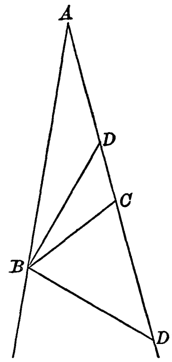

FIG. 2.

Case IV. Suppose $BD = BD'$.

This case can be very easily disposed of by noting that, since BD and BD' (Fig. 2) are at right angles to each other the triangle DBD' must be one half the square on BD with DD' as diagonal. Extend the diagonal in the direction $D'D$. Let any two lines from the point B, making equal angles with BD, meet $D'D$ in the points C and A. The triangle ABC evidently satisfies the conditions of the theorem and is not generally isosceles.

III. Relating to the Law of Cosines for a Polygon.

By F. M. Morgan, Dartmouth College, Hanover, N. H.

The proof of the law of cosines for a plane triangle, as generally given in the texts on trigonometry, does not lend itself readily to a generalization that will

apply to a polygon either plane or skew. If, however, we make use of the theorems on projection in proving the law for a plane triangle, we can readily extend our proof to apply to the above configuration. Moreover, from the equations used in proving this general law, an identical relation among the angles of the polygon can be deduced.

Let us first derive the law of cosines for a plane triangle using the theory of projection. If we denote the sides of the triangle by the letters a_1, a_2, a_3, respectively, we have

$$\begin{aligned} a_1 &= a_2 \cos(a_1a_2) + a_3 \cos(a_1a_3), \\ a_2 &= a_1 \cos(a_2a_1) + a_3 \cos(a_2a_3), \\ a_3 &= a_1 \cos(a_3a_1) + a_2 \cos(a_3a_2). \end{aligned} \tag{1}$$

Let us now multiply these equations by $a_1, -a_2, -a_3$, respectively, and add, remembering that $\cos(a_ia_j) = \cos(a_ja_i)$. We then obtain

$$a_1^2 = a_2^2 + a_3^2 - 2a_2a_3 \cos(a_2a_3)$$

which is the well-known law of cosines.

If we eliminate a_1, a_2, a_3 from equations (1) we obtain

$$\begin{vmatrix} -1 & \cos(a_1a_2) & \cos(a_1a_3) \\ \cos(a_1a_2) & -1 & \cos(a_2a_3) \\ \cos(a_1a_3) & \cos(a_2a_3) & -1 \end{vmatrix} = 0, \tag{2}$$

which when simplified gives

$$\cos^2(a_1a_2) + \cos^2(a_1a_3) + \cos^2(a_2a_3) + 2\cos(a_1a_2)\cos(a_1a_3)\cos(a_2a_3) = 1$$

as an identical relation between the angles of the triangle.[1]

Let us now consider a general polygon whose sides we will denote by $a_1, a_2, \cdots, a_n$ respectively. Then

$$\begin{aligned} a_1 &= a_2 \cos(a_1a_2) + a_3 \cos(a_1a_3) + \cdots + a_n \cos(a_1a_n), \\ a_2 &= a_1 \cos(a_2a_1) + a_3 \cos(a_2a_3) + \cdots + a_n \cos(a_2a_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n &= a_1 \cos(a_na_1) + a_2 \cos(a_na_2) + \cdots + a_{n-1} \cos(a_na_{n-1}). \end{aligned} \tag{3}$$

If we multiply these equations by $a_1, -a_2, -a_3, \cdots, -a_n$ respectively and add we obtain

$$a_1^2 = a_2^2 + a_3^2 + \cdots + a_n^2 - 2[a_2a_3 \cos(a_2a_3) + a_2a_4 \cos(a_2a_4) + \cdots + a_{n-1}a_n \cos(a_{n-1}a_n)],$$

which we will call the law of cosines for a polygon. Moreover it should be noted that the proof is perfectly general, *i. e.*, it applies to a polygon that is either concave or convex, plane or skew.

[1] Paterson, *Elementary Trigonometry*, Oxford, p. 168. Prove $\Sigma \cos^2 A = 1 - 2\Pi \cos A$.

If we eliminate the a's from equations (3) we obtain

$$\begin{vmatrix} -1 & \cos(a_1a_2) & \cos(a_1a_3) & \cdots & \cos(a_1a_n) \\ \cos(a_1a_2) & -1 & \cdots\cdots & \cdots & \cos(a_2a_n) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cos(a_1a_n) & \cdot & \cdot & \cdot & -1 \end{vmatrix} = 0 \qquad (4)$$

as an identical relation between the cosines of the angles. For a quadrilateral (4) becomes, if we denote $\cos(a_ia_j)$ by (i, j),

$$1 - \Sigma(ij)^2 + [(12)(34) - (13)(24) - (14)(23)]^2 - 2(12)[(13)(23) + (14)(24)] - 2(34)[(13)(14) + (23)(24)] = 0.$$

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB, University of Colorado, Boulder, Colo.

This club was organized in October, 1915, "to stimulate interest in mathematics among those who have had calculus." The total membership this year is 41 and the average attendance about 30. Professor George H. Light acts as chairman of the meetings and the following program for 1917–18 was arranged by him "with the assistance of club members," and issued in printed form.

November 20: "Non-Euclidean Geometry" by Leroy A. MacColl '19;
December 4: "Discovery of Logarithms" by Leona E. Vincent '19;
December 18: "Squaring the Hyperbola" by Ada G. Hall '18; "Probability in Arithmetic" by Henry A. Howell '18;
January 15: "Condition that $f(x, y, z)$ can be factored" by Agnes M. Wright '20;
February 5: "Applications for Vectors" by Claribell Kendall, instructor in mathematics;
February 19: "Nth Dimensions" by Lauren C. Hand '19;
March 5: "Relativity in Astronomy" by Edgar W. Wollard '20;
March 19: "American Mathematicians" by Dorothy Bair '20, and Alfreda Alenius '21;
April 2: "Proofs of Pythagoras's Theorem" by Lila Nelson '20; "Geometric Proof that $\sin 3A = 3 \sin A - 4 \sin^3 A$" by Oliver De Motte Sp.;
April 16: "Certain Definite Integrals" by Mildred McMillen '19;
May 7: "Curve Tracing" by Anthony J. Killgore '20;
May 21: "Famous Problems in Mathematics" by Gussie Wellman '21.

THE MATHEMATICS CLUB, Harvard University, Cambridge, Mass.

At least as far back as 1898 there flourished at Harvard an organization known as The Mathematical Conference, in which all students pursuing advanced studies in mathematics were invited to take part. The conference was "intended for the presentation and discussion of work done in courses of reading and research, of articles in the mathematical journals and of other suitable matter, and for the meeting of instructors and students." The meetings were held twice a month. In 1902 it was announced that "the direction of the Conference is in the hands of a committee of graduate and undergraduate students, acting with the advice and assistance of the Division of Mathematics." In the following year the number of meetings was reduced to one a month and it was officially published that "pains will be taken to provide for the discussion of a fair proportion of subjects within the capacity of members of the younger undergraduate classes. An important object aimed at is the free and general interchange of views between students and instructors. . . . Students of Radcliffe College are cordially asked to be present and to contribute to the discussions."

In the autumn of 1904 the Mathematical Conference resolved itself into The Mathematical Club, of which the first officers were: President, William H. Roever; secretary and treasurer, Ralph B. Stone. These officers together with Dr. Julian L. Coolidge, instructor in mathematics, constituted the executive committee. The guiding principles now are as when reorganized thirteen years ago. "Meetings are held once a fortnight, in the evenings, and simple refreshments are served after the more formal part of the meetings. Any student who has taken or is taking a course in mathematics not regularly open to freshmen is eligible for membership in the club; and a special effort is made to have a good proportion of papers which shall be of interest to students taking courses of intermediate grade. A small membership fee is charged[1] to defray expenses. Radcliffe students are not eligible for membership." ·

The officers for 1917–18 are: President, Ralph Keffer Gr. (resigned at mid-years to enter war service), succeeded by John P. Ballantine '18; secretary and treasurer, Joseph L. Walsh Gr. (resigned at mid-years to enter war service), succeeded by D. S. Morse Gr. These officers, together with the faculty adviser, Dr. Gabriel M. Green, constitute the executive committee.

The following programs were given in 1917–18:

October 17: "History of Elementary Trigonometry" by Professor Maxime Bôcher;

November 7: "Curve Smoothing" by Lester R. Ford, instructor in actuarial mathematics;

November 21: "Solution of linear algebraic Equations in infinitely many Variables" by Joseph L. Walsh Gr.;

December 5: "On the Consistency and Equivalence of regular Transforma-

[1] The "small" fee is three dollars, an amount in excess of that charged at any of 40 other similar clubs in America.

tions" by Louis L. Silverman, instructor in mathematics at Cornell University;

December 19: "Rigid Motions in Space" by Professor Dunham Jackson;

January 9: "Some Methods of Interpolation" by John P. Ballantine '18;

February 20: "Finite Mathematics" by Henry M. Sheffer, instructor in philosophy.

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City.

This club of young women was organized in 1910 "as the result of a desire on the part of both the teaching and student bodies to investigate matters connected with mathematics, to study the phases of mathematical development which are crowded out of classroom work, and to keep the students in touch with the best thoughts of the times. It aims to be a source of profitable pleasure." Motto: Honor habet onus.

All students "matriculated in the mathematics department" are eligible for membership, which now totals 80.

Officers 1917–18: President, Rose Sigal '18; vice-president, Dorothea Boves '19; secretary, Anita Rosenthal '19; treasurer, Miriam Werner, instructor in mathematics. Program committee: Professor Louisa M. Webster,[1] Rose Sigal '18, Kathryn McSorley '19, Mildred C. Zwinge '20, Jessie Krosovitch '21.

The programs for 1917 were as follows:

January 12: "The Slide Rule" by Adelheit Steeneck Gr.; "Origin of Geometrical Terms—a Sketch" written by Rose Sigal '18 and presented by 14 members of the club;

February 19: Reception to Freshmen;

March 9: "Mathematics of Physics" by Rosetta Chess '16; "Facts about Figures" by Jessie Krosovitch '21;

April 13: "Resources of Research" by Henryk Arctowski, chief of the science division, New York Public Library; "The Horn Book" by Grace H. Davis '20 (Illustrative photographs taken from Andrew Tuer's History of the Horn Book);

May 11: Guests of the Classical Club; "Three M's, Mysterious, Mystic, Magic" by Emory B. Lease, professor of Latin in the College of the City of New York; election of officers.

September 14: Reception to Freshmen;

October 19: "History of Trigonometry" by 13 Freshmen;

November 9: Report, by the club representative, of the meeting of the New York Division of the Association of Mathematics Teachers of the Middle States and Maryland; "The Mathematician at the Breakfast Table—a paper on mathematical Fallacies" by Mildred C. Zwinge '20;

December 13: "Personal Equation solved by judicial Process or Labor Problem

[1] Miss Webster's article on "Mathematics Clubs" in *The Mathematics Teacher* for June, 1917 (Vol. 9, pp. 203–208), contains information not given in these notes concerning the club of Hunter College.

solved" written by Kathryn McSorley '19 and presented by the Junior Class members. (The sketch was based on "*A, B* and *C*" in S. Leacock's *Literary Lapses.*)

In 1918 there was no meeting during January. At the one in February, twelve members showed the relation between mathematics and costuming; eight geometrically constructed gowns were on exhibition.

Mathematical Club of Rockford College, Rockford, Ill.

So far as the editor knows this is the youngest of the undergraduate mathematics clubs, since it was organized only last October. It has 24 members.

Officers 1917–18: President, Estle Russell '18; vice-president, Dorothy Mandeville '20; secretary-treasurer, Aline Bartholomew '20.

The following programs have been given:

November 7: Opening address on the purpose of the club by the president;
November 21: "The History of Limits" by Marie S. Allen, instructor in mathematics;
December 21: A social meeting with readings from *Flatland, A Romance of Many Dimensions with Illustrations by the Author, A Square*;[1]
February 6: "Fourth Dimension" by Professor Bessie I. Miller, head of the department of mathematics.

The Junior Mathematical Club, University of Wisconsin, Madison, Wis.

As long ago as 1893 a Mathematics Club was organized at the University of Wisconsin for instructors, graduates, and seniors making mathematics their major. Its object was to follow important recent developments in mathematics. In March, 1912, the Junior Mathematical Club was founded and its meetings opened to all interested in mathematics. Most of the members, which now number 25, are Juniors and Seniors majoring in mathematics. The membership of the older organization, now called the "Senior Mathematical Club," is limited to members of the faculty and to graduate students.

Officers 1917–18: President, Florence Krieger '18; vice-president, Hilda Kieckhefer '19; secretary-treasurer, Kathryn Geiger '18. Program committee: Barbara Pearsall '19 (chairman), Frances McKay '18, Raymond Suchy '19.

Normally, meetings are held twice each month and are limited in length to one hour. The average attendance is about 18. In order that members may become better acquainted with each other and with members of the faculty during the year, there are a few social events such as picnics and hikes.

The following programs have been given in 1917–18:

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[1] That is, A^2 = Abbott Abbott? Edwin Abbott Abbott was born in London in 1838 and he was still living there when the material for *Who's Who, 1918* was being collected. Educated at St. John's College, Cambridge, he graduated as senior in the classical tripos. He is the author of over 40 works, many of which are volumes of sermons or deal with the Gospels. Nearly a score of books including *A Shakespearean Grammar* (1870) and *How to Write Clearly* (1872) appeared before the first, and best, edition of *Flatland* (in small 4to) was published at London in 1884. This is the only mathematical work which Mr. Abbott has acknowledged as his.

November 7: Minutes; business; "General Survey of the History of Mathematics: (*a*) Greek, (*b*) Arabic, (*c*) Renaissance and Modern" by Professor Arnold Dresden; "Archimedes, His Life and His Work" by Albert Kohlman '18;

November 19: Minutes; business; "Eudoxus and his Method of Exhaustions which compares to our Method of Limits" by Professor Linnæus W. Dowling; "Euclid, his Life, Anecdotes, and a General Description of his Works" by Barbara Pearsall '19;

December 5: Minutes; business; "Pappus and his modern Elements as we find them in Greek" by Raymond Suchy '19; "Ptolemy and Greek Trigonometry" by Anne Clark '18; "What we forgot to mention concerning Mathematics of the Greeks" by Professor Edward B. Van Vleck; discussion;

January 23: Minutes; business; "What is a straight Line?"—two-minute discussions (including various definitions of a straight line) by every one present.

TOPICS FOR CLUB PROGRAMS.

Perusal of the recent very remarkable book by D'Arcy Wentworth Thompson, *On Growth and Form*,[1] has suggested to the editor that his next selection of topics be made from those which come up in the study of certain fields of botany and biology. The logarithmic spiral which bulks large in such discussion (pages 493–586 in Thompson's book) was chosen as one topic. While the others, Golden Section, and A Fibonacci Series, are less known to the average mathematician, there is much of interest, historical and otherwise, connected with their consideration.

8. The Logarithmic Spiral.[2]

The first discussion of this spiral, in a letter written by Descartes to Mersenne in 1638, was based upon the consideration of a curve cutting radii vectores, ρ (drawn from a certain fixed point, O), under a constant angle, ϕ. Descartes made the very remarkable discovery that if B and C are two points on the curve its length from O to B is to the radius vector OB as the length of the curve from O to C is to OC;[3] whence $s = a\rho$,[4] where s is the length measured along the curve from the pole to the point (ρ, θ), and $a = \sec \phi$.[5] This leads to the polar equa-

[1] Cambridge: at the University Press, 1917. 16 + 793 pp.

[2] Historical sketches and some of the properties of the curve are given in F. Gomes Teixeira, *Traité des courbes spéciales remarquables*, tome 2, Coïmbre, Imprimerie de l'université, 1909, pp. 76–86, 396–399, etc.; in G. Loria, *Spezielle algebraische und transzendente ebene Kurven*, Band 2, 2. Auflage, Leipzig, Teubner, 1911, pp. 60 ff.; in *Mathematisches Wörterbuch* . . . angefangen von G. S. Klügel . . . fortgesetzt von C. B. Mollweide, Leipzig, Band 4_1, 1823, pp. 429–440.

[3] *Oeuvres de Descartes*, tome 2, publiées par C. Adam et P. Tannery. Paris, Cerf, 1898, p. 360. *Cf.* I. Barrow, *Lectiones Geometricae*, Londini, 1670, p. 124; or English edition by J. M. Child, London, Open Court, 1916, pp. 136–9.

[4] The intrinsic equation $s^m R = K$ represents a logarithmic spiral when $m = -1$, a clothoïde when $m = 1$, a circle when $m = 0$, the involute of a circle when $m = -\frac{1}{2}$ and a straight line when $m = \infty$.

[5] That is, the length of the arc measured from the pole is equal to the length of the tangent drawn at the extremity of the arc and terminated by the line drawn perpendicular to the radius vector at the pole.

tion (1) $\rho = ke^{c\theta}$, where k is a constant and $c = \cot\phi$. The pole O is an asymptotic point. The pole and any two points on the spiral determine the curve; for, the bisector of the angle made by the radii vectores of the points is a mean proportional between the radii. If $c = 1$ the ratio of two radii vectores corresponds to a number, and the angle between them to its logarithm; whence the name of the curve.

The logarithmic spiral has been called also the proportional spiral[1] (E. Halley, 1696) but more commonly, because of the property observed by Descartes, the equiangular spiral (Whitworth, 1862). Johann Bernoulli employed the term loxodrome which is now reserved for the spherical curve which cuts all meridians under a constant angle. To Edmund Halley is usually ascribed the discovery that the loxodrome is the stereographic projection of a logarithmic spiral.[2]

Torricelli studied the logarithmic spiral about the same time that Descartes did. He gave a definition which may be immediately translated into equation (1), and from it he obtained expressions for areas and lengths of arcs. These results were rediscovered by John Wallis[3] and published in 1659.

During 1691–93 Jacob Bernoulli gave the following results among others: (*a*) Logarithmic spirals defined by equations (1) for different values of k are equal and have the same asymptotic point; (*b*) the evolute of a logarithmic spiral is another equal logarithmic spiral having the same asymptotic point;[4] (*c*) the pedal of a logarithmic spiral with respect to its pole is an equal logarithmic spiral;[5] (*d*) the caustics by reflection and refraction of a logarithmic spiral for rays emanating from the pole as a luminous point are equal logarithmic spirals.

The discovery of such "perpetual renascence" of the spiral delighted Bernoulli. "Warmed with the enthusiasm of genius he desired, in imitation of Archimedes,

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[1] The lengths of segments cut off from a radius vector between successive whorls of the spiral form a geometric progression.

[2] *Philosophical Transactions*, 1696; but see two letters of Collins, one undated and the other dated Sept. 30, 1675, in *Correspondence of Scientific Men of the Seventeenth Century* . . . Vol. 1, Oxford, University Press, 1841, pp. 144, 218–19.

Cf. F. G. M., *Exercices de géométrie descriptive*, 4e éd., Paris, Mame, 1909, pp. 824–6. Chasles showed (*Aperçu historique*, etc., . . . 2e éd., Paris, 1875, p. 299) that if the logarithmic curve generates a surface by revolving about its asymptote, and if this asymptote is the axis of a helicoidal surface, the two surfaces cut in a skew curve whose orthogonal projection on a plane perpendicular to the asymptote is a logarithmic spiral. See also H. Molins, *Mémoires de l'académie des sciences inscriptions et belles-lettres de Toulouse*, tome 7 (sem. 2), 1885, p. 293 f.; tome 8, 1886, pp. 426. That the logarithmic spiral is a projection of a certain "elliptic logarithmic spiral" was shown in W. R. Hamilton, *Elements of Quaternions*, London, 1866, pp. 382–3. For other quaternion discussion of the logarithmic spiral see H. W. L. Hime, *The Outlines of Quaternions*, London, 1894, pp. 171–3.

[3] *Cf.* Turquan, "Démonstrations élémentaires de plusieurs propriétés de la spiral logarithmique," *Nouvelles Annales de Mathématiques*, tome 5, 1846, pp. 88–97.

[4] The center of curvature at a point on a logarithmic spiral is the extremity of the polar subnormal of the point.

[5] The nth positive pedal of the spiral $\rho = ke^{c\theta}$ with respect to the pole is

$$\rho = k \sin^n \phi e^{cn\left(\frac{\pi}{2}-\phi\right)} e^{c\theta}.$$

(J. Edwards, *Elementary Treatise on the Differential Calculus*, 3d edition, London, Macmillan, 1896, p. 167.)

to have the logarithmic spiral engraved on his tomb, and directed, in allusion to the sublime tenet of the resurrection of the body, this emphatic inscription be affixed—Eadem mutata resurgo."[1]

The logarithmic spiral appears in two propositions of Newton's *Principia* (1687).[2] From the first there develops that if the force of gravity had been inversely as the cube, instead of the square, of the distance, the planets would have all shot off from the sun in "diffusive logarithmic spirals."[3] In the second proposition Newton showed that the logarithmic spiral would also be described by a particle attracted to the pole by a force proportional to the square of the density of the medium in which it moves, while this density is at each point inversely proportional to its distance from the fixed center. This latter proposition was generalized by Jacob Bernoulli.

Cremona's discussion of the logarithmic spiral, and how it may serve, when drawn, for the solution of problems involving extraction of roots[4] (higher than the second) should not be forgotten. Then, too, there is the little known but notable paper, published by James Clerk Maxwell when only 18 years of age,[5] which contains several properties of logarithmic spirals. Some quotations follow:

Page 524 [10]: "The involute of the curve traced by the pole of a logarithmic spiral which rolls upon any curve is the curve traced by the pole of the same logarithmic spiral when rolled on the involute of the primary curve."

Page 529 [16]: "The method of finding the curve which must be rolled on a circle to trace a given curve is mentioned here because it generally leads to a double result, for the normal to the traced curve cuts the circle in two points, either of which may be a point in the rolled curve.

"Thus, if the traced curve be the involute of a circle concentric with the given

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[1] *Cf. Acta eruditorum*, 1692, p. 212.

[2] Book I, proposition 9, and book II, proposition 15.

[3] The hodograph of an equiangular spiral is an equiangular spiral (W. Walton, *Collection of Problems in Illustration of the Principles of Theoretical Mechanics*, 3d ed., Cambridge, 1876, p. 296). In a chapter on electromagnetic observations in J. C. Maxwell's *Treatise on Electricity and Magnetism* (Vol. 2, Oxford, Clarendon Press, 1873, pp. 336–8) the discussion calls for the investigation of the motion of a body subject to an attraction varying as the distance and to a resistance varying as the velocity. This leads to the reproduction of Tait's application (*Proc. Royal Society of Edinburgh*, Vol. 6, 1869, p. 221 f.) of the principle of the hodograph to investigate this kind of motion by means of the logarithmic spiral.

"If a particle be describing a logarithmic spiral under the action of a force to the pole, and simultaneously the law of force be altered to the inverse biquadrate and the velocity to $\sqrt{\frac{2}{3}}\times$ its previous value, the particle will proceed to describe a cardioide." Purkis's *Messenger of Mathematics*, Vol. 2, 1864. For other results of this type, involving the spiral, see Newton's *Principia*, first book, Sections I–III, with notes and illustrations by P. Frost, London, 1880, p. 203.

[4] L. Cremona, *Graphical Statics*. Translated by T. H. Beare, Oxford, Clarendon Press, 1890, pp. 59–64. Italian edition, Torino, 1874, pp. 39–42. The xylonite logarithmic curve (eight inches in width) sold by Keuffel & Esser Co., New York, furnishes the means for accurately and rapidly drawing the curve. The curvature gradually changing it is peculiarly adapted for fitting to any part of a given curve. It assists in the rapid determination of the center of curvature of a given part of the curve, and, hence, in drawing evolutes and equidistant curves.

[5] "On the Theory of Rolling Curves," *Transactions of the Royal Society of Edinburgh*, Vol. 16, part V, 1849, pp. 519–40. [*The Scientific Papers of J. C. Maxwell*, edited by W. D. Niven, Vol. 1, Cambridge, 1890, pp. 4–29.] Loria, Gomes Teixeira, and Wieleitner seem to be equally ignorant of this paper.

circle, the rolled curve is one of two similar logarithmic spirals." (Often attributed to Haton de la Goupillière.)

Page 532 [19]: "If any curve be rolled on itself, and the operation repeated an infinite number of times, the resulting curve is the logarithmic spiral." The curve which being "rolled on itself traces itself is the logarithmic spiral."

Page 535 [23]: "When a logarithmic spiral rolls on a straight line the pole traces a straight line which cuts the first line at the same angle as the spiral cuts the radius vector." (Often attributed to Catalan.)

Among many other results the following may be noted: If a logarithmic spiral roll on a straight line the locus of the center of curvature of the point of contact is another straight line (A. Mannheim, 1859)—The involutes of a logarithmic spiral are equal spirals—The inverse of a logarithmic spiral with respect to its pole is an equal spiral with the same pole—Coplanar logarithmic spirals and their orthogonal trajectories, which are again coplanar logarithmic spirals, come up (1) in the discussion of loxodromic substitutions[1] and (2) in conformal representations.[2] As a consequence of a general theory relative to linear transformations F. Klein and S. Lie obtained the following result:[3] The logarithmic spiral is its own polar reciprocal with respect to each of the equilateral hyperbolas with center at the pole and tangent to the spiral.

The most practical form of a ship's anchor was discussed in 1796 by F. H. Chapman, vice-admiral in the Swedish Marine.[4] He found that the best form for each of the barbed arms would be an arc of a logarithmic spiral cutting the shank of the anchor at an angle of 67° 30′.

The first definite suggestion connecting the logarithmic spiral with organic spirals seems to have been made by Sir John Leslie in his *Geometrical Analysis and Geometry of Curve Lines*.[5] After proving that the involutes of a logarithmic spiral are logarithmic spirals he remarks: "The figure thus produced by a succession of coalescent arcs described from a series of interior centers exactly resembles the general form and the elegant *septa* of the *Nautilus*."[6] The aptness of this remark has been long since established. One of the earliest mathematical discussions of organic logarithmic spirals was by Canon Moseley, "On the Geometrical Forms of Turbinated and Discoid Shells"[7]—a paper of 80 years ago

[1] F. Klein–R. Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, Band I, Leipzig, Teubner, 1890, p. 168.

[2] G. Holzmüller, *Einführung in die Theorie der isogonalen Verwandtschaften und der conformen Abbildung*, Leipzig, Teubner, 1882, pp. 65, 238–241.

[3] *Mathematische Annalen*, Band 4, 1871, p. 77. *Cf. Encyklopädie der mathematischen Wissenschaften*, Band III_3, Leipzig, 1903, pp. 210, 212; also Clebsch-Lindemann, *Vorlesungen über Geometrie*, Band I, Leipzig, Teubner, 1876, p. 995.

[4] "Om rätta Formen på Skepps-Ankrar," *Svensk. Vetensk. Academ. nya Handl.*, 1796, Vol. 17, pp. 1–24. Abridged and translated in *Annalen der Physik* (Gilbert), Band 6, Halle, 1800: "Von der richtigen Form der Schiffsanker," pp. 81–95.

[5] Edinburgh, 1821, p. 438.

[6] For pictures of the nautilus pompilius see pp. 494, 581, 582 of Thompson's book and also T. A. Cook, *The Curves of Life*, London, Constable, 1914, pp. 57, 457. This latter work contains many beautiful illustrations and logarithmic spiral forms are specially discussed on pages 60–63, 413–421; another work by the same author, *Spirals in Nature and Art*, London, Murray, 1903, has some good illustrations.

[7] *Philosophical Transactions of the Royal Society*, London, 1838, Vol. 128, pp. 351–370.

which is one of the classics of natural history. In "turbinate" shells we are no longer dealing with a plane spiral as in the nautilus but with a gauche spiral on a right circular cone cutting the generators at a constant angle and such that along a generator the line-segments between successive whorls form a geometric progression.[1] For mathematical and other details of Moseley's work as well as of that of many others, on univalve and bivalve shells, Thompson's book, with its many exact references to the literature of the subject, should be consulted. One notable work which Thompson appears to have overlooked is Haton de la Goupillière, "Surfaces Nautiloïdes."[2]

In the field of leaf arrangement or phyllotaxis discussion of the theories of A. H. Church[3] and Cook evolved from observations of arrangements in logarithmic spirals of florets of sunflowers, pine cones, and other growths, should be read in connection with Thompson's criticisms. The fine sunflower photograph by H. Brocard[4] ought to be compared with those by Church.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Assistant Professor H. M. SHOWMAN, of Colorado School of Mines, has been promoted to a professorship of mechanics and engineering.

Assistant Professor S. T. SANDERS has been made head of the department of mathematics at Louisiana State University, and Dr. I. C. NICHOLS has been appointed associate professor of mathematics.

Assistant Professor L. P. SICELOFF, of the department of mathematics of Columbia University, has entered the Y. M. C. A. service, and will shortly embark for France.

Professor J. L. COOLIDGE, of Harvard University, has received a commission as major in the Ordnance Department of the National Army and is in active service. Dr. L. R. FORD, instructor in actuarial mathematics, has also entered the military service.

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[1] As early as 1701 Guido Grandi showed that the orthogonal projection of this spiral on a plane perpendicular to the axis of the cone is a logarithmic spiral. The gauche spiral has been studied by Th. Olivier (who called it the conical logarithmic spiral), *Developpements de géométrie descriptive*, 1843, pp. 56–76; by P. Serret, *Théorie nouvelle géométrique et mécanique des lignes à double courbure*, 1860, p. 101; etc. A number of results are collected by Gomes Teixiera, *l. c.*, pp. 396–400.

For other surfaces involving the logarithmic spirals reference should be given to the very interesting pages 232–313 of G. Holzmüller, *Elemente der Stereometrie*, Dritter Teil, Leipzig, Göschen, 1902, on logarithmic spiral tubular surfaces and their inverses.

[2] This occupies almost the whole of the third volume of *Annaes scientificos da academia polytechnica do Porto*, Coïmbra, 1908. *Cf. L'Intermédiaire des mathématiciens*, 1900, tome 7, p. 40; tome 8, pp. 167, 314; tome 17, p. 155.

[3] A. H. Church, *On the Relation of Phyllotaxis to Mechanical Law*, London, Williams and Norgate, 1904.

[4] In *L'Intermédiaire des mathématiciens*, 1909, and in H. A. Naber, *Das Theorem des Pythagoras*, Haarlem, Visser, 1908, opposite p. 80.

Mr. R. A. HARRIS, for twenty-eight years mathematician to the Coast and Geodetic Survey, died on January 17, 1918, at the age of fifty-four years.

The Rev. A. T. G. APPLE, for the past eleven years professor of mathematics and astronomy in Franklin and Marshall College, died on February 5, at the age of fifty-eight years.

Dr. CHRISTIAN HORNUNG, for many years professor of mathematics at Heidelberg University, Tiffin, Ohio, and a charter member of the Association, died on January 31, 1918, aged seventy-three years. Following Professor Hornung's death, Mr. H. L. OLSON was appointed instructor in mathematics.

During the absence of Director R. H. CHITTENDEN, of the Sheffield Scientific School, who has gone to Europe on an important Government Commission, Professor P. F. SMITH, of the department of mathematics, will be acting director.

The *Colorado College Publications*, issued in November, 1917, contains a valuable historical paper on "Newton's solution of numerical equations by means of slide rules," by Professor F. CAJORI, former president of the Association.

UNIVERSITY OF ILLINOIS. Summer Session, June 18 to August 9.—By Professor SISAM: Differential geometry, Solid analytic geometry.—By Professor SHAW: Vector methods, Differential equations.—By Dr. KEMPNER: Advanced algebra. The usual elementary courses are also offered.

INDIANA UNIVERSITY. Summer Session, June 18 to August 9.—By Professor DAVISSON: Non-Euclidean geometry, Mathematical theory of investment.—By Professor ROTHROCK: Advanced integral calculus, Analytical and spherical trigonometry.—By Professor HANNA: Differential equations, Calculus. The usual elementary courses in college algebra, trigonometry and analytic geometry will also be offered.

At the Summer Session of the University of Colorado work in mathematics will be offered by Professor G. H. LIGHT, Professor B. F. FINKEL, of Drury College, and Professor A. COHEN, of Johns Hopkins University. Courses are offered in the elementary mathematics, teachers' course, calculus, differential equations, least squares, Fourier series, projective geometry, differential geometry, Galois' theory of equations, and definite integrals.

UNIVERSITY OF WISCONSIN. Summer Session.—By Professor E. B. VAN VLECK: Linear substitutions, Survey of elementary mathematics, and Analytical geometry.—By Professor E. B. SKINNER: Theory of surfaces and twisted curves (3 hrs.), The teaching of secondary mathematics (5 hrs.), and Algebra.—By Professor H. W. MARCH: Theory of probabilities (3 hrs.), Mechanics (5 hrs.), Calculus (5 hrs.).—By Dr. T. M. SIMPSON: Equations of third and higher degrees (3 hrs.), Integral calculus (12 hrs.).—By Mr. MOORE: Differential equations (5 hrs.), Trigonometry, and Solid geometry.

THE UNIVERSITY OF CHICAGO. Summer Quarter, June 17–August 31, 1918. —By Professor E. H. MOORE: Differential equations in general analysis (first half), four hours, College algebra (first half), five hours.—By Professor G. A. BLISS: Calculus of variations, four hours, Higher plane curves, four hours.—

By Professor L. E. DICKSON: Solution of numerical equations (first half), four hours, Determinants and symmetric functions (second half), four hours, Algebraic invariants, four hours.—By Professor H. E. SLAUGHT: Definite integrals, four hours, Integral calculus, four hours.—By Professor A. C. LUNN: Units and dimensions, four hours, Electromagnetic theory, four hours.—By Professor J. W. A. YOUNG: Topics in geometry, four hours, Plane analytic geometry, five hours.—By Professor R. G. D. RICHARDSON (Brown University): Theory of functions of a complex variable, four hours, Differential calculus, five hours.—By Professor W. H. ROEVER (Washington University, St. Louis): Descriptive geometry, five hours, Plane trigonometry, five hours.—By Professor G. W. MYERS: The teaching of secondary mathematics, School of Education, five hours.

At the meeting of the North Carolina Association of Teachers of Secondary Mathematics, held at Greensboro, February 1 and 2, addresses were made by Professor D. E. SMITH, of Columbia University, on "The origin of mathematics," on "Deficiencies in present preparatory mathematics," and a round-table discussion on "A proper approach to elementary mathematics." Papers were also presented by Miss FANNIE S. MITCHELL, of the Raleigh High School, by Miss VIRGINIA RAGSDALE, of Normal College, by Miss ELLA BRADLEY, of the Gastonia High School, and by Mr. J. W. LASLEY, of the University of North Carolina.

The annual Register of the Officers and Members of the American Mathematical Society appeared in January, showing the total membership of the Society to be 735, exactly the membership on January 1, 1917. During the year 1917, there were 29 persons admitted to membership, and also 29 withdrawals. Twenty-five members of the Society are reported as being in some form of government service connected with the war. The membership extends to every state of the Union except Delaware, Louisiana and South Carolina. There are 57 foreign members, of whom 46 reside in the British Empire.

Professor C. N. MOORE, University of Cincinnati, has been appointed a member of the National Committee on Mathematical Requirements in place of Professor O. VEBLEN, who has resigned on account of his duties in connection with government service.

The Mathematics Club of Chicago is a body of men who are teachers in the high schools of Chicago and neighboring towns, including also some teachers from Lewis Institute, Armour Institute, and the University of Chicago. They meet on the second Friday evening of each month for dinner at the City Club and spend the evening in discussions led by some chosen speaker. The Club has recently been addressed by Professors E. J. WILCZYNSKI and H. E. SLAUGHT. The April meeting will be addressed by Principal W. B. OWEN, of the Chicago Normal College, who is chairman of the committee of the National Education Association that is charged with reviewing the reports of the various departmental committees working under the Commission on the Reorganization of the High School Curriculum.

The departments of Chemistry, Physics and Mathematics of the University of Chicago are uniting through their representatives to give some special courses in preparation for war service. Professor A. C. LUNN is the representative of the department of mathematics in this work. Professor F. R. MOULTON will also give during the Spring Quarter a special course of this kind. Professor Moulton is spending a part of the Winter Quarter in advisory service at Fort Sill, Texas.

The Mathematical Association of Great Britain held its annual meeting at the London Day Training College on January 9 and 10, 1918. Papers were presented by Dr. W. P. MILNE, on "Graphical treatment of power series"; by Mr. G. GOODWILL, on "Suggestions for presenting mathematics in closer touch with reality"; and the presidential address by Professor P. T. NUNN, on "Mathematics and individuality." The meeting was closed by a general discussion on the position of mathematics in the proposed new course of study for secondary schools. The *Mathematical Gazette*, edited by Professor W. G. GREENSTREET, is the official organ of the Association; it appears six times a year from the publishing house of G. Bell & Sons, London. A number of valuable reports have been issued by the Association during the past ten years; these are enumerated below, with the price of each as announced by the publishers: (1) Reports on the teaching of elementary mathematics (geometry, arithmetic and algebra, mechanics, advanced school mathematics, entrance scholarship mathematics, mathematics in preparatory schools) (6d. net); (2) Report on the teaching of elementary algebra and numerical trigonometry (3d. net); (3) Report on the teaching of mathematics in preparatory schools (3d. net); (4) Report on the correlation of mathematical and science teaching (6d. net); (5) A general mathematical syllabus for non-specialists in the public schools (2d. net); and (6) Elementary mathematics in girls' schools (1s. net).

The Council of the Mathematical Association of America has elected the following to membership, on applications duly certified:

To individual membership.

Rev. Peter Archer, S.J. Director, Georgetown Coll. Observ. and Prof. of Math., Georgetown Univ., Washington, D. C.

Sir George Greenhill, M.A. Formerly Prof. of Math., Artillery Coll., Woolwich, Eng. 1 Staple Inn, London, W. C., Eng.

Y. L. Hsia, Ph.D. (Berlin). Director, Science Dept., Government Univ., Peking, China.

Carroll Johnson Ramage, Ph.D. (Grove City Coll.). Lawyer, Saluda, S. C.

Clarence Cornelius Van Nuys, A.M. (Columbia), E.M. (S.D. School of Mines). Prof. of Physics, Col. School of Mines, Golden, Col.

Gilbert F. Winslow, Jr., B.S. in Min. Engg. (Ore. Agric. Coll.). Computer, Coast and Geodetic Survey, Washington, D. C.

To institutional membership.

Georgetown Univ., Washington, D. C.

Knox Coll., Galesburg, Ill.

Denison Univ., Granville, Ohio.

W. D. CAIRNS, *Secy.-Treas.*

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Sample Problem

To solve $\frac{\partial^2\varphi}{\partial t^2} = \sqrt{1+(\Delta h)^2}\,\frac{\partial^2\varphi}{\partial x^2}$, put $m^2 = \sqrt{1+(\Delta h)^2}$ and assume $\varphi = \tau(t)\cdot\xi(x)$; so that

$\frac{\partial^2\varphi}{\partial t^2} = \frac{d^2\tau}{dt^2}\cdot\xi(x)$, and $\frac{\partial^2\varphi}{\partial x^2} = \tau(t)\,\frac{d^2\xi}{dx^2}$ Substituting these into the original equation, we find that the variables, t and x, can be separated by dividing through by $\tau\cdot\xi$ whereupon we have $\frac{d^2\tau}{dt^2}\div\tau = m^2\frac{d^2\xi}{dx^2}\div\xi$. Since the first of these two equal members cannot vary when t changes nor the second when x changes, both must remain equal to some constant, say $-m^2n^2$. The two resulting equations yield the solutions

$$\xi = K_1\cdot\sin[nx+\beta_1], \qquad \tau = K_2\cdot\sin[mnt+\beta_2]$$

whence $\varphi = K_1K_2\operatorname{Sin}[nx+\beta_1]\,\sin[mnt+\beta_2]$

which we may then reduce to a more useful form :

$$\varphi = \sum_{n=o}^{n=\infty} A_n \sin[n(x\pm mt)+\delta_n].$$

An interesting fallacy results from applying the method of integration by parts, $\int u\cdot dv = uv-\int v\cdot du$, to a case where $u=1/x$ and $dv=dx$: we get

$$\int\frac{dx}{x} = \frac{1}{x}\cdot\int dx - \int x\cdot[-1/x^2]$$

$$= \quad 1 \quad + \int dx/x \quad \text{whence } 0=1\ !!$$

$$\int\frac{dx}{5+7x^2} = 1/5\int\frac{dx}{1+{}^7/_5x^2} = \frac{1}{5}\sqrt{5/7}\int\frac{\sqrt{7/5}\,dx}{1+[\sqrt{7/5}\,x]^2}$$

$$= \frac{1}{35}\ \arctan\ [\sqrt{7/5}\ x].$$

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THE AMERICAN MATHEMATICAL MONTHLY

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THE MATHEMATICAL ASSOCIATION OF AMERICA

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

PUBLISHED BY THE ASSOCIATION

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1912. FROM 1912 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

ISSUED MONTHLY EXCEPT IN JULY AND AUGUST

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is uncommon. The usual German expression is "vollständige Induktion." In criticism of this term Federigo Enriques[1] says: "We should not confound *mathematical induction*, namely the argument from n to $n+1 \cdots$ with the *complete induction* of Aristotle." In this Aristotelian sense the term "vollständige Induktion" is used in 1840 in the article "Induction" in Ersch and Gruber's *Encyklopädie*, where we find the example: If two sides are found to be greater than the third side in plane triangles with a right angle, and with an obtuse angle, and also with only acute angles, and this inequality is shown to be true likewise of spherical triangles, then the inequality can be asserted to be true of *all* triangles. Here a "vollständige Induktion" is quite different from the argument from n to $n + 1$. The use of the same name for two different types of induction is objectionable. The name "vollständige Induktion" was used by R. Dedekind in his *Was sind und was sollen die Zahlen*, 1887, §§ 59 and 80. Through him the method received great emphasis in Germany.[2] The English equivalent of "vollständige Induktion," namely, "complete induction," is seldom used.[3] According to A. Haas[4] the designation "höhere Induktion" is also employed. Poincaré, in his *Science et Hypothèse*, does not restrict himself to any one name, but is partial to the phrases "démonstration par récurrence," "raisonnement par récurrence."

MISCELLANEA.

By AUBREY J. KEMPNER, University of Illinois.

§ I. Concerning Greatest and Least Absolute Values of Analytic Functions.

The following theorems are known:

I*a*. If $f(z)$ is an analytic (single-valued) function of the complex variable $z = x + iy$, regular in the interior and on the boundary of a circle C about a point a of the complex plane as center, then $|f(a)| \leqq M$, where M is the largest value which $|f(z)|$ assumes on the boundary of C.

I*b*. Under the same conditions for $f(z)$, $|f(c)| \leqq M$, where c is any interior point of the circle C.[5]

I*c*. In I*b* the circular region may be replaced by a region S, not necessarily simply connected, but the boundary of which we shall assume, in this as in the

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[1] F. Enriques, *Principles of Science*, transl. by K. Royce, Chicago and London, 1914, p. 133.

[2] An excellent article in the English language on "Mathematical Induction" is that by C. J. Keyser in the *Americana*.

[3] See, however, W. H. Bussey, "The Origin of Mathematical Induction," Am. Math. Monthly, Vol. 24, 1917, p. 199.

[4] A. Haas, *Lehrbuch über den binomischen und polynomischen Lehrsatz*, Bremerhaven, 1906, p. 6.

[5] For I*a* and I*b* compare, among others: Fouët, *Leçons sur la théorie des fonctions analytiques*, 1910, Vol. II, p. 79; Osgood, *Lehrbuch der Funktionentheorie*, 1912, Vol. I, p. 300; Vivanti-Gutzmer, *Theorie der eindeutigen analytischen Funktionen*, 1906, p. 81.

following theorems, to consist of a finite number of "regular" arcs of curves.[1] For any interior point c we shall again have $|f(c)| \leqq M$, where M is the largest value which $|f(z)|$ assumes on the boundary of S.

In I*a*, I*b*, I*c*, the sign of equality holds only when $f(z) \equiv$ const.

Of the three theorems, the last two follow immediately from the first, because it would otherwise be possible to select an interior point of the region where $|f(z)|$ assumes its largest value and to consider a small circle about this point as center, thus leading to a contradiction of I*a*.

Corollary. If the absolute value of a single-valued analytic function $f(z)$ is regular along the whole boundary of a region S, and if it is possible to find in the interior of S any point for which $|f(z)|$ exceeds the largest value of $|f(z)|$ along the boundary, $f(z)$ must have at least one singular point in the interior.

It is well known that I*a*, I*b*, I*c* also hold when in the theorems $|f(z)|$ is replaced by the real part u or by the imaginary part v of

$$f(z) = f(x + iy) = u(x, y) + i \cdot v(x, y).$$ [2]

As a direct consequence of I*a*, we also mention the following theorem, which is of great importance in many modern investigations in function theory:

Let $C(r_1)$, $C(r_2)$, $\cdots$, $C(r_n)$ denote concentric circles about the point $z = 0$ of radii $r_1 < r_2 < \cdots < r_n$, respectively, and such that the single-valued analytic function $f(z)$ is regular in the interior and on the boundary of $C(r_n)$; let $M(r_\nu)$ denote the largest value which $|f(z)|$ assumes on the boundary of $C(r_\nu)$; then $M(r_1) \leqq M(r_2) \leqq \cdots \leqq M(r_n)$ (no signs of equality to be admitted unless $f(z) \equiv$ const.), and by a limiting process it is shown that $M(r)$, considered as a function of the continuously increasing radius r, is a monotone increasing function.[3]

Some further consequences of I*a* are developed in this note. They are such obvious extensions that they must have occurred to many mathematicians, but (with the exception of II) I do not recall seeing them in print. However, no thorough study of the literature was attempted.

II. Assume $f(z)$ single-valued and regular in the interior of a region S and on the boundary, and let $m > 0$ be the smallest value which $|f(z)|$ assumes on the boundary. The existence of a point c in the interior of S such that $|f(c)| < m$ constitutes a necessary and sufficient condition that $f(z)$ have at least one zero in the interior of S.[4]

Proof: The condition is of course *necessary*. We still have to show that in case $f(z)$ has no zero in S, $|f(z)|$ assumes for some point of the boundary a smaller value than for any interior point. This is proved by applying I*c* to the

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[1] For the definition of "regular arc"—"reguläres Bogenstück"—see Osgood, *loc. cit.*, p. 51.

[2] See for I*c* Osgood, *loc. cit.*, p. 622; Burkhardt, *Einführung in die Theorie der analytischen Funktionen* . . ., 1908, p. 125, and many others.

[3] Vivanti-Gutzmer, *loc. cit.*, p. 81; Fouët, *loc. cit.*, p. 78; Borel, *Leçons sur les fonctions entières*, 1900, p. 107; Blumenthal, *Principes de la théorie des fonctions . . . d'ordre infini*, 1910, p. 5; and others.

[4] Fouët, *loc. cit.*, p. 79 (stated for a circular region).

function $\varphi(z) = 1/f(z)$; since $f(z)$ has no zero in S or on the boundary, $\varphi(z)$ is regular in S and on the boundary, and $|\varphi(z)|$ assumes its *largest*, $|f(z)|$ its *smallest*, value, for a point on the boundary. The analogon of the corollary given above is obvious.

So far, the terms "maximum of $|f(z)|$" and "minimum of $|f(z)|$" have been avoided, because $|f(z)|$ may very well have several maxima and minima as z moves along the boundary of S. We were concerned only with the "largest" value of $|f(z)|$, the "maximum maximorum," and the "smallest" value, the "minimum minimorum." However, by applying exactly the reasoning applied above, the truth of the following theorem is seen, which comprises I*c* and II:

III. *Assume $f(z)$ regular in a region S and on its boundary. Take the z-plane as the horizontal plane of a system of rectangular space coördinates, and erect in each point z of S and its boundary a perpendicular of length $|f(z)|$. The (continuous) surface generated by the end points of these perpendiculars has no maximum in the interior of S, and where it has a minimum in the interior, it reaches down to the z-plane.*

All of these theorems hold also for branches of multiple-valued analytic functions under proper restrictions for the region S; it is only necessary to stipulate that S shall be a closed region on the Riemann surface belonging to the multiple-valued function, and shall not contain a branch point of the Riemann surface in its interior or on its boundary. This follows from the fact that in such a region the corresponding branch of the multiple-valued function is nothing but an ordinary single-valued analytic function, to which therefore I*c* is applicable.

When one remembers: (1) that $df(z)/dz = f'(z)$ is an analytic function of z, regular everywhere where the single-valued analytic function $f(z)$ is regular, and that therefore I*c* may be applied to $f'(z)$; (2) that $|f'(z)|$ represents the factor of magnification at every point of the conformal mapping of a region in the z-plane upon the corresponding region of the $f(z)$-plane, we obtain the theorems IV and V:

IV. *For a region S, in every interior and boundary point of which the single-valued analytic function $f(z)$ is regular, the factor of magnification of the conformal mapping of the region S upon the corresponding region of the $f(z)$-plane has its largest value on the boundary; or it is constant for all points of the region S, and the function is of the form $f(z) = az + b$.*

The very last statement follows from the fact that $f'(z)$ is constant in this case. The coefficient a may be assumed different from zero, since $f(z) \equiv b$ does not yield any conformal representation. From the standpoint of conformal mapping, $f(z) = z$ is the simplest function.

V. *For a region S, in every interior and boundary point of which the single-valued analytic function $f(z)$ is regular and has its derivative different from zero, the factor of magnification has its smallest value on the boundary. It is constant only for $f(z) = az + b$.*

Since the values of z, for which $f'(z) = 0$, are points for which the conformal representation breaks down, we may combine IV and V by saying:

For any region S in which the conformal representation is not disturbed for any interior or boundary point, the factor of magnification has its largest and also its smallest value for points on the boundary of S; unless the function be of the type $f(z) = az + b$, in which case the factor of magnification is constant.

Any picture of a conformal mapping will serve to illustrate these theorems.

VI. In proving I*a* or I*b*, from which the other theorems are all derived, use is always made, as far as I know, of the analytic character of the function $f(z)$. That the analyticity is, in a certain sense, not essential, is easily established as follows:

Assume $f(z)$ to be a function of the complex variable $z = x + iy$ possessing the properties:

(1) There exists in the z-plane a two-dimensional region S for which $f(z)$ is single-valued and continuous;

(2) Every circle s lying entirely in S is transformed by $f(z)$ into a region s' of the $f(z)$-plane, so that interior points correspond to interior points, and boundary points to boundary points. (In case overlapping regions s' in the $f(z)$-plane are admitted, a slight modification in the statement of the correspondence of points is required, to avoid ambiguity.)

It is possible in various ways to impose these conditions on the function without making it analytic. They are satisfied, for example, in the whole finite complex plane, by $f(x + iy) = \alpha x + i\beta y$, where $\alpha \neq \beta$ are two real numbers, different from zero; this function represents stretching along the x-axis in a certain ratio and stretching along the y-axis in a different ratio. The function is not analytic.

After having chosen a region s in the z-plane, assume the corresponding s' mapped in the $f(z)$-plane, and describe in the $f(z)$-plane the circle of largest radius about the origin having points in common with s'; mark this point, which we denote by α (or one of these points, in case there are more than one), and let a be the corresponding point in s. Then a is a point on the boundary of s, while $|\alpha| = M$, the distance of α from the origin in the $f(z)$-plane, is the largest value attained by $|f(z)|$ for any point z lying in the interior or on the boundary of s. Therefore, *at least for a region in the z-plane satisfying* (1) *and* (2), I*a* and I*b* hold whether $f(z)$ is analytic or not.

To derive the corresponding theorem II on the smallest value of $|f(z)|$, two cases must be considered, according to whether s' contains or does not contain the origin of the $f(z)$-plane as an interior point.

The last discussion might equally well have been based on known theorems of two real functions of two real variables, making use, for example, of a theorem given in Osgood's "*Lehrbuch der Funktionentheorie,*" 1912, p. 71.

The theorems on the real and imaginary parts u and v of $f(x + iy) = u + iv$ may be just as easily derived under the assumptions (1) and (2).

§ II. Concerning the Smallest Integer $m!$ Divisible by a Given Integer n.

In connection with an investigation in number theory the solution of the following problem was needed:

I. For a given integer n to find the smallest integer $\mu(n) = m$ such that $m!$ is divisible by n. A direct method of determining m is required, by which all trials are eliminated.

This problem is closely related to the problem which is treated in many textbooks of elementary number theory:

I*a*. For a given integer m and a given prime p to find the highest power of p contained as a factor in $m!$.[1]

The simple considerations leading to the solutions of these problems are of the same character, in the two cases, but the formulae which give the solution of the second problem do not yield a method of determining without trials the m of the first problem for a given n, even when n is restricted to be of the form p^α, p a prime.

Solution of I. All letters used denote positive integers or zero. For several forms of n the solution is obvious.

1. $\mu(1) = 1$; $\mu(n!) = n$.
2. $\mu(p) = p$, *p a prime.*
3. $\mu(p_1 \cdot p_2 \cdots p_\rho) = p_\rho$ *when* $p_1 < p_2 < \cdots < p_\rho$ *are distinct primes.*
4. $\mu(p^\alpha) = p\alpha$, *for p prime and* $\alpha \leqq p$.

The solution is not trivial for

5. $n = p^\alpha$, $\alpha \gtreqless p$ (thus including 2 and 4 as special cases).

In this case there are complications for the following reason. Let

$$m! = 1 \cdot 2 \cdots p \cdots (2p) \cdots (p \cdot p) \cdots ((p+1)p) \cdots (2p \cdot p) \cdots (p^2 \cdot p) \cdots m;$$

it is clear that every pth number contains p as a factor; but at the same time every (p^2)th number contains p a second time as a factor, every (p^3)th number contains p^3 as a factor, and so on. Denoting by $[a/b]$ the greatest integer k such that $bk \leqq a$, we see that $m!$ contains exactly p raised to the power

$$\left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \left[\frac{m}{p^3}\right] + \cdots$$

as a factor, where the $[m/p^\nu]$ are integers decreasing with ν and of which only a certain number are different from zero.

This is the method by which I*a* is usually proved. Taking $m = p^\alpha$, α any integer, we see that $(p^\alpha)!$ contains exactly the power $p^{1+p+p^2+\cdots+p^{\alpha-1}}$.[2] Therefore

$$(a) \qquad \mu(p^{1+p+p^2+\cdots+p^{\alpha-1}}) = p^\alpha.$$

The truth of the following statement will be obvious after writing it out for a special case, say $p = 5$, $\alpha = 4$, $\beta = 2$:

(*b*) Let g be any positive integer or zero, and $\alpha > \beta > 0$, p any prime; then in the product

[1] Compare, for example, L. W. Reid, *The Elements of the Theory of Algebraic Numbers*, 1910, p. 26, or R. D. Carmichael, *The Theory of Numbers*, 1914, p. 24.

[2] The notation $(p^\alpha)! \equiv 0 \bmod (p^{1+p+\cdots+p p^{\alpha-1}})$ has purposely been avoided throughout.

$$(g\cdot p^{\alpha}+1)\cdot(g\cdot p^{\alpha}+2)\cdots(g\cdot p^{\alpha}+p^{\beta})\cdot(g\cdot p^{\alpha}+p^{\beta}+1)$$
$$\cdots(g\cdot p^{\alpha}+2\cdot p^{\beta})\cdot(g\cdot p^{\alpha}+2\cdot p^{\beta}+1)$$
$$\cdots(g\cdot p^{\alpha}+3p^{\beta})\cdots\cdots(g\cdot p^{\alpha}+\overline{p-2}\cdot p^{\beta}+1)$$
$$\cdots(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta})\cdot(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta}+1)\cdots(g\cdot p^{\alpha}+p^{\beta+1})$$

each one of the $p-1$ products of p^{β} numbers each

$$(g\cdot p^{\alpha}+1)\cdots(g\cdot p^{\alpha}+p^{\beta});\ \cdots;\ (g\cdot p^{\alpha}+\overline{p-2}\cdot p^{\beta}+1)\cdots(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta})$$

contains as a factor exactly the same power of p as does the first group, that is $p^{1+p+\dots+p^{\beta-1}}$, while the last group of p factors, $(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta}+1)\cdots(g\cdot p^{\alpha}+p^{\beta+1})$, contains the factor $p\cdot p^{1+p+\dots+p^{\beta-1}}$, on account of the $p^{\beta+1}$ of the last factor.

Any positive integer may be written in the form $c_1\cdot p^{\delta}+c_2\cdot p^{\delta-1}+\cdots c_{\delta}\cdot p+c_{\delta+1}$, where for each $c: 0\leqq c<p$. It is clear that if m is to be the smallest integer for which $m!$ is divisible by a power of p, m will be divisible by p. Therefore m may be written

$$m=\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\cdots+\gamma_{\nu}\cdot p^{\alpha_{\nu}},$$

where

$$\alpha_1>\alpha_2>\cdots>\alpha_{\nu}>0$$

and

$$0<\gamma_i<p.$$

Then

$$m!=1\cdot2\cdots(\gamma_1\cdot p^{\alpha_1})\cdot(\gamma_1\cdot p^{\alpha_1}+1)\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2})\cdot(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+1)$$
$$\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\gamma_3\cdot p^{\alpha_3})\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+1)$$
$$\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+\gamma_{\nu}\cdot p^{\alpha_{\nu}}).$$

In $1\cdot2\cdots(\gamma_1\cdot p^{\alpha_1})$ exactly $p^{\gamma_1(1+p+p^2+\dots+p^{\alpha_1-1})}$ is contained as a factor; in $(\gamma_1\cdot p^{\alpha_1}+1)\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2})$ exactly $p^{\gamma_2(1+p+p^2+\dots+p^{\alpha_2-1})}$ is contained, and so on; finally, in

$$(\gamma_1\cdot p^{\alpha_1}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+1)\cdots(\gamma_1\cdot p^{\alpha_1}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+\gamma_{\nu}\cdot p^{\alpha_{\nu}})$$

we have exactly the factor $p^{\gamma_{\nu}(1+p+p^2+\dots+p^{\alpha_{\nu}-1})}$.

Therefore $m!$ contains exactly p raised to the power

$$\sum_{\rho=1}^{\nu}\gamma_{\rho}(1+p+p^2+\cdots+p^{\alpha_{\rho}-1}).$$

We have thus obtained (with a slight change in notation) the following formula which solves 5., *provided we can always write the given exponent α in the form*

$$\alpha = \sum_{\rho=1}^{\nu} \gamma_\rho(1 + p + p^2 + \cdots + p^{\beta_\rho}),$$

with all γ *satisfying* $0 < \gamma < p$ *and* $\beta_1 > \beta_2 > \cdots > \beta_\rho \geqq 0$:

$$(c) \qquad \mu(p^{\sum_{\rho=1}^{\nu} \gamma_\rho(1+p+p^2+\cdots+p^{\beta_\rho})}) = \sum_{\rho=1}^{\nu} \gamma_\rho \cdot p^{\beta_\rho+1}.$$

We must next examine whether an integer α may always be written in the form $\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + \cdots + \gamma_\nu(1 + p + \cdots + p^{\beta_\nu})$, where $\beta_1 > \beta_2 > \cdots > \beta_\nu \geqq 0$ and where each γ satisfies: $0 < \gamma < p$. It will be seen that there are exceptional cases to be taken care of.[1]

Let

$$a_1 = 1, \quad a_2 = 1 + p, \quad \cdots, \quad a_{\rho+1} = 1 + p + p^2 + \cdots + p^\rho, \quad \cdots,$$

then we have the recurrence relation

$$a_{\rho+1} = p \cdot a_\rho + 1, \quad a_1 = 1.$$

Assume $a_\nu \leqq \alpha < a_{\nu+1}$, then

$$\alpha = k_\nu a_\nu + r_\nu,$$

where

$$0 < k_\nu < p,$$

except for the possible combination $k_\nu = p$, $r_\nu = 0$. Assume for the moment $r_\nu > 0$, then there is an a_σ such that $a_\sigma \leqq r_\nu < a_{\sigma+1}$, and

$$r_\nu = k_{\nu-1}a_{\nu-1} + r_{\nu-1},$$

where

$$0 < k_{\nu-1} < p,$$

except for the combination $k_{\nu-1} = p$, $r_{\nu-1} = 0$. Continuing in this fashion, we see that for every integer α

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + \cdots + \gamma_\nu(1 + p + \cdots + p^{\beta_\nu}),$$

where $\beta_1 > \beta_2 > \cdots > \beta_\nu \geqq 0$, but $0 < \gamma < p$ only with the restriction that the last γ may be equal to p. This exception arises from the fact that while for every $m!$ a certain p^α exists which is the highest power of p contained as a factor in $m!$, for a given p^α there may not exist an m such that $m!$ is divisible by p^α and by no higher power of p (for example $p = 2$, $\alpha = 6$).

To show that our method of determining $\mu(p^\alpha)$ by means of (c) holds also in case the last coefficient γ is equal to p, we proceed as follows:

First assume

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + p(1 + p + \cdots + p^{\beta_1-1})$$

[1] These exceptional cases were overlooked by J. Neuberg, *Mathesis*, 1887, Vol. VII, p. 68: But for them, our problem would be solved in a very few lines from the known methods of solving problem I*a*.

or

$$\alpha + 1 = (\gamma_1 + 1)(1 + p + \cdots + p^{\beta_1}),$$

assuming for the moment $\gamma_1 + 1 < p$. This increases α by one unit, but we shall show that

$$\mu(p^{\alpha+1}) = \mu(p^\alpha).$$

Clearly $\mu(p^{\alpha+1}) = (\gamma_1 + 1)\cdot p^{\beta_1+1}$, and $[(\gamma_1 + 1)\cdot p^{\beta_1+1}]!$ contains exactly the factor $p^{\alpha+1}$. Now let m_1 be the smallest number such that $m_1!$ has the factor p^α; then $m_1 = \mu(p^{\alpha+1})$ because if we omit the last factor of $[\mu(p^{\alpha+1})]!$, that is $(\gamma_1 + 1)\cdot p^{\beta_1+1}$, the exponent of the highest power of p contained as a factor in the product is reduced by $\beta_1 + 1 \geqq 2$ units at least. It is easy to see that if α contains terms preceding $\gamma_1(1 + p + \cdots + p^{\beta_1})$ the argument holds unchanged, on account of the statement (*b*).

We must still free ourselves from the assumption $\gamma_1 + 1 < p$. But when $\gamma_1 + 1 = p$, our new last coefficient is equal to p and we may repeat the process just indicated; the original α is now increased to $\alpha + 2$, but still $\mu(p^{\alpha+2}) = \mu(p^\alpha)$.

An example will suffice to illustrate conditions: Take $p = 3$, $\alpha = 37$. We have $37 = 2\cdot(1 + 3 + 9) + 2\cdot(1 + 3) + 3\cdot 1$; replace this by $2\cdot(1 + 3 + 9) + 3\cdot(1 + 3) = 38$; and again by $3\cdot(1 + 3 + 9) = 39$; and finally by $1\cdot(1 + 3 + 9 + 27) = 40$. Then $(1\cdot 3^4)! = (81)!$ is the smallest number $m!$ containing 3^{37} as a factor.

It contains also 3^{40} as a factor, since $\mu(3^{37}) = \mu(3^{38}) = \mu(3^{39}) = \mu(3^{40}) = 81$, as can be immediately verified.

However, since the relation

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + p\cdot(1 + p + \cdots + p^{\beta_1 - 1}),$$

when treated as if the rule given by (*c*) still held although $\gamma_2 = p$, will give us $\gamma_1\cdot p^{\beta_1+1} + p\cdot p^{\beta_1} = (\gamma_1 + 1)\cdot p^{\beta_1+1}$, that is, the same value which we get by applying (*c*) to $\alpha + 1 = (\gamma_1 + 1)\cdot(1 + p + \cdots + p^{\beta_1})$, we shall have the theorem:

THEOREM: *Assume α written in the form*

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + \gamma_2(1 + p + \cdots + p^{\beta_2}) + \cdots + \gamma_\nu(1 + p + \cdots + p^{\beta_\nu}),$$

where $\beta_1 > \beta_2 > \cdots > \beta_\nu \geqq 0$, $0 < \gamma_\rho < p$ *for* $\rho = 1, 2, \cdots, \nu - 1$, *but* $0 < \gamma_\nu \leqq p$. *Then*

$$\mu(p^\alpha) = \gamma_1\cdot p^{\beta_1+1} + \gamma_2\cdot p^{\beta_2+1} + \cdots + \gamma_\nu\cdot p^{\beta_\nu+1} = (p - 1)\cdot\alpha + (\gamma_1 + \gamma_2 + \cdots + \gamma_\nu).$$

The very last equality is verified immediately by substituting for α the expression given in the theorem.

Example: $p^\alpha = 5^{1092}$.

$$\alpha = 1\cdot(1 + 5 + \cdots + 5^4) + 1\cdot(1 + 5 + 5^2 + 5^3) + 5\cdot(1 + 5 + 5^2).$$

$$\mu(5^{1092}) = 1.5^5 + 1\cdot 5^4 + 5\cdot 5^3 = 4.1092 + (1 + 1 + 5) = 4375.$$

It may be left to the reader to show that the last theorem is equivalent to the following

RULE: *To find* $\mu(p^\alpha)$, *form a sequence of numbers by means of the recurrence formula*

$$a_{\rho+1} = p\cdot a_\rho + 1, \quad a_1 = 1.$$

Assume $a_\nu \leqq \alpha < a_{\nu+1}$ *and carry out the following divisions until we meet a remainder zero:*

$$\begin{aligned} \alpha &= k_\nu \cdot a_\nu + r_\nu, & 0 < r_\nu &< a_\nu, \\ r_\nu &= k_{\nu-1}\cdot a_{\nu-1} + r_{\nu-1}, & 0 < r_{\nu-1} &< a_{\nu-1}, \\ &\cdots\cdots\cdots & & \\ r_{\lambda+2} &= k_{\lambda+1}\cdot a_{\lambda+1} + r_{\lambda+1}, & 0 < r_{\lambda+1} &< a_{\lambda+1}, \\ r_{\lambda+1} &= k_\lambda \cdot a_\lambda. & & \end{aligned}$$

Then

$$\mu(p^\alpha) = (p-1)\cdot\alpha + (k_\nu + k_{\nu-1} + \cdots + k_{\lambda+1} + k_\lambda).$$

This should be compared with the following rule for solving problem I*a* (see Carmichael, *loc. cit.*, p. 26):

To find the highest power p^α of a prime p contained in $m!$, write m in the form $m = q_1\cdot p^{\alpha_1} + q_2\cdot p^{\alpha_2} + \cdots + q_\tau\cdot p^{\alpha_\tau}$, $\alpha_1 > \alpha_2 > \cdots > \alpha_\tau \geqq 0$, all q satisfying $0 < q < p$; then

$$\alpha = \frac{m - (q_1 + q_2 + \cdots + q_\tau)}{p - 1}.$$

Example 1: $p^\alpha = 3^{1000}$.

$$a_1 = 1;\ a_2 = 3\cdot a_1 + 1 = 4;\ a_3 = 3\cdot a_2 + 1 = 13;\ a_4 = 3\cdot a_3 + 1 = 40;$$

$$a_5 = 3\cdot a_4 + 1 = 121;\ a_6 = 3\cdot a_5 + 1 = 364;\ a_7 = 3\cdot a_6 + 1 = 1093,$$

and

$$\alpha = 2\cdot a_6 + 272,\ 272 = 2\cdot a_5 + 30,\ 30 = 2\cdot a_3 + 4,\ 4 = 1\cdot a_2.$$

Hence

$$\mu(3^{1000}) = 2\cdot 1000 + (2 + 2 + 2 + 1) = 2007.$$

Example 2: $p^\alpha = 5^{100000}$.

$$a_1 = 1;\ a_2 = 6;\ a_3 = 31;\ a_4 = 156;\ a_5 = 781;\ a_6 = 3906;\ a_7 = 19531;\ a_8 = 97651.$$

Then

$$100000 = 1\cdot 97651 + 2349,\ 2349 = 3\cdot 781 + 6,\ 6 = 1\cdot 6;$$

$$\mu(5^{100000}) = 4\cdot 100000 + (1 + 3 + 1) = 400005.$$

6. $n = p_1^{\epsilon_1}\cdot p_2^{\epsilon_2}\cdots p_\nu^{\epsilon_\nu}$, *where the* p'*s are distinct primes.* (General case.)

This case reduces immediately to 5., because if $m!$ contains the factors $p_1^{\epsilon_1}, \cdots, p_\nu^{\epsilon_\nu}$, it must also have their product as a factor.

Denoting by max $[\mu(p_1^{\epsilon_1}), \mu(p_2^{\epsilon_\nu}), \cdots, \mu(p_\nu^{\epsilon_\nu})]$ the largest of the numbers $\mu(p_1^{\epsilon_1}), \cdots, \mu(p_\nu^{\epsilon_\nu})$, we have the theorem (which of course includes 3. as a special case):

THEOREM: $\mu(p_1^{\epsilon_1}\cdot p_2^{\epsilon_2}\cdots p_\nu^{\epsilon_\nu}) = \max\,[\mu(p_1^{\epsilon_1}), \mu(p_2^{\epsilon_2}), \cdots, \mu(p_\nu^{\epsilon_\nu})]$.

Example: $n = 3^2\cdot 5^{29}\cdot 11^{19}\cdot 113$.

$\mu(3^2) = 2\cdot 3 = 6$; $\mu(5^{29}) = 4\cdot 5^2 + 5\cdot 5 = 125$, because $29 = 4(1 + 5) + 5.1$; $\mu(11^{19}) = 1\cdot 11^2 + 7\cdot 11 = 198$, because $19 = 1\cdot(1 + 11) + 7\cdot 1$; $\mu(113) = 113$, because 113 is a prime number; therefore $\mu(3^2\cdot 5^{29}\cdot 11^{19}\cdot 113) = 198$.

Considering $\mu(n)$ as a function of n, one sees that it has, like most number theoretic functions, a very irregular behavior:

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24,$$

$$\mu(n) = 1, 2, 3, 4, 5, 3, 7, 4, 6, \ \ 5, 11, \ \ 4, 13, \ \ 7, \ \ 5, \ \ 6, 17, \ \ 6, 19, \ \ 5, \ \ 7, 11, 23, \ \ 4.$$

BOOK REVIEWS AND NOTICES.

SEND ALL COMMUNICATIONS ABOUT BOOKS TO W. H. BUSSEY, University of Minnesota.

Descriptive Geometry. By ERVIN KENISON and HARRY CYRUS BRADLEY. The Macmillan Company, New York, 1917. x + 287 pages. $2.00.

The book under review confines itself to a treatment of that branch of descriptive geometry which is known as the method of double orthographic projection or more simply as the Mongean method.

The very brief introduction which precedes Chapter I is scarcely sufficient to give the reader an adequate notion of the purposes and nature of descriptive geometry. Exception might be taken to the statement: "Its operations are not strictly mathematical." For, aside from the more or less precise mechanical operation involved in the use of the drawing instruments, the methods which descriptive geometry continually applies are, according to Loria, only such as are taught by the old Euclidean geometry and the modern projective geometry, and its processes are so very exact that it is comparable with algebra and analysis.[1]

The first seventeen chapters are concerned principally with the method of representing points, lines, and planes of space, and of solving, by means of plane constructions, the problems of space which involve points, lines, and planes. The great number of problems which have been solved illustrate the more important processes of the Mongean method and give the student a very good notion of the operations of this method. The division of problems into chapters might suggest an attempt at classification. However, no explicit statements have been made which would lead the student to suspect that all the problems

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[1] See Loria, *Vorlesungen über Darstellende Geometrie*, preface to Vol. I.

which arise can be divided into a few groups in each of which the solution of a few fundamental ones is sufficient for the solution of all the problems of the group.

In these chapters there are a few inaccuracies to which attention should be called. Thus in Section 34 the statement is made under *case* (*a*) that there is no line in space to correspond to two lines of the picture plane of which one is perpendicular to the ground line. That this statement is not correct can easily be seen if we think of the lines A^h and A^v of Fig. 48 as the traces of planes perpendicular to the horizontal and vertical planes of projection, respectively. These two planes surely intersect in a line, and this is the line represented by the given projections A^h and A^v. Under *case* (*b*) it is stated that there is no line in space corresponding to two lines of the picture plane which are perpendicular to the ground line at different points. Here, again, the lines B^h and B^v of Fig. 49 may be regarded as the traces of planes perpendicular to the horizontal and vertical planes of projection, respectively. These two planes are parallel and hence intersect in a line at infinity.[1] On page 88 the statement is made that a straight line is determined when one of its points and its direction (such as parallel or perpendicular to another line) are known. The words "or perpendicular" should be omitted, for through a point an infinite number of lines can be passed perpendicular to another line.

The last nine chapters deal with curves and surfaces, tangent planes to surfaces, intersections of surfaces by planes, and intersections of surfaces. The surfaces considered are principally cones, cylinders, spheres, and such simple surfaces of revolution as the torus. The problem of developing cylinders and cones is also treated. The lack, or brevity, of definitions in these chapters makes impossible a clear notion of some of the terms used. For instance, no definition is given of a tangent line to a curve, or of a tangent line to a surface. Likewise, the terms surface, curved surface, double curved surface are used without being defined. In Section 163 the statement is made that a double curved surface of revolution is formed by the revolution of any curve about any straight line as an axis, provided the resulting surface is such that no straight line can be drawn upon it. That this definition contains contradictions is evident from the fact that the surface obtained by revolving an hyperbola about its transverse axis is a double curved surface of revolution, and yet it is composed of two one-parameter families of straight lines. To the reviewer it seems that the statement that a general developable surface is composed of the tangents to a space curve is much more helpful to the student than the statement made in Section 179 that every two consecutive elements lie in the same plane. In the chapters on curves and surfaces, as in those on the point, line, and plane, the large number of problems which have been solved give the student a very good notion of the method of procedure.

In addition to the numerous figures giving the Mongean solutions of the various problems considered, the authors have used figures which they call "pictorial representations," such for instance as Figs. 11, 17, 96. They are

[1] For a discussion of these cases, see Loria, *loc. cit.*, Vol. 1, Section 8.

intended to convey to the mind a clearer notion of the space relations than do the Mongean pictures. The authors have not attempted to explain the methods of constructing such figures and there can be no objection to this in a book which concerns itself primarily with the Mongean method. However, such pictures, when used, should be properly drawn. In Fig. 96, for instance, the ellipses representing the circles should not have their principal axes horizontal and vertical, and the lines cc^v and bb^v should be tangent to the upper ellipse and would be if that ellipse were properly drawn. The same remarks apply to the lower ellipse and to certain other figures.

Wm. H. Roever.

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Plane and Spherical Trigonometry, with Tables. By G. N. Bauer and W. E. Brooke. Second Revised Edition. D. C. Heath and Co., Boston, 1917. xi + 174 + v + 139 pages.

A writer of textbooks may approach trigonometry from two different standpoints; he may wish to give merely the formulas and processes needed in the solution of triangles, or he may regard trigonometry as a chapter of mathematics dealing with a certain well-defined class of functions whose use in solving triangles is incidental. If in American colleges we could separate our engineering students from the specialists in mathematics, two different types of textbook could be devised to meet these two tendencies; this being as a rule impossible, an attempt must be made to meet both requirements at once. A glance at recent texts shows that we have apparently arrived at a sort of standard in the choice of subject matter; to summarize briefly, an acceptable textbook on trigonometry should contain, it would seem, the following topics:

(*a*) The definitions and elementary relations of the six functions;
(*b*) The addition-formulas and factor-formulas for changing sums of sines or cosines into products;
(*c*) The formulas for $2x$ and $x/2$, or even $3x$, but with no reference to wider formulas of which these might be the simplest cases;
(*d*) The graphs of the functions, either in a separate chapter or throughout;
(*e*) The treatment of trigonometric equations, at least among the exercises, and preferably with ample use of graphic methods;
(*f*) The application of formulas to the solution of plane and spherical triangles.

Besides these absolute essentials, we usually find chapters on

(*g*) The inverse functions;
(*h*) De Moivre's Theorem, and its simpler consequences; such as for instance the formulas for sin (nx), and the definitions of the hyperbolic functions, with their graphs perhaps.

Since trigonometry is usually a 3-hour branch for one semester, these matters can not be handled satisfactorily to any extent; a choice must be made, and here is where the difference in textbooks comes in.

Our present text, under (*a*), defines the 6 functions once and for all by means of the coördinates of the endpoint P of a rotating line-segment pivoted at the origin; the first 50 pages develop this topic splendidly, after which 5 more pages are given to the line-values of the functions in connection with a unit circle. The exercises are well selected; nothing but praise can be bestowed upon this treatment (Chap. 1–5).

Concerning (*b*), we find the addition-formulas proved by the trick device

$$\sin (x + y) = \frac{BQ + RP}{OP} = \frac{BQ}{OQ}\frac{OQ}{OP} + \frac{RP}{QP}\frac{QP}{OP},$$

and we are moved to enter a mild protest; such a proof is of no value to the student. There are many ways of proving this relation without having recourse to such artificial transformations. In extending the formula to angles beyond 90°, the authors have followed the usual plan of first putting

$$\sin (x + y + 90°) = \cos (x + y),$$

and then, in the expansion, replacing $\cos x$ by $\sin (90° + x)$, etc. The exercises include the formulas for the functions of $3x$. The reduction of $\sin (mx) \cos (nx)$ and similar forms to a sum of sines or cosines is mentioned briefly, but clearly. The cases of trigonometric equations which are presented for solution are to be found here and there scattered among the other exercises; they are sufficient in number and variety, perhaps; but would it not have been better to concentrate this entire matter in one chapter, with more explanation, and with emphasis on graphical methods?

Speaking of graphs, our authors have conceded to them the utmost minimum of space and importance; out of 125 pages devoted to plane trigonometry, only two pages are assigned to the graphs of the functions, with a grand total of six examples to be used as material for exercises! Surely this procedure is open to grave criticism and is counter to the spirit of the times. Skillful handling of equations necessarily demands graphic methods; and when we reflect that popular magazines nowadays are furnishing plans and diagrams for home-made harmonic-motion machines, when we think of alternating currents with their trigonometric equations, or when we consider the solution of mixed equations like $x^2 - 1 + 2 \sin x = 0$, we can only wonder why the authors have practically excluded graphic methods from their text. Where shall we find a treatment of trigonometric graphs if not in a trigonometry? This omission gives the book an old-fashioned appearance, and might prejudice a casual reader at the first glance. In this connection we may note that the unshaded, light-line diagrams used on pp. 128, 135 in illustration of spherical triangles are unsatisfactory to the eye through want of perspective.

The treatment of spherical triangles is adequate for practical purposes; the only area-formula mentioned is the one employing the angles as such, and no allusion is made to the existence of sine or tangent formulas for the area. Here and elsewhere the authors have not realized how the dry and didactic tone can

be relieved and how the student's horizon can be widened by occasional references to other methods and formulas, as well as by slight historical excursions now and then. The presentation of the triangle-solutions, both plane and spherical, is carried out well; the pupil will learn how to conduct his work neatly and economically by studying the typical solutions of the text.

So much for the essential contents of this little volume. A rather advanced chapter on inverse notation and functions is inserted immediately after the addition-formulas; the average student will not master this, but it is there if needed. The chapter on De Moivre's theorem concerns itself chiefly with the nth roots of unity and of complex numbers generally, and with the development of the sine and cosine series; these are found by first getting sin (nx) from De Moivre's theorem, putting $nx = y$, and then keeping y constant while x approaches zero. There is ample authority for this method; but will it make an appeal to the beginner? The matter of roots belongs rather to algebra; a more trigonometric use of the theorem might no doubt have been made by developing the formulas given in Chrystal's *Algebra*, Vol. 2, pp. 275–278, which are moreover of great importance in trigonometric integration.

The tables at the end give the natural functions to 3 and then to 4 places; and their logarithms, as also those of the natural numbers, to 3, 4, and 5 places in succession. All these are very clear and legible, with the exception of Table IX; here the columns of difference-numbers are placed in between the vertical columns for sine, cosine, and tangent, and moreover they are placed halfway between the logarithmic numbers. Since there are no horizontal or vertical guidelines to lead the eye, a zigzag effect is produced which is very annoying to one's eyesight and patience. This ought to be corrected when the time comes for another edition.

A. F. Frumveller.

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BOOK NOTICES.

It is now rather usual for a textbook on "Analytic Geometry" to contain a chapter on "Empirical Equations." The new *Analytic Geometry* just published by Edwin S. Crawley and Henry B. Evans of the University of Pennsylvania is no exception. It contains such a chapter of fourteen pages. This is about the usual length.

A more extensive treatment of the subject of empirical equations is to be found in the book entitled *Empirical Formulas* by Theodore R. Running, associate professor of mathematics in the University of Michigan, recently published by John Wiley and Sons as No. 19 of the series of Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward.

Doctors' theses which are not printed in mathematical periodicals are likely to be overlooked by some who would be interested in them. Two such theses submitted to the faculty of the Catholic University of America have recently

been printed in pamphlet form at Washington, D. C. They are: "On the cardioids fulfilling certain assigned conditions" (44 pages) by Sister Mary Gervase, and "On the in-and-circumscribed triangles of the plane rational quartic curve" (31 pages) by Joseph Nelson Rice.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS AND SOLUTIONS TO B. F. FINKEL, Springfield, Mo.

2699. Proposed by ROGER E. MOORE, University of Wisconsin.

Show that if $a_k^{(r)}$ denotes the kth term of an arithmetic progression of order r, and c_k denotes the kth binomial coefficient in the expansion of $(a - b)^n$, n being a positive integer,

$$s \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \qquad \text{if} \qquad n > r.$$

2700. Proposed by ARTEMAS MARTIN, Washington, D. C.

In a factory 250 men are paid an average wage of \$15 each per week. The men are paid unequally, the wages being \$20, \$16, \$10 and \$8 per week, respectively, for different classes of work. How many are employed at each rate of pay?

NOTE. I am told that this question was set in a Civil Service Examination Paper to be worked by Arithmetic. 2,896 answers have been found. Are there any more?

2701. Proposed by EDWARD H. WORTHINGTON, University of Pennsylvania.

Find the sum of the infinite series

$$\frac{1}{5} + \frac{1\cdot 2}{5\cdot 7}r + \frac{1\cdot 2\cdot 3}{5\cdot 7\cdot 9}r^2 + \cdots + \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}r^{n-1} + \cdots.$$

Verify your result for $r = 0$ and $r = 1$.

2702. Proposed by N. P. PANDYA, Sojitra, India.

A conic of variable eccentricity has a focus and corresponding directrix fixed. The latus rectum cuts a fixed circle in A' and B'. If A be the vertex of the conic, find the locus of the centroid of the triangle $AA'B'$.

2703. Proposed by S. A. COREY, Albia, Iowa.

Let A_1, A_2, A_3, A_4, and $-(A_1 + A_2 + A_3 + A_4)$ be the vector sides of a pentagon, plane or gauche. Let B_1, B_2, B_3, B_4, and $-(B_1 + B_2 + B_3 + B_4)$ be the sides of a second pentagon, where

$$B_1 = (c_1 + c_2)A_1 + (c_1 - c_2)A_3,$$
$$B_2 = (c_4 + c_3)A_4 + (c_4 - c_3)A_2,$$
$$B_3 = (c_2 + c_1)A_2 + (c_2 - c_1)A_4,$$
$$B_4 = (c_3 + c_4)A_3 + (c_3 - c_4)A_1,$$

c_1, c_2, c_3, and c_4 being ordinary scalars.

Then, if a_r and b_s are the lengths of the sides A_r and B_s, respectively, and if $\cos A_rA_s$ = cosine of the angle included between the sides A_r and A_s, and if $\cos B_rB_s$ = cosine of the angle included between the sides B_r and B_s, prove that

$$2(c_1c_4 + c_2c_3)(a_1a_4 \cos A_1A_4 + a_2a_3 \cos A_2A_3) = b_1b_2 \cos B_1B_2 + b_3b_4 \cos B_3B_4.$$

2704. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A particle moves from rest under gravity down the arc of a parabola with the axis vertical and concavity upward; express the time to the vertex in terms of an elliptic integral of the second kind.

2705. Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the area of a loop of the trochoid

$$x = \tfrac{1}{3}a(3\phi - \pi \sin \phi), \qquad y = \tfrac{1}{3}a(3 - \pi \cos \phi).$$

2706. Proposed by HARRIS F. MACNEISH, New York City.

Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the given point and the given line may be equal to the part within the given circle.

2707. Proposed by S. A. COREY, Albia, Iowa.

Let a, b and c be the vector sides of a triangle. Construct another triangle with vector sides r, s and t, where

$$r = ma - dnb, \qquad s = na + (m + en)b.$$

Then prove that

$$(m^2 + emn + dn^2)(a^2 + eab \cos (ab) + db^2) = r^2 + ers \cos (rs) + ds^2,$$

where d, e, m and n are any scalar quantities; a, b, r and s are the tensors, or lengths, of the sides a, b, r and s, respectively; and cos (rs) is the cosine of the angle between r and s when placed coinitially.

2708. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform plank of length $2a$ and thickness $2h$ rests in equilibrium on a fixed rough horizontal cylinder of radius c, so that a vertical plane containing the dimension $2a$ and the center of gravity of the plank is at right angles to the axis of the cylinder; find the period of a complete small oscillation of the plank.

SOLUTIONS OF PROBLEMS.

492. (Algebra). Proposed by ARTEMAS MARTIN, Washington, D. C.

If two numbers, A and B, $B > A$, be selected at random, what is the probability that they have no common divisor?

SOLUTION BY WARREN WEAVER, Throop College of Technology, Pasadena, Cal.

The following treatment of this question is due to Tchebycheff, and has been taken from Fisher's *Mathematical Theory of Probability*.

If $p_2, p_3, \cdots, p_n$ denote respectively the probability that each of the primes 2, 3, 5, $\cdots$, n is not a common factor of A and B, then the probability that no prime number is a common factor is:

$$P = p_2 \cdot p_3 \cdot p_5 \cdot \;\cdots\; \cdot p_n \cdot \;\cdots\; \cdot \text{ ad. inf.}$$

This follows from the multiplication theorem and the fact that the sequence of the prime numbers is infinite.

By dividing any integer by the prime n we obtain besides the quotient a certain remainder that must be one of the following numbers, namely,

$$0, 1, 2, 3, \cdots, (n - 1).$$

Each of the remainders may be considered as a possible event. The probability of obtaining 0 as a remainder is then $1/n$. This same quantity is the probability that n is a factor of B. The probability that both A and B *are* divisible by n is therefore $1/n^2$. The probability that the numbers A and B *do not* both have the prime factor n is then:

$$p_n = 1 - 1/n^2.$$

Hence,

$$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \text{ ad. inf.} \tag{1}$$

This infinite product is shown to have the value:

$$P = \frac{6}{\pi^2} = 0.60793 \cdots.$$

It seems that there may perhaps be in this method a slight fallacy, however. For it considers the probability that two numbers A and B ($A < B$) do not each have the factor n, where n is any one of the infinite sequence of prime numbers. However large A be, it is finite, and there are therefore an infinite number of primes larger than A none of which, obviously, can be a factor of both A and B. It would seem, then, that after a certain finite point the terms of the infinite product (1) should all be unity. The point at which the terms should begin to be unity would depend in some complicated and unknown way upon the value of A—unknown, of course, because we do not have an analytical expression for the rth prime, to say nothing of an expression for the largest prime smaller than a given number.

If we consider a slightly different example this becomes clearer. Suppose we have given a positive integer A. What is the probability that the proper fraction A/B is in its lowest terms, B being chosen at random to have any positive integral value larger than or equal to A? By the previous reasoning it will be:

$$p = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \left(1 - \frac{1}{s^2}\right),$$

where $s = A$ if A be prime or s is the largest prime $< A$ if A be not prime. For example, when $A = 15$

$$p = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\left(1 - \frac{1}{13^2}\right) = 0.61807,$$

a value which differs from P by only 1.67 per cent.

Concerning the discussion here given to the result of Tchebycheff it is worth while to note that one may be led to incorrect intuitive conclusions because of the fact that we are unable to conceive of the process of taking a positive integer "at random." We are likely to think of taking one out of the interval from 0 to, say, 10^{50}, considering it from the biased point of view of practical experience exceedingly unlikely that we will get an integer from beyond this interval; while, as a matter of fact, the probability of obtaining by chance a positive integer within this interval is, of course, infinitely small as compared to the probability of the number coming from beyond this interval.

Since, therefore, when A is picked at random the probability that it will be less than, say, 100 is very small indeed and since the result $6/\pi^2$ is a very close approximation to the correct result for all cases when A turns out to be greater than 100 we may conclude that it is exceedingly likely that the result $6/\pi^2$ is close to the true probability. A more exact statement or calculation of the *a priori* probability of this event seems beyond the possibilities of the theory.

REMARKS BY ROGER A. JOHNSON, St. Paul, Minnesota.

This problem is known in the literature of the science of probability as Tchebycheff's Problem.

The chance that an integer A has as factor a given prime m is $1/m$. The chance that both A and B, two integers, contain m as a factor is $1/m^2$. The chance that they have not this factor is $p_m = 1 - (1/m^2)$. The chance that they have no common prime factors is

$$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right) \cdots = 6/\pi^2,$$

a result evaluated by Euler.

Some years ago, I observed this fact stated in a brief note in the *Philosophical Magazine*, and made an experimental verification. About 200 numbers were obtained, having three to five digits each, the last digits being distributed uniformly among the values 0, 1, 2, $\cdots$, 9. These were placed in a box, drawn two at a time, and compared. When all were drawn, they were

replaced, mixed thoroughly, and the process repeated. It was previously decided to stop at the end of 600 trials. Throughout the work, the ratio of relatively prime pairs to total trials oscillated near the value .6, and at the conclusion, the number of relatively prime pairs was 364, a value surprisingly close to the correct theoretical value 364.8. Of course, an experiment involving, say, 10,000 trials would be more satisfying, but this result is perhaps not without interest.

493 (Algebra). Proposed by ALBERT BABBITT, University of Nebraska.

Determine the coefficients b, c, d of the equation $x^3 + bx^2 + cx + d = 0$ so that they shall be roots of the same equation. [From *Supplemento a Periodico di Matematica.*]

SOLUTION BY H. S. UHLER, Yale University.

Since the coefficient of x^3 is $+1$ and as b, c, d are to be roots, we have the following conditions:

$$-b = b + c + d \ (1), \qquad c = bc + cd + db \ (2), \qquad -d = bcd \ (3).$$

Equation (3) gives

$$d = 0 \ (4), \qquad \text{or} \qquad bc = -1 \ (5).$$

Combining (4) with (1) and (2) we get

$$2b + c = 0 \ (1'), \qquad bc - c = 0 \ (2').$$

Combining (5) with (1) and (2) we find $2b^2 + bd - 1 = 0$ (1″) and $b^2d - b$ -

Equations (1′) and (2′) lead at once to the two pairs of values $b = 0$, $c = 0$, and $b = 1$. $c = -2$.

The factors of (2″) give $b = 1$ and $d = (b + 1)^{-1}$.

Substituting $b = 1$ in (5) and (1″) we find respectively $c = -1$ and $d = -1$. Replacing d by $(b + 1)^{-1}$ in (1″) gives

$$2b^3 + 2b^2 - 1 = 0. \tag{6}$$

The discriminant of (6) is positive so that this cubic has two complex roots and one positive real root. These roots may be expressed as

$$b = \tfrac{1}{6}[(46 + 6\sqrt{57})^{1/3} + (46 - 6\sqrt{57})^{1/3}] - \tfrac{1}{3}.$$

The approximate value of the real root, b_1, is

$$b_1 = +\,0.565,197,717,38.$$

Accordingly, the complex roots are approximately

$$b = \tfrac{1}{2}[-(b_1 + 1) + i\sqrt{(b_1 + 1)(3b_1 - 1)}],$$

or

$$b_2 = -\,0.782,598,858,69 + i0.521,713,717,94,$$

$$b_3 = -\,0.782,598,858,69 - i0.521,713,717,94.$$

The corresponding values of c and d may be found by the aid of (5) and $d = (b_1 + 1)^{-1}$, respectively. Finally, collecting all the results

b	c	d
0	0	0
1	-2	0
1	-1	-1
b_1	$-b_1^{-1}$	$(b_1+1)^{-1}$
b_2	$-b_2^{-1}$	$(b_2+1)^{-1}$
b_3	$-b_3^{-1}$	$(b_3+1)^{-1}$

Also solved by J. E. ROWE and the Proposer.

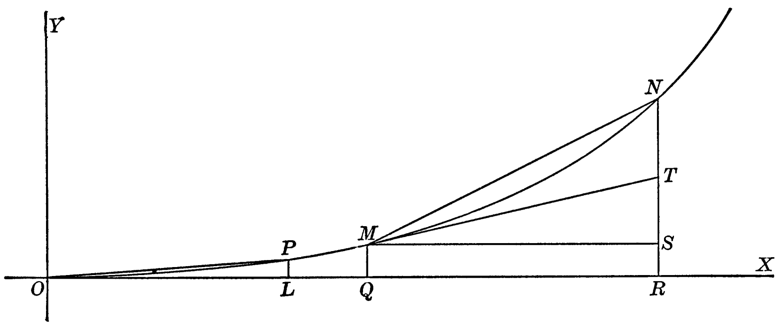
524 (Geometry). Proposed by NORMAN ANNING, Somewhere in France.

Many railways use as "easement curve" the cubic parabola. If points on such a curve are named by their distances measured along the curve from the point of inflection ("flat end")

show that, within the limits of ordinary practice, *i. e.*, for angles so small that the difference between arc and sine is inappreciable, the deflection from tangent at m to set n is $(n - m)(n + 2m)$ times the deflection from tangent at 0 to 1.

Solution by W. J. Thome, Detroit, Michigan.

Let the cubic parabola $y = ax^3$ be constructed, and on it let the points named 0, 1, m, and n be represented by O, P, M, and N respectively. Through these points draw the lines parallel to the axes as shown in the figure. Also the chords OP and MN, and MT, the tangent at M.



Since the angles are so small that $\theta = \sin\theta$, we also have $\theta = \tan\theta$ and $\cos\theta = 1$. Also, an arc of the curve, its chord, and the projection of either on the X-axis are all equal. Hence we have

$$\measuredangle LOP = \tan LOP = \frac{LP}{OL} = \frac{a(OL)^3}{OL} = a(OL)^2 = a(OP)^2 = a(1)^2 = a,$$

$$\measuredangle SMN = \tan SMN = \frac{SN}{MS} = \frac{RN - RS}{QR} = \frac{RN - QM}{OR - OQ} = \frac{a(OR)^3 - a(OQ)^3}{OR - OQ} = \frac{a(ON)^3 - a(OM)^3}{ON - OM}$$

$$= \frac{a(n)^3 - a(m)^3}{n - m} = \frac{a(n^3 - m^3)}{n - m} = a(n^2 + nm + m^2),$$

$$\measuredangle SMT = \tan SMT = \frac{dy}{dx} = 3ax^2 = 3a(OQ)^2 = 3a(OM)^2 = 3am^2,$$

$$\begin{aligned}\measuredangle TMN &= \measuredangle SMN - \measuredangle SMT \\ &= a(n^2 + nm + m^2) - 3am^2 = a(n^2 + nm - 2m^2) \\ &= (n - m)(n + 2m)a \\ &= (n - m)(n + 2m)\measuredangle LOP.\end{aligned}$$

525 (Geometry). Proposed by C. N. SCHMALL, New York City.

Given a quadrant of a circle AOB, where OA and OB are bounding radii, and a semicircle ACO having OA as a diameter and lying on the same side as the quadrant. Describe a circle which shall touch the two arcs and the radius OB.

I. Solution by W. J. Thome, Detroit, Mich.

Suppose the problem solved and the figure constructed.

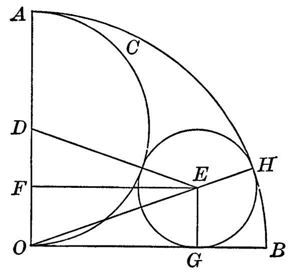
Let O and D be the centers of the two given circles and let E be the center of the required circle. Draw DE. Draw OE and extend it until it intersects the given quadrant at H. Through E draw EF and EG perpendicular to OA and to OB, respectively.

The problem may be considered solved if we can obtain a value of EG, the radius of the required circle. Let $OA = 2R$, $OD = R$, and $EG = r$. Then $OE = 2R - r$, $DE = R + r$, and $DF = R - r$. Now $\overline{OE}^2 - \overline{EG}^2 = \overline{DE}^2 - \overline{DF}^2$, since $OG^2 = FE^2$, or

$$(2R - r)^2 - r^2 = (R + r)^2 - (R - r)^2.$$

Whence,

$$r = \frac{R}{2}, \quad \text{or} \quad EG = \tfrac{1}{2}OD.$$



II. Solution by Horace Olson, Chicago, Illinois.

Let x and y be the coördinates of the center of the required circle, referred to OB and OA as the positive directions on the x and y axes, respectively. Then by the conditions of the problem:

$$y = r - \sqrt{x^2 + y^2}, \qquad y = \sqrt{\left(\frac{r}{2} - y\right)^2 + x^2} - \frac{r}{2}; \qquad r = OA;$$

whence

$$r^2 - 2ry - x^2 = 0, \quad \text{and} \quad 2ry - x^2 = 0.$$

Subtracting, we have $r^2 - 4ry = 0$; whence $y = \frac{1}{4}r$ and $x = \pm \frac{1}{2}\sqrt{2}r$. The value $x = +(r/2)\sqrt{2}$ is evidently the only one applying to this problem. The radius of the required circle is therefore $\frac{1}{4}r$.

A solution similar to the second above was received from Elijah Swift.

435 (Calculus). Proposed by B. F. FINKEL, Drury College.

Show that

$$\int_0^\infty e^{-x^2-(a^2/x^2)}dx = \frac{\sqrt{\pi}}{2e^{2a}}$$

by a transformation rather than by the usual method of differentiating under the sign of integration, as, for example, in Byerly's *Integral Calculus*, pages 106–107.

Note by William Hoover, Columbus, Ohio.

On this interesting equation, reference may be made to an article by William Walton in the *Quarterly Journal of Mathematics*, Vol. XII, p. 181, On the Evaluation of a Pair of Definite Integrals, a few lines opening it being as follows:

"The evaluation of the two definite integrals

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]\cos\alpha} \cdot \cos\left\{\left(x^2 + \frac{c^2}{x^2}\right)\sin\alpha\right\}dx,$$

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]\cos\alpha} \cdot \sin\left\{\left(x^2 + \frac{c^2}{x^2}\right)\sin\alpha\right\}dx$$

is effected in textbooks on the integral calculus by the substitution of an impossible expression for k in the known relation

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]k}dx = \frac{\pi^{1/2}}{2k^{1/2}} \cdot e^{-2ck}.$$

"As such a method of arriving at the evaluation of definite integrals is regarded by eminent writers on the subject as suggestive rather than demonstrative, I think that the following evaluations of the two integrals in question may be of some interest to students."

He proceeds with the demonstrations, occupying two and one half pages of the *Quarterly*, avoiding the methods he criticizes and that by dlfferentiation referred to in Mr. Finkel's problem, closing with the words "the results here obtained coincide, it may be remarked, with those given in Todhunter's *Int. Cal.*"

To solve Professor Finkel's problem we need only put $\alpha = 0$ in the first of the integrals mentioned by Walton.

437 (Calculus). Proposed by LEIGH PAGE, Yale University.

Integrate

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx$$

without the use of the gamma functions.

I. Solution by Oscar S. Adams, Coast and Geodetic Survey, Washington, D. C.

By direct integration, we have

$$\int_0^\infty e^{-ax}dx = \frac{1}{a}.$$

Let $a = \alpha - i\beta$. Then

$$\int_0^\infty e^{-\alpha x+i\beta x}dx = \frac{1}{\alpha - i\beta} = \frac{\alpha + i\beta}{\alpha^2 + \beta^2},$$

or

$$\int_0^\infty e^{-\alpha x}(\cos \beta x + i \sin \beta x)dx = \frac{\alpha + i\beta}{\alpha^2 + \beta^2}.$$

By equating real and imaginary parts we obtain the two definite integrals

$$\int_0^\infty e^{-\alpha x} \cos \beta x dx = \frac{\alpha}{\alpha^2 + \beta^2}, \qquad \text{and} \qquad \int_0^\infty e^{-\alpha x} \sin \beta x dx = \frac{\beta}{\alpha^2 + \beta^2}.$$

Since the first of these integrals is a uniform function of β, we have the relation

$$\int_0^\beta d\beta \int_0^\infty e^{-\alpha x} \cos \beta x dx = \int_0^\infty dx \int_0^\beta e^{-\alpha x} \cos \beta x d\beta = \int_0^\beta \frac{\alpha\, d\beta}{\alpha^2 + \beta^2};$$

or

$$\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x}\, dx = \tan^{-1} \frac{\beta}{\alpha}.$$

This integral is also a uniform function of β in the region $+\frac{\pi}{2} \geqq \tan^{-1} \frac{\beta}{\alpha} \geqq -\frac{\pi}{2}$.

Hence, in this domain, we have

$$\int_0^\infty dx \int_0^2 e^{-\alpha x} \frac{\sin \beta x}{x}\, d\beta = \int_0^2 \tan^{-1} \frac{\beta}{\alpha}\, d\beta$$

or

$$\int_0^\infty \frac{e^{-\alpha x}(1 - \cos 2x)}{x^2}\, dx = 2 \tan^{-1} \frac{2}{\alpha} - \frac{\alpha}{2} \log (\alpha^2 + 4).$$

This is a uniform function of α. Hence, α can converge to zero. This gives

$$\int_0^\infty \frac{2 \sin^2 x}{x^2}\, dx = 2 \cdot \frac{\pi}{2} = \pi, \qquad \text{or} \qquad \int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \tfrac{1}{2}\pi.$$

Since $\sin^2 x/x^2$ is an even function of x, we have

$$\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\, dx = 2 \int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \pi.$$

II. Solution by G. Paaswell, N. Y. City.

Since $\sin^2 x/x^2$ is an even function of x we have

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = 2 \int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$$

Place

$$\int_0^{\infty} \frac{\sin^2 ax}{x^2} dx = u; \qquad \text{then} \qquad \frac{du}{da} = \int_0^{\infty} \frac{\sin 2ax}{x} dx = \frac{\pi}{2}.$$

Hence, $u = \pi a/2$. Upon placing $a = 1$, we have

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi.$$

III. Solution by the Proposer.

The definite integral

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2} dx$$

is one of frequent occurrence in theoretical physics. As the text-books invariably seem to use gamma functions in evaluating this integral, the following integration without the use of other than the elementary functions may be of interest to lecturers in theoretical physics and to their students. We have

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2} dx = 2 \int_0^{\infty} \frac{\sin^2 x}{x^2} dx$$

since the integrand is an even function of x.

Consider

$$\int_0^{\infty} \int_0^{\infty} \int_y^{\infty} e^{-yx} \sin^2 x dx dy dy = \int_0^{\infty} \int_0^{\infty} e^{-yx} \frac{\sin^2 x dx dy}{x} = \int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$$

Since the limits for the y integration are not functions of x;

$$\begin{aligned} \int_0^{\infty} \int_0^{\infty} \int_y^{\infty} e^{-yx} \sin^2 x dx dy dy &= \int_0^{\infty} \int_y^{\infty} \int_0^{\infty} e^{-yx} \sin^2 x dy dy dx \\ &= \frac{1}{2} \int_0^{\infty} \int_y^{\infty} dy dy \left[\int_0^{\infty} e^{-yx} dx - \int_0^{\infty} e^{-yx} \cos 2x dx \right]. \end{aligned}$$

Now

$$\int_0^{\infty} e^{-yx} dx = \frac{1}{y} \qquad \text{and} \qquad \int_0^{\infty} e^{-yx} \cos 2x dx = \frac{y}{y^2 + 4}.$$

Hence,

$$\int_0^{\infty} \int_0^{\infty} \int_y^{\infty} e^{-yx} \sin^2 x dx dy dy = 2 \int_0^{\infty} \int_y^{\infty} \frac{dy dy}{y(y^2 + 4)} = \frac{1}{4} \int_0^{\infty} \log\left(1 + \frac{4}{y^2}\right) dy.$$

Integrating the last expression by parts,

$$\frac{1}{4} \int_0^{\infty} \log\left(1 + \frac{4}{y^2}\right) dy = \frac{1}{4}\left[y \log\left(1 + \frac{4}{y^2}\right)\right]_0^{\infty} + \frac{1}{2} \int_0^{\infty} \frac{dy}{1 + \frac{y^2}{4}} = \frac{1}{4}\left[y \log\left(1 + \frac{4}{y^2}\right)\right]_0^{\infty} + \frac{\pi}{2}.$$

The expression $y \log [1 + (4/y^2)]$ is indeterminate for both limits. Evaluating it by the usual methods, it becomes zero for each. Hence,

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2} \qquad \text{or} \qquad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi.$$

Solutions similar to the second above were received from William Hoover and Paul Capron.

Solutions were received too late for credit in a previous issue of the Monthly as follows: From Paul Capron, 490 and 491 Algebra and 522 Geometry; from C. P. Sousley, 486 and 488 Algebra and 518, 520, 521, and 522 Geometry; from William Hoover, 521 and 522 Geometry; and from C. C. Yen, 490 Algebra and 522 Geometry.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

DISCUSSIONS.

I. Relating to Generalizations of the Witch and the Cissoid.

By Frederick H. Hodge, Franklin College.

Suppose a circle given of radius a tangent to the x-axis at the origin and a second circle of radius $2a + k$ tangent to the first at the point where the y-axis cuts it and, hence, having $(0, -k)$ as the coördinates of its center. We draw a radius vector (see figure), OS, through the origin making an angle ϕ with the x-axis and cutting the circles in the points R and S respectively. We draw a line through R parallel to the x-axis and a line through S parallel to the y-axis. We wish to determine the equation of the locus of P the point of intersection of these two lines as the radius vector revolves about the origin.

From the figure it is readily seen that the parametric equations of the locus are

$$x = \overline{OS} \cos \phi, \qquad y = \overline{OR} \sin \phi.$$

Since $\overline{OR} = 2a \sin \phi$, we have $y = 2a \sin^2 \phi$ and $\sin^2 \phi = y/2a$. To determine the value of $\overline{OS}$ we connect C with S and in the triangle OSC we have, by the law of cosines, $\overline{CS}^2 = (2a + k)^2 = k^2 + \overline{OS}^2 - 2k\, \overline{OS} \cos (90^\circ + \phi) = k^2 + \overline{OS}^2 + 2k\, \overline{OS} \sin \phi$. Hence, $\overline{OS}^2 + 2k\, \overline{OS} \sin \phi - 4a^2 - 4ak = 0$. This gives the desired value for $\overline{OS}$,

$$\overline{OS} = -k \sin \phi \pm \sqrt{4a^2 + 4ak + k^2 \sin^2 \phi},$$

from which we have

$$x = (-k \sin \phi \pm \sqrt{4a^2 + 4ak + k^2 \sin^2 \phi}) \cos \phi.$$

Transposing the $-k \sin \phi \cos \phi$, squaring and simplifying by means of the relation $y = 2a \sin^2 \phi$, we obtain the desired equation:

$$a^2[x^2 - 2(a + k)(2a - y)]^2 = k^2x^2(2ay - y^2).$$

Except for particular values of k this is a fourth-degree equation and the locus passes through the points $(0, 2a)$ and $(\pm 2 \sqrt{a(a + k)}, 0)$.

If $k = 0$ the equation reduces to the parabola $x^2 = -2ay + 4a^2$.

If $k = \infty$, the larger circle becomes a straight line and the construction is identical with the well-known construction giving the witch of Agnesi. To verify the fact that the witch is a member of the family of curves we have under consideration we take the equation in the form

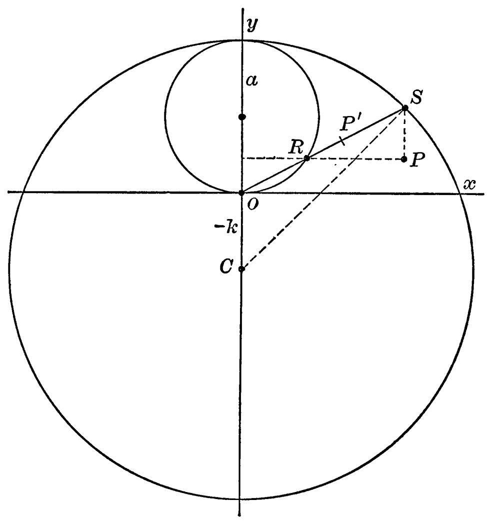
$$x^2 - \frac{kx}{a} \sqrt{2ay - y^2} = 2(a + k)(2a - y),$$

divide by k and then let k approach ∞. The equation then reduces to

$$-x\sqrt{2ay-y^2} = 2a(2a-y) \quad \text{or} \quad x^2y = 4a^2(2a-y),$$

which is the equation of the witch.

To obtain a generalization of the cissoid, take a radius equal to OR and with S as center cut SO at P' (see figure). We proceed to show that the locus of the point P' as the radius vector revolves about the point O is such a curve.



From the figure,

$$OP' = OS - P'S = OS - OR.$$

From the work above we have

$$OS = -k\sin\varphi \pm \sqrt{4a^2+4ak+k^2\sin^2\varphi},$$

$$OR = 2a\sin\varphi.$$

Hence, the locus of P' is given in polar coördinates by the equation

$$= -(k+2a)\sin\varphi \pm \sqrt{4a^2+4ak+k^2\sin^2\varphi}.$$

Transforming from polar to rectangular coördinates and simplifying, the equation becomes

$$(x^2+y^2)^2 + 2(k+2a)y(x^2+y^2) = 4a(a+k)x^2.$$

This gives a quartic curve very much like the cardioid in appearance. It passes through the points $(0, 0)$, $(2\sqrt{a(a+k)}, 0)$ and $(0, -2\sqrt{k+2a})$. If both sides of the equation be divided by k and k allowed to approach infinity the equation approaches the limiting form $y^3 = x^2(2a-y)$, which is the equation of the cissoid (x and y being interchanged from the usual form). Obviously as k

approaches infinity the larger circle approaches a straight line and the construction merges into the classic construction for the cissoid.

II. Relating to a Geometric Proof of a Theorem on Collineations.

By James H. Weaver, Hilliard, Ohio.

The relations between double elements of a collineation in space (by which is meant a three-space, unless otherwise stated) have been discussed analytically, use being made of the characteristic equation of the collineation and of the properties of elementary divisors.[1] Enriques, in his *Projective Geometrie* (Leipzig, 1903), has shown geometrically that if a collineation in a plane has a double point it has a double line. It is here proposed to prove the

Theorem. *If a collineation in space has a double point, it has a double plane.*

If the collineation is perspective the theorem is evident. Suppose, then, that the collineation is non-perspective.

Let A be a double point of a non-perspective collineation π and let a be a line through A. Then π sets up the following correspondence

$$a \barwedge a_1 \barwedge a_2 \barwedge a_3 \barwedge a_4.$$

Let B and C be two distinct points on a different from A. Then

$$B \barwedge B_1 \barwedge B_2 \barwedge B_3 \barwedge B_4,$$

where B_i is on a_i $(i = 1, 2, 3, 4)$, and similarly for the C's. Let the intersection $(BB_1, CC_1) = D$. Since A is a self-corresponding point, D will be independent of the points B and C. Also, let $(B_iB_j, C_iC_j) = D_i$, $(i = 1, 2, 3, j = i + 1)$.

If a, a_1 and a_2 are coplanar the theorem is evident and the double plane contains also a double line.

If a, a_1 and a_2 are not coplanar, D_1 is not on plane $[aa_1]$. Let the line DD_1 meet the plane $[aa_1]$ in some point E, and let us assume C so chosen that E lies on CC_1. Let $\pi(E) = E_1$. E_1 is on D_2D_3 and also on the plane $[CC_1, C_1C_2]$. Hence D, D_1, D_2 and D_3 are coplanar and the plane containing them is a double plane.

By duality, *if a collineation in space has a double plane it also has a double point.*

The method here used for establishing the theorem for a three-space is general and can be applied to an n-space giving the theorem that if a collineation in an n-space has a double point it has a double $(n - 1)$-space. This method of proof when applied to collineations in a plane is somewhat simpler than that of Enriques mentioned above.

[1] Segre, "Sulla teoria e sulla classificazione delle omografie in un spazio lineare ad un numero qualunque di dimensioni," *Reale Accad. dei Lincei*, Serie 3*a*, Bd. XIX, S. 6. See also Muth, *Theorie und Anwendung der Elementartheiler*, Leipzig, 1899; Veblen and Young, *Projective Geometry*, Vol. I, Boston, 1910; Bôcher, *Introduction to Higher Algebra*, N. Y., 1912; Newson, "A New Theory of Collineations," *Amer. Jour. Math.*, Vol. XXIV.

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by R. C. Archibald, Brown University, Providence, R. I.

CLUB ACTIVITIES.

The Mathematics and Physics Club of the University of Alabama, University, Ala.

This club was founded in November, 1916, under the name Mathematics Club, but at the second meeting this year the name was changed to that indicated above. The object of the club as stated in the constitution is "to encourage interest in mathematical and physical science at the University of Alabama and to facilitate the acquaintance of persons at the University who are interested in mathematics and physics." The average attendance at meetings now is about 12 although as many as 50 have been present at special meetings.

The officers for 1917–18 are: President, Robert F. Leftwich '18; secretary-treasurer, Donald H. Thornbury '18. The program committee consists of these officers and of Professor Tomlinson Fort, the head of the department of mathematics. The following programs were given during the autumn of 1917:

October 15: "Fourth Dimension" by David Skolnick '19; "Duplicating the Cube" by Robert T. Milne, instructor in mathematics.

November 7: "Curves in the Physical and Chemical Laboratories" by Maurice B. Amise '18; "John Napier" by Reginald C. Smith '19.

December 1: "Mathematics in Artillery Practice" by Robert F. Leftwich '18.

December 7: "Curved Flight of a Base-ball and similar Problems" by Donald H. Thornbury '18.

Barnard College Mathematics Club, Columbia University, New York, N. Y.

This club was founded in 1909, in order "to foster an interest in mathematics." All students who are taking or who have had the equivalent of a course in analytics and elementary calculus are eligible for membership. There are now about 40 members and the average attendance is 25. The club is entirely in the hands of students who elect each year an honorary president from among the members of the mathematics faculty. This officer takes no part in the affairs of the club except in an advisory capacity.

The officers of the club for 1917–18 are: Honorary president, Dr. George W. Mullins; president, Ellen Leut '18; vice-president, Viola Williams '18; treasurer, Sophie Schulman '18; secretary, Mary Wellick '18; program committee: Fannie Rubenstein '18, Sophia Koerner '19 and Veronica Jentz '20.

Each November the club has its annual dance and once a year the alumnæ of the club are called together and the program is in the nature of an alumnæ report. The following programs were given in 1916–18:

October, 1916: "Dynamic Trajectories" by Professor Edward Kasner.

November: "Non-homogeneous Coördinates" by Eugenia Hausle '17.

December: "Fourth-Dimensional Vistas"[1] by Beatrice Walker '17.

February, 1917: "Non-Euclidean Geometry" by Helena Bausch '17; "Measurements of Time" by Sophie Schulman '18.

March: "Mathematics of Navigation" by Harold Jacoby, professor of astronomy in Columbia University.

April: "Paper Folding" by Mary Welleck '18; "Geometrical Puzzles" by Fanny Rubenstein '18.

May: "Women in Science" by Professor Kasner; "Mathematical Fallacies" by Dr. Charles A. Fisher, instructor in mathematics; "Mathematics of Business" by Dr. George W. Mullins.

October 15, 1917: Opening meeting. Informal addresses by Professor Frank N. Cole, and Dr. G. W. Mullins and Dr. Kenneth W. Lamson, instructors in mathematics.

December 7: "Generalizations of Elementary Mathematical Concepts" by Professor Thomas S. Fiske.

January 15, 1918: "An Introduction to Projective Geometry" by Gretchen Torek '19; "Mathematical Card Tricks" by Fanny Rubenstein '18; "Magic Squares" by Dorothy Jacobs '18.

February 19: "Curve Families" by Dr. G. W. Mullins.

March 15: "Correlation and Graphical Methods" by Helena Bausch '17 and Evelyn Davis '17, now in the research department of the American Telephone and Telegraph Company.

April 30: "Infinite Series" by Joyce Buckbee, '18.

Mathematics Club of Columbia University, New York, N. Y.

In order "to stimulate and further the interests of mathematical scholarship amongst undergraduate students" and first-year graduate students at Columbia University the Junior Mathematical Colloquium was organized in November or December, 1910. Three years later this developed, under the supervision of Professor Herbert E. Hawkes, into the present more formal club with a constitution. The moving spirits in the founding of the colloquium were Professor Charles C. Grove and Dr. N. J. Lennes whose activity at the University of Chicago along similar lines has been already referred to in these columns.

The officers for 1917–18 are: President, William L. Schaaf '19; vice-president, Russell D. Burdick '19; secretary-treasurer, Francis W. Rogers '19. The executive committee (which acts as program committee) is composed of "officers of the club and a member of the Faculty of Columbia University nominated by the Mathematics Department of Columbia University." This faculty adviser is now Professor C. C. Grove. The attendance at the meetings last year varied from 9 to 32 with an average of about 19; the average attendance this year is about 14.

The following are programs of meetings 1916–18:

[1] *Four-Dimensional Vistas* is the title of a book by C. Bradgon (New York, A. A. Knopf, 1916).

October 16, 1916: "Algebraic Methods of Solving Quadratic Equations" by Franklin Hollander '19.
October 30: "Graphical Solution of Quadratic Equations" by Emil A. Goerlich '19.
November 13: "Life of Emory McClintock, Columbia's Greatest Graduate in the Field of Mathematics" by Professor Thomas S. Fiske.
November 27: "Geometric Constructions of the Cubic" by Professor Henry B. Mitchell.
December 18: "Some Theorems on the Cubic" by Charles J. Hyman '17.
January 8, 1917: "Discussion of the Kind of Points in a Plane necessary to carry out the Geometry of Euclid" by Lewi Tonks '17.
February 19: "Why is a Conic?" by Richard Wagner, Jr. '18.
March 5: "Logarithms in General of Complex Numbers" by William L. Schaaf '19.
March 19: "Equal Area Maps" byIsrael Koral '20.
April 2: "Rotating Figures" by William Malisoff Gr.
April 23: "The Path of a Projectile" by Dr. Charles A. Fischer, instructor in mathematics.
October 29, 1917: "Descartes and his Theory of Equations" by Professor William B. Fite.
November 12: "Sturm's Function" by George Dean '20; "Perspective Triangles" by Charles P. Davis '20.
November 26: Problems proposed for solution.
December 4: Informal discussion meeting. Solutions of problems set at the last meeting.
December 17: "Quaternions" by Victor Schachtel '19.
February 18, 1918: "Two Proofs of the Fundamental Theorem of Algebra" by Alfred M. Michaelis '21 and William L. Schaaf '19.
March 4: "Theory of Transfinite Numbers" (Cantor) by Franklin Hollander '19.
March 18: "Mathematics in the Science of Physics" by Leon Morris '19.
March 25: "Dimensionality" by Israel Koral '20.
April 8: "General Theory of Relations" by Professor Cassius J. Keyser.
April 22: "Paradoxes of the Infinite" [1] by Moses Davis '20.

THE EUCLIDEAN CIRCLE OF INDIANA UNIVERSITY, Bloomington, Ind.

This club was organized in September, 1907. In 1916–17 there were about 25 members and the attendance at each meeting was 10–12. During the current year it was not found feasible to reorganize the Circle.

Those qualified for membership have been students majoring in mathematics, juniors, seniors, graduates, and members of the mathematical faculty. The object of the organization was to make its members "better acquainted with the history of mathematics." "A club picture has been taken each year."

[1] B. Bolzano's *Paradoxien des Unendlichen* (first published in 1850, after his death; facsimile reprint: Berlin, Mayer und Müller, 1889) has been more than once the basis of lectures by Schwarz at the University of Berlin.

THE MATHEMATICAL AND PHYSICAL SOCIETY OF THE UNIVERSITY OF TORONTO, Toronto, Ontario.

Among the existing undergraduate clubs of America this one is notable in more ways than one. In the first place it is the most venerable, since its thirty-sixth birthday is already a thing of the past. In the second place it has published three slight volumes of its proceedings.[1] The society was founded on January 27, 1882, when the report of a committee appointed to draw up a constitution was adopted and the following officers were elected: President, W. S. Loudon Gr.; vice-president, J. M. Clark; secretary-treasurer, D. Burns; corresponding secretary, T. G. Campbell; councillors: Faculty of Arts, A. H. McDougall (4th year), G. I. Riddell (3d year), T. Mulvey (2d year), G. H. Hogarth (1st year); School of Practical Science, D. Jeffrey (3d year), —— Fotheringham (2d year). Of these officers Mr. Loudon is now professor of mechanics in the University of Toronto; Mr. Clark is a well-known lawyer of Toronto; Mr. McDougall is principal of the Collegiate Institute at Ottawa and Mr. Mulvey is Under Secretary of State at Ottawa.

A few years later when the students in the School of Practical Science became sufficiently numerous they seceded and formed the Engineering Society.

At first a graduate was chosen president, the other officers being undergraduates. Some years ago the constitution was changed, so that a graduate was to be honorary president and undergraduates were to hold the active offices. The officers for 1917–18 are as follows: Honorary president, Dr. John Satterly, assistant professor of physics; president, Norris E. Sheppard '18; vice-president, William W. Shaver '19; secretary-treasurer, Everett O. Hall '19; corresponding secretary, Hazel C. Miller '18; representatives of classes: Harry E. Foreman '18, Mary M. Stephens '19, William S. Vaughn '20, Donald F. Shugart '21.

The object of the society is "the encouragement of study and original research

[1] Since the editions of these publications were very small, and long since exhausted, and since particulars concerning them are lacking in all mathematical bibliographies, it may be well to put some details on record here. The title page of each part: *Papers read before the Mathematical and Physical Society of Toronto University during the year(s) 1890–91* [*1891–92*] (*1892–93, 1893–94*), Toronto: Rowsell and Hutchison, Printers, King Street East, 1891 [1892] (1895). Pages 60 [60] (54).

Contents, *Part 1:* A. Baker, "Poetic Interpretation in Mathematics," pp. 7–21; A. T. De Lury, "On Certain Deductions from the Theorem of Dr. Graves," pp. 22–30; A. C. Chant, "The Structure of Matter," pp. 31–42; F. Sanderson, "The Law of Human Mortality and its Place in Science," pp. 43–54; R. Henderson, "Newton's Laws of Motion," pp. 55–59. *Part 2:* W. J. Loudon, "Musical Scales, Their Origin, Formation, and the Physical Relation Which They Bear to Music," pp. 5–18; I. E. Martin, "The Religion of Algebraic Curves," pp. 19–32; C. A. Chant, "The Wave Theory of Sound," pp. 33–47; W. Gillespie, "Some Trigonometrical Expansions," pp. 48–53; G. R. Anderson, "Measurement of Time," pp. 54–60. *Part 3:* "On the Non-Euclidean Geometry," pp. 6–28 [a translation by A. T. DeLury of the "Note sur la géométrie non-euclidienne" in the *Traité de Géométrie* of Rouché et de Comberousse]; J. C. Glashan, "Elementary Proof of a Theorem in Trigonometry," pp. 29–32; C. A. Chant, "The Development of Electricity," pp. 33–46; A. M. Scott '96, "Geometrical Representation of the Special Roots of Unity" (abstract), pp. 47–49; Miss J. S. Hillock '95, "Laplace" (abstract), p. 50; G. W. Rudlen '94, "The Quadrature of the Circle" (abstract), pp. 51–52; Miss A. Lindsay '93, "Maria Gaetana Agnesi" (abstract), pp. 53–54.

in mathematics and physics, and the preservation of the results of such work." All the special students in mathematics and physics are considered members—about 75 now—and the average attendance at meetings is about 50.

"In earlier years the solution of problems was a feature of each program but that has been discarded and light refreshments have been introduced." The programs[1] for 1916–18 are as follows:

November 2, 1916: Opening meeting, social; "Aircraft," an address by John C. McLennan, professor of physics.

November 16: "Cultural Advantages of the Mathematics and Physics Course" by Herbert R. Rowan '17 ("To strive, to seek, to find, and not to yield");[2] "Science versus Art," Discussion led by Mabel Campbell '19 and James B. Russell '19 ("For, even though vanquished, he could argue still").

November 30: "Our Society's Past" by Clarence A. Chant, associate professor of astrophysics ("Lives of great men all remind us"); "Experiments on Probability" by Norris E. Sheppard '18.

December 14: "Newton" by Aylmer B. Paisley '20 ("And still the wonder grew"); "The Moon: a look at our next neighbor" by Janet M. Halliday '18.

January 11, 1917: "Societies to which mathematics and physics students may aspire" by Professor John C. Fields ("Held from afar, aloft, th' immortal prize And urged the rest, by equal steps to rise"); "Experiments on Light" by Albert R. Self '17.

January 25: "Euclid" by William W. Shaver '19 ("Let us begin and carry up this corpse, Singing together"); "Mathematical Symbolism" by James C. Thompson '18 ("Earth hath not anything to show more fair").

February 8: Open Meeting ("Up, up my friend and quit your books").

February 22: "Life Assurance" by William A. Jackson '17 ("Yet the strong man must go"); "Experiments in Heat and Mechanics" by Professor John Satterly ("While we sat wrapt in wonder").

March 8: Annual Business Meeting ("The old order changeth, Yielding place to new").

November 8, 1917: Opening meeting. "George Russell" by Professor Alfred T. DeLury.

November 22: "The ancient Mathematicians" by Douglas H. Blatchford '18; "Germany and the Germans" by Professor John C. Fields.

December 6: Graduates' meeting. "Life in an Actuary's Office" by Janet Holmes '17; "Mathematicians of the Middle Ages" by William A. Jackson '17; "Spectra" by Florence M. Quinlan '17.

December 13: Social meeting.

January 10, 1918: "Mathematics of Modern Times" by Mabel C. Childs '18; "J. J. Thompson" by Ernest R. I. Pratt '18.

January 27: Debate between First and Second Year Students.

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[1] For many years programs have been issued annually in printed form.

[2] After most of the titles on the printed program for 1916–17 were quotations, such as this, in small type.

February 9: Open meeting. "Eclipses" by Professor Clarence A. Chant.
February 21: "Faraday" by Ila B. Giles '19; "The Mathematics and Physics Courses at the Royal College of Science, London" by Professor John Satterly.
March 7: Annual Business Meeting.

MATHEMATICS CLUB, The Western College for Women, Oxford, Ohio.

This club was organized in 1905 "to stimulate interest in certain phases of mathematics which, while closely related to class work, do not fall directly under it. The members include those in the elective mathematics classes and the teaching staff of the department. For the current year there are 25 student and 3 faculty members. There are no officers; the seniors take turns in looking after the meetings. The programs for the past three years are as follows:
December 6, 1915: "Some Constructions leading to Conic Sections" by Frances Orr, instructor in mathematics; "A Problem in Factoring" by Kathleen Banker '16; "Parallelograms inscribed in a Rectangle" by Mary C. Little '16; "A few Classic Unknowns in Mathematics," report of G. A. Miller's article in *Scientific Monthly*, October, 1915, by Helen McBride '16.
February 14, 1916: "Higher Plane Curves, their use in Mechanics" by Harriet Rice '16; "Higher Plane Curves, their Application to Geometrical Constructions" by Ginevra E. McCoy '17; "Mathematics in Dr. Eliot's Five-Foot Shelf of Books," report of W. H. Bussey's article in this MONTHLY, June, 1915, by Norrine DeLaney '17; Current Topics.
March 13: "Number Systems of the North American Indians"[1] by Ethel S. Sebald '17; "Early History of Mathematics in the United States" by Marie Pearson '17.
October 23, 1916: Picnic.
November 20: "The Triangle and its Circles"[2] by Edna P. Pepper '18; "A Circle Theorem"[3] by Anna S. Armstrong '17; "Elementary Proof of a Theorem on the Circle due to F. Morley," report of T. Dantzig's article in this MONTHLY, September, 1916, by Mary C. Little, assistant in mathematics; "Recent additions to the mathematics library" by Annie C. Crane '19 and Edna Berkele '19.
January 22, 1917: "Greek Methods of Solving Quadratic Equations"[4] by Ethel S. Sebald '17; "Some modern Solutions of the Quadratic" by Anna S. Armstrong '17; "Graphic Solution of $y = x^n$" by Mary R. Shipp '18; Report on

[1] W. C. Eells's paper with this title appeared in this MONTHLY, November and December, 1913.

[2] The following pamphlet on this subject has been published: W. H. Bruce, *Some Noteworthy Properties of the Triangle and its Circles* (Heath's Mathematical Monographs, No. 8). Boston, Heath, 1903, pp. 28.

[3] R. A. Johnson's article with this title appeared in this MONTHLY for May, 1916. It was followed by A. Emch's "Remarks on the Foregoing Circle Theorem." The theorem in question ("If three equal circles are drawn through a point, the circle through their intersections is equal to each of them") was announced by Michail Manoïlescu in *L'Education Mathématique*, 1 Mai, 1910, 12 année, p. 128.

[4] W. C. Eells's paper with this title was published in this MONTHLY, January, 1911.

"Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi" [1] by Ginevra E. McCoy '17; Current Topics by Mary E. Spencer '19, Rhoda E. Trook and Dorothy M. Wilkinson '19.

March 5: "On the Representation of Large Numbers and Infinite Processes," report of A. Emch's article in *Scientific Monthly*, March, 1916, by Jessie P. Wise '18; "Rithmomachia and other number games" [2] by Sarah M. Sloan '18; "Perfect Numbers" by Ginevra E. McCoy '17; "A Curious Convergent Series" [3] by Mary C. Little, assistant in mathematics.

October 22, 1917: Reception for Dr. Ettalene Grice '08, who is assisting in reading some of the Babylonian tablets in the Yale Museum; she gave a talk on the "Mathematics of the Babylonians."

October 29: "Magic Squares" by Helen T. Anger '19 and Sena M. Sutherland '18; "Some Peculiar Properties of Numbers" by Edna M. Sebald '18.

November 26: "Common Geometric Forms in Art" by Annie C. Crane '19 and Jessie P. Wise '18; Current Topics by Helen B. Griesmer '20.

February 11, 1918: "The Three Famous Problems of Antiquity": (*a*) "The Trisection of an Angle" by Mary E. Thomas '18; (*b*) "The Duplication of the Cube" by Mary R. Shipp '18; (*c*) "The Quadrature of the Circle" by Sarah M. Sloan '18.

March 4: "Certain Typical Problems, their Origin and History" by Edna P. Pepper '18 and Mary E. Spencer '19; Current Topics by Edith M. Sawin '19.

April 15: "The Fourth Dimension" by M. Lucile Brown, instructor in mathematics.

"At the close of each program a little time is devoted to informal discussion and a cup of tea." At Western College it is felt that even the few meetings of the club held each year "give a tremendous impetus to the genuine interest in the work of the department."

TOPICS FOR CLUB PROGRAMS.

Addendum: In the last line of Topic 5 add, after *L'Education Mathématique*, 1913, Vol. 15, pp. 161–163; Vol. 16, pp. 1–3, 13–15.

9.[4] Golden Section.

In the *Elements* of Euclid (who flourished about 300 B. C.), the following propositions occur: (1) "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment "(Book II, proposition 11); (2) "To cut a given finite line in extreme

[1] With an Introduction, Critical Notes and an English Version by L. C. Karpinski, New York, Macmillan, 1915.

[2] The *Teachers College Record* for November, 1912 (Vol. 13, pp. 385–495), is devoted to articles on "Number Games and Number Rhymes" by D. E. Smith, C. W. Hunt, F. J. Flynn, C. C. Eaton, R. K. Atwell and F. B. Selkin. Chapter 3 (pp. 413–422) on "Rithmomachia, the great medieval number game" by D. E. Smith and C. C. Eaton is reprinted, with a few modifications, from the American Mathematical Monthly, April, 1911. A bibliography is given on page 495.

[3] F. Irwin's paper with this title appeared in this Monthly, May, 1916.

[4] The reader is reminded of the statement introducing Topic 8 in the last issue of the Monthly; topics 8, 9 and 10 were selected from those which arise in the discussion of various forms of growth.

and mean ratio" (Book VI, proposition 30).[1] While these propositions are equivalent in statement the methods of construction given by Euclid are quite different. There can be little doubt that the construction in the second is due to Euclid and in the first to the Pythagoreans (fifth century B. C.). The result is used "To construct an isosceles triangle having each of the angles at the base double of the remaining one" (*Elements*, Book IV, 10) and this leads to the construction of a regular pentagon (Book IV, 11).

In the *Elements*, book XIII, the first five propositions, which are preliminary to the construction and comparison of the five regular solids, and deal with properties of a line segment divided in extreme and mean ratio, are usually attributed to Eudoxus who flourished about 365 B. C. Proclus tells us that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section"; scholars agree that "the section" refers to the division in extreme and mean ratio.

The so-called book XIV of Euclid's *Elements*, written by Hypsicles of Alexandria between 200 and 100 B. C., contains some results concerning "the section."

In recent times the name golden section has been applied to the division of a line segment as above[2] in the ratio $(\sqrt{5} - 1) : 2$. Terquem believed that the expression "extreme and mean ratio" (which is an exact translation of Euclid's Greek phrase) is "une réunion de mots ne présentant aucun sens," [3] and following J. F. Lorenz (1781) employed the term "continued section." Terquem has also suggested:[4] "diviser une droite décagonalement." Leslie introduced the term "medial section."[5] "Divine proportion" was used by Fra Luca Pacioli in 1509[6] and possibly earlier by Pier della Francesca;[7] "sectio divina" and "proportio divina" occur in the writings of Kepler.

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[1] These enunciations are taken from *The Thirteen Books of Euclid's Elements* translated with introduction and commentary by T. L. Heath, 3 vols., Cambridge, at the University Press, 1908. For statements in connection with our discussion see particularly, Vol. 1, pp. 137, 403; Vol. 2, p. 99; Vol. 3, p. 441.

[2] The earliest instances which I find of the use of the term golden section are in J. Helmes, "Eine einfachere, auf einer neuen Analyse beruhende Auflösung der sectio aurea, nebst einer kritischen Beleuchtung der gewöhnlichen Auflösung und der Betrachtung ihres pädagogischen Werthes." *Archiv der Mathematik*, Grunert, Band 4, 1844, pp. 15–22; in A. Wiegand, *Geometrische Lehrsätze und Aufgaben*, Band 2, 1. Abtheilung, Halle, 1847, p. 142; and also in A. Wiegand, *Der allgemeine goldene Schnitt und sein Zusammenhang mit der harmonischen Theilung* . . Halle, 1849.

[3] *Nouvelles annales de mathématiques*, Paris, tome 12, 1853, p. 38.

[4] *Journal de mathématiques pures et appliquées*, Paris, tome 3, 1838, p. 98.

[5] J. Leslie, *Elements of geometry, geometrical analysis and plane trigonometry*, Edinburgh, 1809, p. 66.

[6] *Divina proportione opera a tutti gli ingegni perspicaci e curiosi necessaria que ciaseum studioso di philosophia: prospettiva, pictura, sculptura, architectura: musica: e altre matematice . . . Venetiis . . . 1509.* Although not printed till 1509 the manuscript of this work was completed in 1497. The geometrical drawings were made by Leonardo da Vinci. Another edition of the Latin text "herausgegehen, übersetzt und erläutert von C. Winterberg" appeared at Vienna (Gräser) 1889. Another edition 1896, 6 + 367 pp. A full analysis of Pacioli's work is to be found in A. G. Kästner, *Geschichte der Mathematik* . . . Band I, Göttingen, 1796, pp. 417–449. See also M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Band 2, 2. Auflage, Leipzig, 1900, pp. 341 ff., 347.

[7] It has been shown by G. Mancini that parts of Pacioli's *Divina proportione* were taken from a Vatican manuscript by Pier della Francesca. See (1) G. Pittarelli, *Atti del IV. Congresso*

Pacioli's work was doubtless influential in inspiring a certain amount of mysticism in the consideration of golden section by later writers. In a work published in 1569, P. Ramus associates the Trinity with the three parts of golden section. A little later Clavius wrote of its "god-like proportions." As noted above Kepler declared himself similarly. He said also: "Geometry has two great treasures, one is the Theorem of Pythagoras, the other the division of a line into extreme and mean ratio; the first we may compare to a measure of gold, the second we may name a precious jewel." [1]

There is an interesting passage on golden section by Albert Girard in his edition of Stevin's works.[2] Girard gives a method of expressing the ratio of the segments of a line (cut in golden section) in rational numbers that converge to the true ratio. For this purpose he takes the sequence

$$(1) \qquad 0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots,$$

every term of which (after the second) is equal to the sum of the two terms that precede it, and says, after Kepler, any number in this progression has to the following the same ratios (nearly) that any other has to that which follows it. Thus 5 has to 8 nearly the same ratio that 8 has to 13; [illegible] consecutive numbers such as 8, 13, 21 nearly express the [illegible] in golden section. Since the fractions

$$(2) \qquad \tfrac{1}{1}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{5}, \tfrac{5}{8}, \tfrac{8}{13}, \tfrac{13}{21}, \cdots$$

are the various convergents of the continued fraction

$$\frac{\sqrt{5}-1}{2} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1\cdots}}},$$

Maupin reasons with force (after taking into account all which follows in the note) that Girard was probably familiar with the elements of continued fractions. Simson interprets Girard's reasoning differently. In notes on the next topic I shall have occasion to return to the remarkable series (1) and (2).

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dei matematici, tomo 3, Roma, 1909; (2) G. Mancini, "L'opera 'De Corporibus Regularibus' di Pietro Franceschi detto Francesca usurpata da Fra Luca Pacioli" (con dodici tavole) *Reale accademia dei Lincei*, 1915. See review by F. Cajori in this MONTHLY, Vol. 23, 1916, p. 384. (3) G. B. de Toni, "Intorno al codice sforzesco 'De divina proportione' di Luca Pacioli e i disegni geometrici di quest' opera attributi a Leonardo da Vinci," *Modena Soc. dei naturalisti e matematici, atti*, 13_4, 1911, pp. 52–79.

[1] Exact references to sources, and some quotations from originals, are given in (1) J. Tropfke, Geschichte der Elementar-Mathematik, Band 2, Leipzig, Veil, 1903; (2) F. Sonnenburg, *Der goldne Schnitt. Beitrag zur Geschichte der Mathematik und ihre Anwendung.* (Progr.). Bonn, 1881. (Not always reliable).

[2] *Les œuvres mathématique de Simon Stevin* revues, corrigées et augmentées par A. Girard. Leyde, 1634, pp. 169–170. The passage in question is reprinted with commentary in G. Maupin, *Opinions et curiosités touchant la mathématique* (deuxième série), Paris, 1902, pp. 203–209. It has been discussed also by R. Simson, *Philosophical Transactions*, 1753, Vol. 48, pp. 368–377.

In the nineteenth century the literature of golden section is by no means inconsiderable. It includes at least a score of separate pamphlets and books and many times that number of papers. In numerous, voluminous and rather unscientific writings A. Zeising[1] finds golden section the key to all morphology and contends, among other things, that it dominates both architecture and music. A distinctly new line was set under way by Fechner who applied scientific experimental method to the study of æsthetic objects.[2] He was led to the conclusion that the rectangle of most pleasing proportions was one in which the adjacent sides are in the ratio of parts of a line segment divided in golden section.[3]

Sir Theodore Cook discusses[4] golden section from some new points of view in connection with art and anatomy, and the writings of F. X. Pfeifer[5] remind one both in subject matter and style of treatment of Zeising's publications.

For mathematical treatment of problems in golden section, in ordinary or generalized form, see also the papers by C. Thiry[6] and R. E. Anderson,[7] E. Catalan's *Théorèmes et Problèmes de géométrie élémentaire*[8] and Emsman's program[9] containing more than 350 relations and problems.

10. A Fibonacci Series.

Foremost among mathematicians of his time was Leonardo Pisano (also known as Fibonacci) who flourished in the early part of the thirteenth century. His greatest work is Liber abbaci "a Leonardo filio Bonacci compositus, anno 1202 et correctus ab eodem anno 1228." It was first printed in 1857.[10]

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[1] For example (1) *Neue Lehre von den Proportionen des menschlichen Körpers aus einem bisher unerkannt gebliebenen, die ganze Natur und Kunst durchdringenden morphologischen Grundgesetze entwickelt*, Leipzig, 1854, 457 pp.; (2) *Das Normalverhältnis der chemischen und morphologischen Proportionen*, Leipzig, 1856, 114 pp. and the posthumous work: (3) *Der goldene Schnitt*, Leipzig, 1884, 28 pp. *Cf.* S. Günther, "Adolph Zeising als Mathematiker" *Zeitschrift für Mathematik und Physik*, Historisch-literarische Abtheilung, Band 21, 1876, pp. 157–165.

[2] G. T. Fechner, *Zur experimentalen Aesthetik*, Leipzig, 1871.

[3] C. L. A. Kunze speaks of "Rechteck der schönsten Form" in his *Lehrbuch der Planimetrie*, Weimar, 1839, p. 124. A reference may be given to a recent discussion of "printer's oblong" and "golden oblong" in H. L. Koopman, "Printing Page Problems with Geometric Solutions," *The Printing Art*, Cambridge, Mass., 1911, Vol. 16, pp. 353–356.

[4] T. A. Cook, *The Curves of Life*, London, Constable, 1914.

[5] (*a*) "Die Proportion des goldenen Schnittes an den Blättern und Stengeln der Pflanzen," *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 1885, Vol. 15, pp. 325–338; (*b*) *Der goldene Schnitt und dessen Erscheinungsformen in Mathematik Natur und Kunst*, Augsburg, [1885], 230 pp. A resumé of this work given by O. Willman in *Lehrproben und Lehrgänge aus der Praxis der Gymnasien und Realschulen*, 1892 was the basis of E. C. Ackermann, "The Golden Section," American Mathematical Monthly, 1895, Vol. 2, pp. 260–264. *Cf. Zeitschrift f. math. und naturwiss. Unterricht*, 1887, Vol. 18, pp. 44–47, 605–612.

[6] C. Thiry, "Quelques propriétés d'une droite partagée en moyenne et extrême raison," *Mathesis*, 1894, Vol. 14, pp. 22–24.

[7] "Extension of the Medial Section Problem and Derivation of a Hyperbolic Graph," *Proceedings of the Edinburgh Mathematical Society*, 1897, Vol. 15, pp. 65–69.

[8] 6e éd., Paris, 1879, pp. 261–263. Some of these properties are given in the first edition of this work by H. C. de La Frémoire, Paris, 1844.

[9] *Zur sectio aura.* Progr. Stettin, 1874 (*Cf. Zeitschrift f. math. und naturw. Unterricht*, Vol. 5, pp. 289–291).

[10] Il liber Abbaci di Leonardo Pisano pubblicato da Baldassure Boncompagni, Roma, MDCCCLVII. For an analysis of this work see M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Band II, 3. Auflage, Leipzig, Teubner, 1900, pp. 5–35.

Among miscellaneous arithmetical problems of the twelfth edition is one entitled "How many pairs of rabbits can be produced from a single pair in a year." [1] It is supposed (1) that every month each pair begets a new pair which, from the second month on, becomes productive; and (2) that deaths do not occur. From these data it is found that the number of pairs in successive months would be as follows:

(3) 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377.

These numbers follow the law that every term after the second is equal to the sum of the two preceding and form, according to Cantor, the first known recurring series in a mathematical work. The doubtful accuracy of this latter statement has been pointed out by Günther.[2]

The series (3) was well known to Kepler who discusses and connects it with golden section and growth, in a passage of his *De nive sexangula*, 1611.[3] Commentaries of Girard and Simson, and the relation of the series to a certain continued fraction, have been noted above in connection with Topic 9. But the literature of the subject is very extensive and reaches out in a number of directions. In what follows u_n will be regarded as the $(n + 1)$st term of what we shall call the Fibonacci Series (1); so that $u_0 = 0$ and $u_1 = 1$. For reasons which shall appear later the terms Lamé series, and Braun or Schimper-Braun series have been employed in this connection. Girard observed that the three numbers u_n, u_{n+1}, u_{n+1}[4] may be regarded as corresponding to lengths which form an isosceles triangle of which the angle at the vertex is very nearly equal to the angle of the regular pentagon.

The relation $u_{n-1}u_{n+1} - u_n^2 = (-1)^{n+1}$ while implied in the passage quoted from Kepler is more explicitly used by Simson (1753). It was to this relation, and hence to the Fibonacci series that Schlegel[5] was led when he sought to generalize the well-known geometrical paradox of dividing a square 8×8 into four parts which fitted together form a rectangle 5×13.[6] Catalan found (1879) the more

[1] Pages 283–284.

[2] S. Günther, *Geschichte der Mathematik*, 1. Teil, Leipzig, Göschen, 1908, p. 137.

[3] J. Kepler, *Opera*, ed. Frisch, tome 7, pp. 722–3. After discussions of the form of the bees' cells and of the rhombo-dodecahedral form of the seeds of the pomegranite (caused by equalizing pressure) he turns to the structure of flowers whose peculiarities, especially in connection with quincuncial arrangement he looks upon as an emanation of sense of form, and feeling for beauty, from the soul of the plant. He then "unfolds some other reflections" on two regular solids the dodecagon and icosahedron "the former of which is made up entirely of pentagons, the latter of triangles arranged in pentagonal form. The structure of these solids in a form so strikingly pentagonal could not come to pass apart from that proportion which geometers to-day pronounce divine." In discussing this divine proportion he arrives at the series of numbers 1, 1, 2, 3, 5, 8, 13, 21 and concludes: "For we will always have as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost. I think that the seminal faculty is developed in a way analogous to this proportion which perpetuates itself, and so in the flower is displayed a pentagonal standard, so to speak. I let pass all other considerations which might be adduced by the most delightful study to establish this truth."

[4] There is a typographical error (13 for 21) in Girard's discussion in this connection.

[5] V. Schlegel, "Verallgemeinerung eines geometrischen Paradoxons," *Zeitschrift für Mathematik und Physik*, 24. Jahrgang, 1879, pp. 123–128.

[6] This paradox was given at least as early as 1868 in *Zeitschrift für Mathematik und Physik*, Vol. 13, p. 162. *Cf.* W. W. R. Ball, *Mathematical Recreations and Essays*, 5th edition, London, Macmillan, 1911, p. 53; and E. B. Escott, "Geometric Puzzles," *Open Court Magazine*, Vol. 21, 1907, pp. 502–5.

general relation[4] $u_{n+1-p}u_{n+1+p} - u_{n+1}^2 = (-1)^{n+2-p}(u_p)$,[1] from which may be derived $u_{n+1}^2 + u_n^2 = u_{2n+1}$ first given, along with many other properties, by Lucas,[2] in a paper showing the relation between the Fibonacci series and Pascal's arithmetical triangle. It was Binet[3] who showed that

$$2^n \sqrt{5}\, u_n = (1 + \sqrt{5})^n - (1 - \sqrt{5})^n,$$

and Catalan gave the following result a few years later:[4]

$$2^{n-1}u_n = \frac{n}{1} + 5\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + 5^2\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots.$$

Lucas showed the importance of the Fibonacci series in discussions of (*a*) the decomposition of large numbers into factors and (*b*) the law of distribution of prime numbers.[5] Binet was led to the series in his memoir on linear difference equations (*l. c.*), and Lamé indicated its application[6] in determining an upper limit to the number of operations made in seeking the greatest common divisor of two integers. Landau evaluated the series $\Sigma(1/u_{2h})$ and $\Sigma(1/u_{2h+1})$, and found that the first was related to Lambert's series and the second to the theta séries.[7]

For further references and mathematical discussions one may consult (1) *L'Intermédiaire des mathématiciens*, 1900, pp. 172–7; 1902, p. 43; 1915, pp. 39–40; (2) "Sur une généralisation des progressions géométriques," *L'Education mathématique*, 1914, pp. 149–151, 157–158; and (3) V. Schlegel, "Séries de Lamé supérieurs," *El progreso matematico*, 1894, año 4, pp. 171–174.

As to growths it is particularly in connection with older chapters on leaf

[1] E. Catalan, *Mélanges mathématiques*, tome 2, [Liège, 1887], p. 319.

[2] E. Lucas, "Note sur la triangle arithmétique de Pascal et sur la série de Lamé," *Nouvelle correspondance mathématique*, tome 2, 1876, p. 74.

[3] J. P. M. Binet, "Mémoire sur l'intégration des equations linéaires aux différences finies d'un ordre quelconque, à coefficients variables," *Comptes rendus de l'académie des sciences de Paris*, tome 17, 1843, p. 563.

[4] *Manual des candidats a l'École Polytechnique*, tome 1, Paris, 1857, p. 86.

[5] E. Lucas, (*a*) "Recherches sur plusieurs ouvrages de Léonard de Pise et sur divers es questions d'arithmétique supérieure. Chapter 1. Sur les séries recurrentes," *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche*, tome 10, pp. 129–170, Marzo, 1877; (*b*) Théorie des fonctions numériques simplement périodiques," *American Journal of Mathematics*, Vol.1, 1878, pp. 184–229, 289–321 [on p. 299 are given the first 61 terms of the Fibonacci series and the factors of every term]; (*c*) "Sur la théorie des nombres premiers" [dated mai 1876], *Atti della r. Accademia delle Scienze di Torino*, Vol. 11, 1875–76, pp. 928–937; (*d*) "Note sur l'application des séries recurrentes a la recherche de la loi de distribution des nombres premiers, "*Comptes rendus de l'académie des sciences*, Vol. 82, 1876, pp. 165–167. See also A. Aubry, "Sur divers procédés de factorisation," *L'Enseignement Mathématique*, 1913, especially §§ 11, 16 and 17, pp. 219–223.

[6] B. Lamé, "Note sur la limite du nombre des divisions dans la recherche du plus grand commun diviseur entre deux nombres entiers." *Comptes rendus de l'académie des sciences*, tome 19, 1844, pp. 867–870. See also J. P. M. Binet, idem, pp. 939–941.

Because of results obtained in the above-mentioned memoir the Fibonacci series is frequently called the Lamé series. I can find no verification of Thompson's statement (*On Growth and Form*, p. 643) that the series 2/3, 3/5, 5/8, 8/13, 13/21, ⋯ "is called Lami's series by some, after Father Bernard Lami, a contemporary of Newton's, and one of the co-discoverers of the parallelogram of forces." Indeed the statement is doubtless incorrect.

[7] E. Landau, "Sur la série des inverses des nombres de Fibonacci," *Bulletin de la société mathématique de France*, tome 27, 1899, pp. 298–300.

arrangement or phyllotaxis that the Fibonacci Series comes up. Among the earliest and most important of these are the memoirs of Braun (based on researches of Schimper and himself),[1] and L. et A. Bravais.[3] Of later papers there are those by Ellis,[3] Dickson,[4] Wright,[5] Airy,[6] Günther,[7] and Ludwig.[8] Much that was fanciful and mysterious was swept away by the publication of P. G. Tait's note "On Phyllotaxis." [9] Of recent books on the subject the most notable are those by Church,[10] Cook,[11] and Thompson.[12] The two former are beautifully illustrated. The latter reproduces Tait's discussion in an appreciative manner.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Mr. E. S. LANE, fellow at the University of Chicago, has been appointed instructor in mathematics at Rice Institute.

Mr. W. G. SIMON, fellow at the University of Chicago, has been appointed instructor in mathematics at Western Reserve University.

Dr. OLIVE C. HAZLETT, instructor at Bryn Mawr College, has been appointed to an assistant professorship of mathematics at Mount Holyoke College.

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[1] A. Braun, "Vergleichende Untersuchung über die Ordnung der Schuppen an den Tannenzapfen als Einleitung zur Untersuchung der Blätterstellung überhaupt," *Nova Acta Acad. Caes. Leopoldina*, Vol. 15, 1830, pp. 199–401.

[2] L. et A. Bravais, (1) "Sur la disposition des feuilles curviseriées," *Ann. des sc. nat.*, 2e série Vol. 7, 1837, pp. 42–110; (2) *Mémoire sur la disposition géométrique des feuilles et des inflorescenses*, Paris, 1838.

[3] R. L. Ellis, *Mathematical and Other Writings*, Cambridge, 1863; "On the Theory of Vegetable Spirals," pp. 358–372.

[4] A. Dickson, "On some abnormal cases of pinus pinaster," *Transactions of the Royal Society of Edinburgh*, Vol. 26, 1871, pp. 505–520.

[5] C. Wright, "The uses and origin of the arrangements of leaves in plants" (read 1871), *Memoirs of the American Academy*, Vol. 9, part 2, Cambridge, Mass., p. 384f.

[6] H. Airy, "On Leaf Arrangement," *Proceedings of the Royal Society of London*, Vol. 21, 1873, pp. 176–179.

[7] S. Günther, "Das mathematische Grundgesetz im Bau des Pflanzenkörpers," *Kosmos*, II. Jahrgang, Band 4, 1879, pp. 270–284.

[8] F. Ludwig, "Einige wichtige Abschnitte aus der mathematischen Botanik," *Zeitschrift für mathematischen und naturwiss. Unterricht*, Band 14, 1883, p. 161f.

[9] P. G. Tait, *Proc. Royal Society Edinburgh*, Vol. 7, 1872, pp. 391–4.

[10] A. H. Church, *On the Relation of Phyllotaxis to Mechanical Laws*, London, Williams and Norgate, 1904. On page 5 Church writes: "The properties of the Schimper-Braun series 1, 2, 3, 5, 8, 13, $\cdots$, had long been recognized by mathematicians (Gerhardt, Lamé). . . ." In *Botanisches Centralblatt*, Band 68, 1896, F. Ludwig writes (on p. 7) that the numbers of this series "werden vielfach von Botanikern als Braun'sche, von Mathematikern als Gerhardt'sche oder Lamé'sche Reihe bezeichnet." I have not been able to verify that any mathematician used the term Gerhardt series in this connection, or that anyone by the name of Gerhardt wrote about the Fibonacci series. From what has been indicated above it seems certain that "Gerhard'sche" should be "Girard'sche."

[11] T. A. Cook, *The Curves of Life*, London, Constable, 1914.

[12] D'A. W. Thompson, *On Growth and Form*, Cambridge: at the University Press, 1917.

Dr. E. A. ENGLER, professor of mathematics at Washington University from 1881 to 1901, president of Worcester Polytechnic Institute from 1901 to 1911, and since 1911 secretary-treasurer of Washington University, died on January 16, at the age of sixty-one years.

Assistant Professor WARREN WEAVER, of Throop College of Technology, is on leave of absence, serving in the Science and Research Division of the Signal Corps at Washington, D. C.

Professor F. R. MOULTON, of the University of Chicago, has been commissioned a major in the Ordnance Department and assigned to service at Washington where he will have important duties in connection with the mathematical phases of testing field pieces.

Professor J. N. VAN DER VRIES, of the University of Kansas, who has been connected with the war work of the Chamber of Commerce of the U. S. at Washington, has been transferred to Chicago to take charge of the newly established Chicago branch of the Chamber, having control of the work in the central west.

Mr. ALFRED DAVIS, a charter member of the Association, and for a number of years teacher of mathematics at the Francis W. Parker School, Chicago, has accepted the professorship and head of the department of mathematics at William and Mary College, Williamsburg, Virginia.

Professor FLORIAN CAJORI has resigned his position of professor of mathematics at Colorado College to accept the chair of professor of history of mathematics in the University of California. Professor Cajori has been connected with Colorado College for twenty-nine years.

Professor C. A. WALDO, who retired last June from the Thayer professorship of mathematics and applied mechanics at Washington University, St. Louis, Mo., is now living at 401 West 118th Street, New York City. Professor Waldo's educational career extends over a period of forty years, beginning as an instructor in mathematics at Wesleyan University in 1877. He occupied in succession the professorship and head of the department of mathematics at Rose Polytechnic Institute, DePauw, Purdue and Washington Universities. He was retired from Washington University as professor emeritus.

The Association of Teachers of Secondary Mathematics of North Carolina met at Greenville, on March 8 and 9. Professor C. B. UPTON of Teachers College, Columbia University, was the principal speaker; other addresses were made by Mr. W. W. RANKIN, and Mr. J. W. LASLEY, of the University of North Carolina. The sessions of the conference were primarily devoted to discussions of the humanizing of mathematics, the addresses of Professor UPTON being upon the subjects: "Mathematics as an aid to the interpretation of life about us," "Recent tendencies to visualize the beginnings of geometry," and "The modern methods of teaching arithmetic."

University of Kansas. Summer session, June 3 to July 12.—By Professor C. H. ASHTON: College algebra; Mechanics.—By Professor E. B. STOUFFER:

Calculus; Modern geometry.—By Professor J. J. WHEELER: Solid geometry; Trigonometry; Analytical geometry.

July 15 to Aug. 9.—By Professor U. G. MITCHELL: History of elementary mathematics; Teachers' course.

Cornell University. Summer session, July 8 to August 16.—By Professor W. B. CARVER: Topics related to geometry, five hours.—By Professor W. A. HURWITZ: Topics related to algebra, five hours. The preceding two courses are primarily for teachers.—By Professor F. W. OWENS: Projective geometry, three hours. Courses in solid geometry, algebra, trigonometry, analytic geometry and elementary calculus will be given.

The tenth regular meeting of the American Mathematical Society and the forty-first meeting of the Chicago Section were held at the University of Chicago on Friday and Saturday, April 12 and 13. The sessions of Friday morning and Saturday morning were devoted to the presentation of original papers, the number of papers read being nineteen. The session of Friday afternoon was devoted to a symposium on the theory of summable series. The attendance at the meetings was nearly normal in number; but several who usually are present were absent owing to duties connected with the prosecution of the war. At the dinner on Friday evening, where some forty-five members enjoyed the usual social intercourse, Professor L. E. DICKSON, President of the Society, presided, and addresses were made by Professor VAN VLECK on his experience in connection with the war registration board at Madison, by Professor D. R. CURTISS on the effect of the war on scientific productiveness, by Professor S. LEFSCHETZ, who came five hundred miles to the meeting, by Professor E. R. HEDRICK on correspondence with prominent mathematicians in France, England, and Germany, and by Professor H. E. SLAUGHT on the work in which Major F. R. MOULTON is engaged in the ordnance department at Washington.

In an article entitled "Medicine and mathematics in the sixteenth century" which appeared in the *Annals of Medical History*, summer number, 1917, Professor D. E. SMITH considers the reasons for the close relations of these two branches then existing and lists a large number of men who were distinguished in both, among whom may be cited LEONARDO DA VINCI, COPERNICUS, CARDAN, GEMMA FRISIUS, and ROBERT RECORDE.

The Committee on Policy of the American Association for the Advancement of Science has changed the place of the next meeting from Boston to Baltimore, one controlling reason being the proximity of Baltimore to Washington, where many prominent scientific men from all parts of the country are now engaged in war work. The dates will be December 27 to 31, 1918, and the program will be restricted very largely to definite working problems related to the war.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the ASSOCIATION, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

Our readers are hereby reminded that this journal is not published in July and August. The next number will appear as early in September as the conditions in the printing industry will permit.

THE AMERICAN MATHEMATICAL MONTHLY

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ON THE I-CENTERS OF A TRIANGLE.

By NATHAN ALTSHILLER, University of Oklahoma.

Introduction. The first two sections below are known. The following two sections are a natural extension of the first two. I believe them to be new. The sections 5, 6, 7, are consequences of the first four applied to the triangle. The rest of the article is an application of the first four sections to the inscribed quadrilateral. The results are undoubtedly new.

Definition. The I-centers of a triangle are the centers of the four circles touching the sides of the triangle.

1. Let ABC (Fig. 1) be a given triangle, I its incenter, I_1 its excenter relative to the vertex A, *i. e.*, the center of the escribed circle touching the side BC, opposite the vertex A, not produced. The lines BI, BI_1 being the bisectors of two adjacent supplementary angles, are perpendicular to each other, hence the segment II_1 subtends a right angle at the point B. The same is true about the point C, for similar reasons. Hence: *The incenter of a triangle and its excenter relative to a given vertex, are the extremities of a diameter of a circle passing through the other two vertices of the triangle.*

2. The center of the circle $BICI_1$ (1) is the point common to II_1 and the perpendicular bisector of BC. Let E, E' be the points where this perpendicular bisector meets the circumcircle (O) of ABC, the points E and A being on opposite sides of BC. Now the line II_1 being the interior bisector of the angle BAC, passes through the mid-point E of the arc BEC of the circumcircle (O), hence: *The center of the circle $BICI_1$ is the mid-point E of the arc of the circumcircle of ABC which is subtended by the side BC and which does not contain the vertex A.*[1]

[1] Julius Petersen, *Méthodes et théories pour la résolution des problèmes de constructions géométriques*, p. 7. Paris, Gauthier-Villars, 1901, third edition. Clement V. Durell, *A Course of Plane Geometry for Advanced Students*, Part I, p. 33, London, Macmillan and Co., 1909.

3. Let I_2, I_3 (Fig. 1) be the excenters of ABC relative to the vertices B, C. The lines BI_2, BI_3 being the bisectors of two adjacent supplementary angles, are perpendicular to each other, hence the segment I_2I_3 subtends a right angle at the point B. The same is true about the point C, for similar reasons. Hence: *The two excenters of a triangle relative to two given vertices are the extremities of a diameter of a circle passing through the two considered vertices of the triangle.*

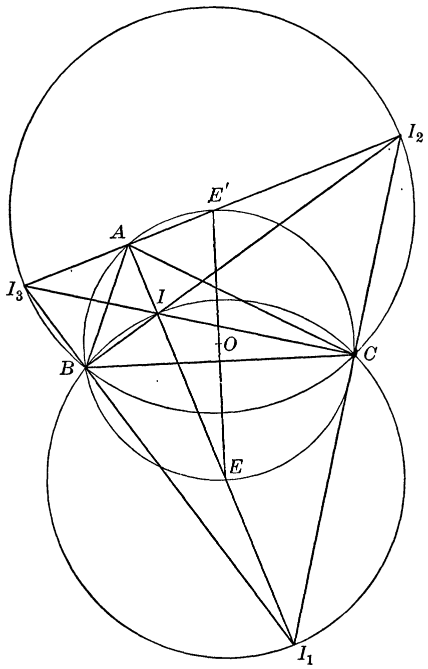


FIG. 1.

4. The center of the circle BI_2CI_3 (3) is the point of intersection of I_2I_3 with the perpendicular bisector EE' of BC. Now the line I_2I_3 being the exterior bisector of the angle BAC, passes through the mid-point E' of the arc $BE'C$ of the circumcircle (O). Hence: *The center of the circle BI_2CI_3 is the mid-point of the arc of the circumcircle of ABC which is subtended by BC and which contains the third vertex of the triangle.*

5. Let FF', GG' be the diameters of the circumcircle (O) of ABC perpendicular to the sides CA, AB respectively, the points F, B lying on opposite sides of CA, and G, C—on opposite sides of AB. The points E, E', F, F', G, G' are the centers of six circles (E), (E'), (F), (F'), (G), (G'), which pass through pairs of vertices of the triangle and through pairs of its I-centers I, I_1, I_2, I_3. We thus obtain the following table (2, 4):

TABLE I.

(E)	I, I_1	I_2, I_3	(E')
(F)	I, I_2	I_3, I_1	(F')
(G)	I, I_3	I_1, I_2	(G')

Hence: *Each I-center of a triangle lies on three of the six circles which pass through the pairs of vertices of the triangle and have for their centers the mid-points of the arcs subtended by the respective sides of the triangle on its circumcircle.*

Each of the six circles contains two I-centers, and the two points are the extremities of a diameter.

6. The six circles (5) may be used for the construction of the I-centers of the triangle. Therefore: *If the mid-points of the six arcs subtended by the sides of a triangle on its circumcircle are given, the I-centers of the triangle may be constructed by use of the compass alone.*

7. The two circles (E), (E') passing through B, C and having for their respective centers E, E' (2, 4), are independent of the position which the vertex A occupies on the circumcircle (O) of the triangle ABC. Consequently: *If a variable triangle has a fixed base and a fixed circumcircle, its four I-centers describe two circles, each passing through the two fixed vertices and having for their respective centers the extremities of the diameter of the circumcircle which is perpendicular to the fixed side.*

8. Let $M_{2,3}$, $M_{2,3}^{1,4}$ (Fig. 2) denote the mid-points of the two arcs of the circle (C) subtended by the side BC of the quadrilateral $ABCD$ inscribed in (C), the point $M_{2,3}^{1,4}$ being the mid-point of the arc which contains the points A, D. A similar notation will be adopted for the mid-points of the arcs into which (C) is divided by each of the other three sides and by each of the two diagonals of the quadrilateral. Thus $M_{1,3}^{2}$, for instance, will denote the mid-point of the arc subtended by the diagonal AC and which contains the point B. The figure thus involves twelve such mid-points M.

The four points A, B, C, D taken three at a time, determine four triangles, each of which has four I-centers, so that we have in all sixteen I-centers. The incenters of the four triangles BCD, CDA, DAB, ABC, will be denoted by P, Q, R, S, and the excenters of the triangle BCD relative to the vertices B, C, D, by P_2, P_3, P_4, respectively. Similarly for the excenters of the other three triangles.

The I-centers P, P_4, of the triangle BCD are the points of intersection of the circle $(M_{2,3})$ having for its center the point $M_{2,3}$ and passing through B, C (2) with the line $DM_{2,3}$, while the I-centers P_2, P_3 are the points of intersection of the line $DM_{2,3}^{1,4}$ with the circle $(M_{2,3}^{1,4})$ (4). Similarly for the I-centers of the other triangles. The circle $(M_{2,3})$ contains the I-centers P, P_4, of the triangle BCD, and the I-centers S, S_1 of the triangle ABC. Similarly for the other eleven (M) circles. Thus we obtain the following table, where the (M) circle and the I-centers lying on it are written in the same horizontal line, on the same side of the vertical line.

TABLE II.

$(M_{2,3})$	$S,\ S_1,\ P,\ P_4$	$S_2,\ S_3,\ P_2,\ P_3$	$(M_{2,3}^{1,4})$
$(M_{1,4})$	$R,\ R_2,\ Q,\ Q_3$	$Q_1,\ Q_4,\ R_1,\ R_4$	$(M_{1,4}^{2,3})$
$(M_{1,2})$	$R,\ R_4,\ S,\ S_3$	$R_1,\ R_2,\ S_1,\ S_2$	$(M_{1,2}^{3,4})$
$(M_{3,4})$	$P,\ P_2,\ Q,\ Q_1$	$P_3,\ P_4,\ Q_3,\ Q_4$	$(M_{3,4}^{1,2})$
$(M_{1,3}^{4})$	$Q_1,\ Q_3,\ S,\ S_2$	$Q,\ Q_4,\ S_1,\ S_3$	$(M_{1,3}^{2})$
$(M_{2,4}^{3})$	$P_2,\ P_4,\ R,\ R_1$	$P,\ P_3,\ R_2,\ R_4$	$(M_{2,4}^{1})$

Hence: *The sixteen I-centers of the four triangles obtained by taking the vertices of an inscribed quadrilateral three at a time, lie on twelve circles. Each I-center lies on three circles. Each circle contains four I-centers. The centers of those twelve circles all lie on the circumcircle of the quadrilateral.*

9. The two pairs of I-centers lying on the same (M) circle (8) are the extremities of two diameters of this circle (2, 4). Therefore: *The four I-centers lying on the same circle form a rectangle.*

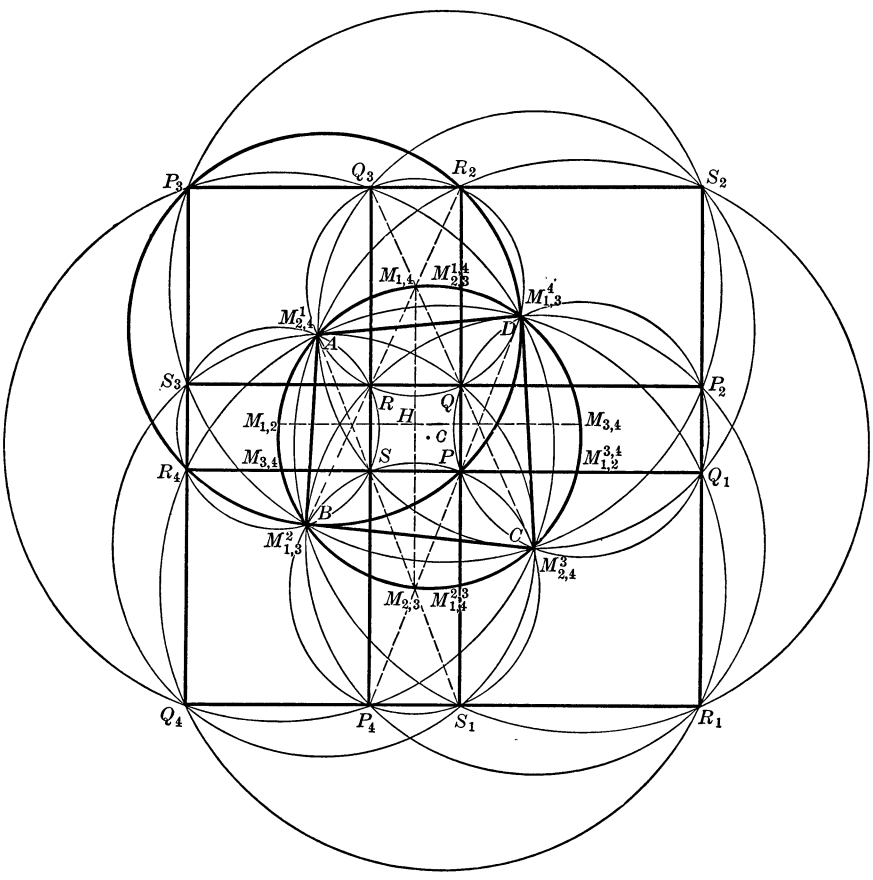


FIG. 2.

10. Let H be the point of intersection of the line $M_{1,2}M_{3,4}$ with the line $M_{2,3}M_{1,4}$. The angle $M_{1,2}HM_{2,3}$ is measured by one half of the sum of the arcs $M_{1,2}BM_{2,3}$ and $M_{3,4}DM_{1,4}$. Now

$$\smile M_{1,2}BM_{2,3} = \smile M_{1,2}B + \smile BM_{2,3} = \tfrac{1}{2} \smile AB + \tfrac{1}{2} \smile BC$$

and

$$\smile M_{3,4}DM_{1,4} = \smile M_{3,4}D + \smile DM_{1,4} = \tfrac{1}{2} \smile CD + \tfrac{1}{2} \smile DA.$$

Hence the angle $M_{1,2}HM_{2,3}$ has for its measure

$$\tfrac{1}{4}\cdot(\smile AB + \smile BC + \smile CD + \smile DA) = \tfrac{1}{4}\cdot 360^\circ = 90^\circ.$$

Therefore: *The lines* $M_{1,2}M_{3,4}$, $M_{2,3}M_{1,4}$ *are perpendicular to each other.*

11. The incenter P of the triangle BCD lies on the same side of BC as the vertex D, and is common to the circle $(M_{2,3})$ (8) and to the bisector $DM_{2,3}$ of the angle BDC inscribed in (C). Similarly the incenter S of the triangle ABC lies on the same side of BC as A, and is common to $(M_{2,3})$ and $AM_{2,3}$. Thus the incenters P, S lie on the same side of BC as the segment AD and are the points of intersection of the circle $(M_{2,3})$ with the lines $M_{2,3}D$, $M_{2,3}A$ respectively.

The line $M_{2,3}M_{1,4}$ is the bisector of the angle $AM_{2,3}D$ inscribed in (C), and therefore also of the angle $SM_{2,3}P \equiv$ angle $AM_{2,3}D$. But the angle $SM_{2,3}P$ is a central angle of the circle $(M_{2,3})$, hence $M_{2,3}M_{1,4}$ is the perpendicular bisector of the chord PS. Similarly the line $M_{1,4}M_{2,3}$ is the perpendicular bisector of the chord QR of the circle $(M_{1,4})$. Thus the two segments PS and QR have the same perpendicular bisector $M_{2,3}M_{1,4}$. By an analogous reasoning it may be shown that the line $M_{1,2}M_{3,4}$ is the common perpendicular bisector of the two segments PQ and RS. The opposite sides of the quadrilateral $PQRS$ are thus perpendicular to the two lines $M_{1,2}M_{3,4}$, $M_{2,3}M_{1,4}$ which are themselves perpendicular to each other. Consequently: *The four incenters of the four triangles determined by the vertices of an inscribed quadrilateral taken three at a time, form a rectangle.*[1]

12. The four points PSP_4S_1 lying on the circle $(M_{2,3})$ (8), form a rectangle (9), hence PS_1 is perpendicular to PS at P. On the other hand $PQRS$ forming a rectangle (11), the perpendicular to PS at P is PQ, hence the three points Q, P, S_1 are collinear. The three points R, S, P_4 are collinear for similar reasons, and the two lines QPS_1 and RSP_4 are parallel, as opposite sides of the rectangle $PQRS$. Considering the rectangles PQP_2Q_1 (8, 9) and $PQRS$ (11), it may be established in a similar way that we have the two parallel lines RQP_2 and SPQ_1.

$RP_4R_1P_2$ and PSP_4S_1 are rectangles inscribed in the circles $(M^3_{2,4})$ and $(M_{2,3})$ respectively (8, 9), hence the lines P_4R_1 and P_4S_1 are the respective perpendiculars to the lines RP_4 and SP_4 at P_4; but the three points R, S, P_4 are collinear (12), hence the lines P_4R_1, P_4S_1 coincide, and the three points P_4, S_1, R_1 are collinear. Considering the two rectangles $RP_4R_1P_2$ and PQP_2Q_1, it may be shown in a similar way that the points P_2, Q_1, R_1 are collinear.

Thus the parallel QPS_1 through P to the side RP_4 of the rectangle $RP_4R_1P_2$ meets the side P_4R_1 of this rectangle in the point S_1, and the parallel SPQ_1 through P to the side RP_2 meets the side P_2R_1 in the point Q_1, hence $PQ_1R_1S_1$ is a rectangle. In an analogous manner it may be shown that we have the rectangles $QP_2S_2R_2$, $RQ_3P_3S_3$, $SP_4Q_4R_4$. Consequently: *If of the four triangles determined by the vertices of an inscribed quadrilateral taken three at a time, the three triangles are taken having a given vertex in common, the three excenters relative to this vertex in the three triangles, are three vertices of a rectangle, the fourth vertex of which is the incenter of the fourth triangle.*

[1] See this MONTHLY, Vol. XXIV, March, 1917, p. 124.

13. The three rectangles $PQ_1R_1S_1$ (12), PQP_2Q_1 (8, 9), and $QP_2S_2R_2$ (12) show that the points R_1, Q_1, P_2, S_2 are collinear, as well as the points S_1, P, Q, R_2, and the two lines are parallel. Similar considerations may be applied to the three rectangles PSP_4S_1 (8, 9), $PQRS$ (11), QRQ_3R_2 (8, 9), and again to the series $SP_4Q_4R_4$ (12), SRS_3R_4 (8, 9), $RS_3P_3Q_3$ (12). Thus we obtain the four parallel lines $R_1Q_1P_2S_2$, S_1PQR_2, P_4SRQ_3, $Q_4R_4S_3P_3$. Similarly for the four parallel lines $R_1S_1P_4Q_4$, Q_1PSR_4, P_2QRS_3, $S_2R_2Q_3P_3$. Consequently: *The sixteen I-centers of the four triangles determined by the vertices of an inscribed quadrilateral taken three at a time, lie by groups of four on eight straight lines. The eight lines consist of two sets of four parallel lines, and the lines of one set are perpendicular to the lines of the other.*

NOTE ON CONTINUOUS FUNCTIONS.

By CAPTAIN K. P. WILLIAMS.

1. The class of continuous functions is at the same time one of the simplest and most important in analysis. On account of the character of physical phenomena, quantities whose variation is of a continuous sort are naturally the first considered. In the more recently developed topics in analysis other classes of functions are considered, and the idea of continuity does not play such a fundamental rôle. The considerations that follow are intended primarily for those who have just become familiar with the properties of continuous functions as developed from a rigorous point of view. As they are of a simple nature they may serve as an easy introduction to the study of classes of functions where one's starting point is no longer the idea of continuity.

When we examine the properties of continuous functions we find that they fall into two rather broad classifications. There are those properties that have to do with the behavior of the function in the immediate vicinity of a point, and those that relate to some character with reference to the interval of definition. We could call them properties "im kleinen," and properties "im grossen." The definition of continuity itself is an example of the first class of properties, and the theorem that every continuous function has actual extrema in a closed interval, an example of the second. Thus we see how strongly properties of a function "im kleinen" effect its behavior "im grossen." Intimately connected with any property "im kleinen" is the question of uniformity; that is we inquire whether the property in question occurs in a uniform manner throughout the interval of definition. And in an analogous manner we can ask whether a property "im grossen" is true in every sub-interval. With such ideas before us we can examine easily how far various properties of continuous functions are uniquely characteristic.

2. We consider a function $f(x)$ which approaches a definite limit at every point, but which is not necessarily continuous at any point in the interval (a, b). Denote by $f'(x)$ the limit approached. We first prove the

Lemma. *The function $f'(x)$ has a definite limit at every point.*

Let x_0 be any point on the interval (a, b), and ϵ any positive number. By hypothesis we can find η such that

$$|f(x') - f(x'')| < \epsilon,$$

when $|x_0 - x'| < \eta$, and $|x_0 - x''| < \eta$ and x' and x'' are both different from x_0. Take x_1 and x_2 any two points in the interval $|x_0 - x| < \eta$, then we can find x' and x'' so that

$$|f'(x_1) - f(x')| < \epsilon, \qquad |f'(x_2) - f(x'')| < \epsilon,$$

while x' and x'' are also both in the interval $|x_0 - x| < \eta$ and are different from x_0. It follows then that

$$|f'(x_1) - f'(x_2)| < 3\epsilon,$$

which establishes the property announced.

We can go a step farther and state

Theorem 1. *The function $f'(x)$ is continuous.*

We have

$$|f'(x_0) - f(\bar{x})| < \epsilon, \qquad \text{for} \qquad |x_0 - \bar{x}| < \delta,$$

and from what we have just proved,

$$|f'(x_1) - f'(x_2)| < \epsilon, \qquad \text{for} \qquad |x_0 - x_1| < \eta, \qquad |x_0 - x_2| < \eta.$$

Let now $\bar{\eta}$ be the smaller of η and δ, and let x_1 and x_2 be in the interval $|x - x_0| < \bar{\eta}$. We can find an $\bar{x}$ in this interval such that

$$|f'(x_1) - f(\bar{x})| < \epsilon,$$

while at the same time from the manner of choice of $\bar{\eta}$,

$$|f'(x_0) - f(\bar{x})| < \epsilon, \qquad |f'(x_1) - f'(x_2)| < \epsilon.$$

We thus have

$$|f'(x_0) - f'(x_2)| < 3\epsilon \qquad \text{for} \qquad |x_0 - x_2| < \bar{\eta},$$

which shows at once that $f'(x)$ is a continuous function.

Theorem 2. *If $f(x)$ approaches a limit uniformly on (a, b) it will be a continuous function.*

Under the hypothesis we can find η such that

$$|f'(x_0) - f(x)| < \epsilon, \qquad \text{for} \qquad |x_0 - x| < \eta,$$

uniformly for x_0 on (a, b), the number ϵ being arbitrary. Take x_1, x_2 any two points in $|x - x_0| < \eta/2$; then

$$|f(x_2) - f'(x_0)| < \epsilon.$$

But we also see that

$$|f(x_2) - f'(x_1)| < \epsilon, \qquad |f(x_0) - f'(x_1)| < \epsilon.$$

It therefore follows that

$$| f(x_0) - f'(x_0) | < 3\epsilon,$$

so that

$$f(x_0) = f'(x_0). \qquad \text{Q. E. D.}$$

Thus in place of the ordinary definition of continuity we could adopt the following: *A continuous function is one that possesses a limit at every point, and in which the limit is approached uniformly.* This gives us a property "im kleinen," if existent uniformly, as completely characteristic of continuity.

3. We now put $\delta(x) = f'(x) - f(x)$ and examine the nature of the function thus defined. Let δ be the greatest lower bound of the function $| \delta(x) |$ on (a, b). We prove

Theorem 3. *The quantity δ is equal to zero.*

Suppose $\delta \neq 0$; then at every point $| \delta(x) | > \eta > 0$. Then for any x' we can find a ξ such that

$$| f(x) - f(x') | > \frac{\eta}{2}, \qquad \text{when} \qquad | x - x' | < \xi, \qquad x \neq x'.$$

Consider now a point x_0. Since $f'(x_0)$ is the limit approached we have

$$| f'(x_0) - f(x) | < \frac{\eta}{8}, \qquad \text{when} \qquad | x - x_0 | < \xi,$$

ξ being a sufficiently small positive quantity. Therefore

$$| f(x') - f(x'') | < \frac{\eta}{4}, \qquad \text{when} \qquad | x' - x_0 | < \xi, \qquad \text{and} \qquad | x'' - x_0 | < \xi.$$

Now consider x' as fixed. We can then find an interval in the interval $| x - x_0 | < \xi$, containing x', and such that when x'' is in this interval we also have

$$| f(x') - f(x'') | > \frac{\eta}{2},$$

and have thus reached a contradiction.

Corollary. *In every sub-interval of (a, b) the greatest lower bound of $\delta(x)$ is zero.*

Denote by $\bar{x}$ a point such that in every vicinity of it the greatest lower bound of $| \delta(x) |$ is zero. It follows that

Theorem 4. *The points $\bar{x}$ are everywhere dense.*

4. We next consider a property "im grossen," which in itself is not sufficient to completely characterize continuity,[1] but which we shall suppose is true in every sub-interval.

Theorem 5. *Let $f(x)$ be a function such that in every interval containing x_0 it assumes all values between any two of its values; further suppose it assumes any value only once; then $f(x)$ is continuous at $x = x_0$.*

[1] K. P. Williams, Concerning a Certain Totally Discontinuous Function, *Bulletin of the American Mathematical Society*, vol. 21, 1914, pp. 117–120.

Let (a_1, b_1) be any interval containing x_0, and let M_1, m_1 be the bounds of the function on (a_1, b_1); then the function assumes all values between m_1 and M_1. Take the case where $m_1 < f(x_0) < M_1$. Continue in this way where a_1, a_2, a_3, $\cdots$ approach x_0, and likewise b_1, b_2, b_3, $\cdots$. Let M_i, m_i be the bounds of the function on (a_i, b_i). Then both M_1, M_2, $\cdots$ and m_1, m_2, $\cdots$ approach $f(x_0)$. Suppose for instance that M_1, M_2, M_3, $\cdots$ do not approach $f(x_0)$, but have $M > f(x_0)$ as greatest lower bound. Take k, where $f(x_0) < k < M$, then there is one value $\bar{x}$ on (a_1, b_1) such that $f(\bar{x}) = k$. Let i be large enough that (a_i, b_i) does not contain $\bar{x}$; then we must have $M_i < k$, or otherwise there would be a root of $f(x) = k$ on (a_i, b_i), which is impossible. But this contradicts the assumption that M is the greatest lower bound of M_1, M_2, $\cdots$. Since M_1, M_2, $\cdots$ and m_1, m_2, $\cdots$ both tend to $f(x_0)$ it follows that $f(x)$ is continuous at $x = x_0$.

The cases other than that where $m_1 < f(x_0) < M_1$ follow in a similar way.

Corollary. *The theorem is true if $f(x)$ assumes any value at most n times, n a constant.*

NOTE ON FUNCTIONS WHICH APPROACH A LIMIT AT EVERY POINT OF AN INTERVAL.

By E. W. CHITTENDEN, University of Illinois.

In the foregoing paper Captain Williams has discussed properties of functions which approach a limit at every point of an interval. It is the purpose of this note to present the following theorem:

THEOREM. *If a function $f(x)$ has a limit $f'(x)$ at every point x of an interval (a, b), then for every positive number σ, however small, the number of points at which the measure of discontinuity (saltus) exceeds σ is finite, and the set of points at which $f(x)$ differs from the continuous function $f'(x)$ is at most enumerably infinite.*

At any point x of the interval (a, b) there is for any small positive number e an open interval (segment) $S_{xe} = (x - h < x' < x + h)$ such that for any point x' in the segment, distinct from x,

$$|f(x') - f'(x)| < \frac{e}{4},$$

and also, since $f'(x)$ is continuous,

$$|f'(x') - f'(x)| < \frac{e}{4}.$$

Hence

$$|\delta(x')| = |f'(x') - f(x')| < \frac{e}{2}.$$

Therefore the oscillation of the function $\delta(x)$ on the set obtained from S_x by omitting the point x is less than e.

Every point of the interval (a, b) is enclosed in a segment S_x except the points a and b, for which $S_a = (a \leqq x < a + h)$, $S_b = (b - h < x \leqq b)$. (It is to be

understood that h depends on both x and e.) From the Heine-Borel theorem (see, for instance, Veblen and Lennes, *Infinitesimal Analysis*, p. 34) it follows that there exists for every e a finite set of points

$$a = x_1 < x_2 < x_3 \cdots, \; x_{m-1} < x_m = b$$

such that every point of the interval (a, b) belongs to some segment S_{x_i}, and that for every i $(= 1, \cdots, m)$ the oscillation of $\delta(x)$ on S_{x_i} is, if we omit the point x_i, less than e. Hence if at any point x the measure of discontinuity of the function $\delta(x)$, and therefore of $f(x)$, exceeds e, x is some one of the points x_i. Assign to e successively the values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, 1/n, \cdots$ and denote the points corresponding to $e = 1/n$ by x_{ni} $(i = 1, 2, 3, \cdots, m_n)$. The set of all points x_{ni} is enumerable and contains the set of all points at which the saltus of $\delta(x)$ is positive, which must therefore be an enumerable set. The function $\delta(x)$, and consequently the function $f(x)$, is continuous for every point x not in the set of points x_{ni} since such a point belongs (for every value of n) to a segment $S_{x_{ni}}$ on which the oscillation of $\delta(x)$ is less than $1/n$.

Consider the classical example of a function continuous at the irrational points of an interval and discontinuous at every rational point. The function $f(x) = 0$, if x is an irrational point of the interval $(0, 1)$, $f(x) = 1/q$ if $x = p/q$ (p and q relatively prime integers and $p < q$). The function $f'(x)$ exists and vanishes identically. Hence $\delta(x) = -f(x)$. $f(x)$ is discontinuous on a dense enumerable set and possesses the maximum degree of discontinuity permissible under the theorem.

The argument of this note can be extended immediately by means of suitable changes of the terminology so as to apply to any abstract set admitting a definition of distance and the generalized Heine-Borel theorem.

THE NINE-POINT CIRCLE OBTAINED BY METHODS OF PROJECTIVE GEOMETRY.

By H. N. WRIGHT, Whittier College.

Place a mirror of zero dimensions, but with a fixed direction d at a point A. Then any line a through A reflects into a line a' through the same point, and a'

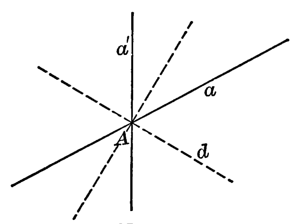


FIG. 1.

reflects back into a. Moreover by considering the angles of reflection it is clear that the pencil described by a is projective to the pencil described by a'. Thus

a and a' are a pair in an involution of lines about A, which we may call a *mirror involution.* The double elements are the line in the direction of the mirror and the line at right angles to it.

Two mirror involutions A and B may be used to set up a quadratic transformation[1] of the plane. Any point P by reflection from A and B goes into a point P' and conversely. In general a straight line transforms into a conic through

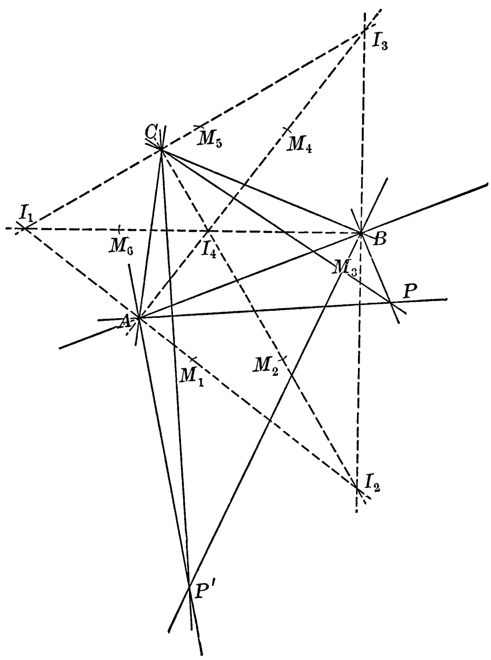


FIG. 2.

A and B. However all points of the line AB, other than the points A and B, go into the same point C. Now if a mirror involution is placed at C with the direction of its mirror bisecting the exterior angle of the triangle ABC at that point, it may be shown that any two of the three involutions yield the same quadratic transformation.[2] Then a conic which is obtained by transforming a line passes through A, B and C.

By following the construction of the transformation it is seen that the vertices and the ortho-center of the triangle, whose sides are the directions of the mirrors, are the four invariant points I_1, I_2, I_3 and I_4. Also we see that a point of any one of the lines I_iI_j transforms into another point of the same line; thus setting up an involution of points on I_iI_j in which I_i and I_j are the self-corresponding

[1] This is a special case of the quadratic transformation discussed in a paper, "On the Combination of Involutions," by D. N. Lehmer in this MONTHLY for March, 1911.

[2] See proof in the general case in paper mentioned. *We note that this proves the proposition that if three lines, one through each of the vertices of a triangle, are concurrent, then their isogonal conjugates are concurrent.*

elements. Hence the midpoint M_k of the inner segment I_iI_j corresponds to the ideal point of the line I_iI_j. From this it follows that *the ideal line of the plane transforms into a conic through each of the midpoints of the six segments* I_iI_j in addition to passing through A, B and C, as noted above.

We may now show that *this conic is a circle.* If P is an ideal point the lines a and b joining it to two of the mirrors are parallel. Then the sum of the angles α, β and γ is four right angles. Now as P moves on the ideal line a change in

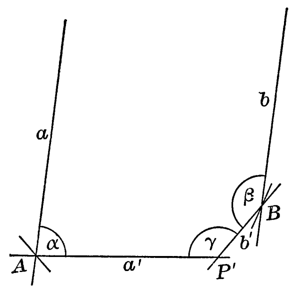


FIG. 3.

the angle α is accompanied by a change in the angle β of the same magnitude but opposite in sign. Then $\alpha + \beta$ is constant and hence $\gamma = 360° - (\alpha + \beta)$ is constant and P' moves on a circle. It follows, then, that *the conic obtained by transforming the ideal line of the plane is the nine-point circle of each of the four triangles* $I_iI_jI_k$.

THE ROCKY MOUNTAIN SECTION.

The second annual meeting of the Rocky Mountain section of the Mathematical Association of America was held at Laramie, Wyoming, under the auspices of the University of Wyoming, March 29 and 30, 1918.

The meeting opened with a dinner in Hoit Hall at 6 P. M., at which the address of welcome by Acting President Nelson of the University of Wyoming and the response by Professor O. C. Lester, of the University of Colorado, were given. After the dinner, an adjournment to the administration building was made and the following program was carried out:

1. Special Courses in Mathematics for Technical Students. PROFESSOR S. L. MACDONALD, Colorado A. & M. College, Ft. Collins.
2. The Theory of the Mercury Arc. PROFESSOR J. W. WOODROW, University of Colorado, Boulder.
3. The Length Integral in the non-Euclidean World of Poincaré. PROFESSOR C. E. STROMQUIST, University of Wyoming, Laramie.

The purpose of this paper was to derive an integral for length in the non-Euclidean world proposed by Poincaré in his "Foundations of Science," English translation by Halsted, page 75. The shortest distance between two points is

assumed to be the circle through these points and perpendicular to the boundary sphere. The general form of the length integral under this assumption is then worked out for the case of the plane. Under the further restriction that transversals are perpendicular to their extremals, the length integral along a curve $y = f(x)$ between two points, $P(x_1, y_1)$ and $P(x_2, y_2)$, reduces to the form

$$\int \frac{\sqrt{1+p^2}}{x^2+y^2-R^2}\,dx,$$

where $p = dy/dx$ and R is the radius of the boundary sphere.

4. The Trend Curve for the Price of Copper. PROFESSOR C. S. SPERRY, University of Colorado, Boulder.

5. A Problem in Geometry. MR. J. Q. MCNATT, Colorado Fuel & Iron Co., Florence.

The author gave a new proof for the relation between the side of a regular inscribed pentagon and the side of a regular inscribed decagon.

6. Some Systems of Coördinates. PROFESSOR G. H. LIGHT, University of Colorado, Boulder.

This paper dealt principally with intrinsic coördinates and showed the extremely simple form that the equations of some well-known curves and their evolutes assume when expressed in terms of these coördinates.

7. Mathematics at the Front. MR. W. H. HILL, Greeley High School, Greeley.

8. Some Functions of Solid Angles. PROFESSOR J. C. FITTERER, University of Wyoming, Laramie.

9. The Origin of the name "Rolle's Curve." The Origin of the name "Mathematical Induction." PROFESSOR FLORIAN CAJORI, Colorado College, Colorado Springs.

The second of these papers by Professor Cajori appeared in the May number of this MONTHLY; the first will appear in a later issue.

10. The Sine and Cosine Integrals $\int \sin x/x\,dx$ and $\int \cos x/x\,dx$ in Electromagnetism. PROFESSOR C. C. VANNUYS, Colorado School of Mines, Golden.

This paper deals with interesting physical applications of the functions known to mathematicians as the sine and cosine integrals. One of the problems dealt with is that of determining the equivalent resistance and inductance due to radiation of electromagnetic waves of a long straight conductor carrying a harmonic alternating current of single frequency such as is employed in the oscillation circuits used in radio telegraphy.

Another problem discussed is that of the electromotive force induced in a straight vertical conductor by an oscillatory current in a parallel conductor at a great distance from it. In each case, the results are obtained in terms of these series. The paper closes with an analysis of the five integrals given below. γ in these series is Euler's constant.

$$Six = \int_0^x \sin x/x\,dx = x - x^3/3!3 + x^5/5!5 - \cdots,$$

$$Cix = \int_\infty^x \cos x/x\,dx = \gamma + \log x - x^2/2!2 + x^4/4!4 - \cdots,$$

$$Eix = \int_\infty^x e^{-x}/x\,dx = \gamma + \log x + x + x^2/2!2 + \cdots,$$

$$Shix = \int_0^x \sinh x/x\,dx = x + x^3/3!3 + x^5/5!5 + \cdots,$$

$$Chix = \int_\infty^x \cosh x/x\,dx = \gamma + \log x + x^2/2!2 + x^4/4!4 + \cdots.$$

On account of the length of the program and the interest shown in the papers it was found necessary to adjourn at 11 P. M. until 8:30 the next morning, when the program was completed and officers were elected for the ensuing year as follows:

CHAIRMAN, C. C. VANNUYS, Professor of Physics, Colorado School of Mines.

VICE-CHAIRMAN, S. L. MACDONALD, Professor of Mathematics, Colorado A. & M. College.

SECRETARY-TREASURER, G. H. LIGHT, Assistant Professor of Mathematics, University of Colorado.

Five visitors were present and the following fifteen members: C. R. Burger, Colorado School of Mines; I. M. DeLong, University of Colorado; J. C. Fitterer, University of Wyoming; W. H. Hill, Greeley High School; O. C. Lester, University of Colorado; G. H. Light, University of Colorado; S. L. Macdonald, Colorado A. & M. College; J. Q. McNatt, Colorado Fuel & Iron Co.; O. A. Randolph, University of Colorado; C. B. Ridgaway, University of Wyoming; H. M. Showman, Colorado School of Mines; C. S. Sperry, University of Colorado; C. E. Stromquist, University of Wyoming; G. P. Unseld, Westminster High School; C. C. VanNuys, Colorado School of Mines.

G. H. LIGHT, *Secretary.*

THIRD ANNUAL MEETING OF THE OHIO SECTION.

The third annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on March 29, 1918, in connection with the meetings of some sections of the Ohio College Association, and the Association of Ohio Teachers of Mathematics and Science. Chairman Forbes B. Wiley occupied the chair, being relieved by Professor R. B. Allen for an interval.

The following thirty persons were registered, all but the last eight being members of the Association:

R. B. Allen, Kenyon College; W. E. Anderson, Wittenberg College; G. N. Armstrong, Ohio Wesleyan University; C. L. Arnold, Ohio State University;

C. B. Austin, Ohio Wesleyan University; Grace M. Bareis, Ohio State University; Ethelwynn R. Beckwith, Western Reserve University; R. L. Borger, Ohio University; R. D. Bohannan, Ohio State University; W. D. Cairns, Oberlin College; William Hoover, Columbus; H. W. Kuhn, Ohio State University; G. W. McCoard, Ohio State University; Charlotte Morningstar, Ohio State University; C. C. Morris, Ohio State University; Hortense Rickard, Ohio State University; S. A. Singer, Capital University; K. D. Swartzel, Ohio State University; J. H. Weaver, Ohio State University; C. J. West, Ohio State University; R. B. Wildermuth, Capital University; Forbes B. Wiley, Denison University. Non-members: H. M. Beatty, Columbus; J. C. Boldt, Dayton (Stivers High School); Helen Carl, Columbus; J. E. Evans, Columbus; Clarice Hobensack, Columbus; J. E. Newell, Columbus; H. O. Rugg, University of Chicago; Miskel Schaeffer, Columbus.

The following program was carried out as arranged by the executive committee:

1. Chairman's Address. An Experiment with Coördinates. PROFESSOR FORBES B. WILEY, Denison University.
2. The Possibilities of a College Course in Investment Mathematics. ETHELWYNN R. BECKWITH, College for Women, Western Reserve University.
3. Discussion of preceding paper. PROFESSOR C. J. WEST, Ohio State University.
4. A Prevalent Hyperbola. PROFESSOR WILLIAM HOOVER, Columbus, O.
5. A non-Commutative and non-Associative Linear Algebra with an Application to Electricity. PROFESSOR R. L. BORGER, Ohio University.
6. The Equipment and Administration of the Mathematics Departments of the Colleges of Ohio. HORTENSE RICKARD, Ohio State University.

At 7:30 P. M. there was an informal meeting at the Ohio Union in the nature of a round table. The subject for discussion was the following:

7. Statistical Tests in Collegiate Mathematics, especially in College Algebra. Leaders of Discussion, PROFESSOR H. O. RUGG, University of Chicago, PROFESSOR W. E. ANDERSON, Wittenberg College.

Twenty of those present dined together at Oxley Hall on Friday evening. Many remained and participated in the meetings of the Association of Ohio Teachers of Mathematics and Science on Saturday. The subject of chief interest to mathematicians on this program was the paper by Professor H. O. Rugg on Normal Tests in High School Algebra.

ABSTRACT OF PAPERS.

1. In his paper on An Experiment with Coördinates, Professor Wiley used as axes of reference, first, parallel lines and later parallel complex planes, plotting pairs of numbers as straight lines. He exhibited the graphs for the linear and special cases of the quadratic functions in two variables. Emphasis was placed upon the fact that this topic had proven to be of interest to undergraduates and had opened problems none too advanced for their investigation. The paper

aroused interest and gave rise to a twenty-minute discussion on questions suggested by it.

2. In speaking on the Possibilities of a College Course in Investment Mathematics, Mrs. Beckwith outlined a year's work for a three-hour course, covering the subjects of compound interest, annuities, bond valuation, depreciation and life insurance. This course is designed to provide the student with a knowledge of the principles of conservative investment which are of personal as well as professional value, and is being given this year in Women's College of Western Reserve University.

3. Professor West, in discussing the paper of Professor Beckwith, called attention to the difficulty of deciding on the contents of a course in the mathematics of investment. Such a course may be developed with the idea of furnishing illustrative material for mathematics, or it may be developed as a component part of a course in business administration. Professor West was of the opinion that a separate and distinct course of the first type was hardly worth while.

4. The "prevalent" hyperbola alluded to by Professor Hoover is of form $axy + bx + ay + c = 0$. A great variety of instances arising in the teaching and reading of pure and mixed mathematics over a considerable range were adduced from geometry, mechanics, etc. To illustrate, the locus of the feet of normals from a fixed point in the plane of a conic is of the general form above.

5. In Professor Borger's paper there is defined a particular non-associative, non-commutative linear algebra in two units and the set of theorems pertaining to it. It furnishes a vector treatment for the theory of alternating current phenomena. Steinmetz has used the complex number system of algebra to represent current and electro-motive force, but his number field was not adapted to that purpose. Professor Borger develops the algebra that is demanded, assuming the Steinmetz postulates as the required conditions for the representation of the entities involved.

6. Miss Rickard read a paper giving information in regard to the equipment and administration of the mathematics departments of colleges throughout the state. Material was obtained by means of a questionnaire sent to the heads of the departments of mathematics in Ohio colleges.

Replies had been received from only a small proportion of the colleges, but the information proved to be of so great interest that the Section asked Miss Rickard to continue her collection of data and to make a complete report next year.

7. At the round table discussion at 7:30, attended by twenty-two people, the leading feature was the talk by Professor Rugg along the lines of his well-known work on statistical standardized tests in teaching. He set forth the aims, results and limitations of standardized tests in elementary and secondary instruction and indicated their extension profitably to college algebra. A clear conception of what he would have the student attain, and ability to see the subject from the student's viewpoint, must be possessed by the successful teacher. Emphasis was laid upon four factors entering into educational tests: the pupil's ability;

the subject of instruction; the system of marking; what we mark, ability *versus* performance.

The discussion was opened by Professor W. E. Anderson, who voiced the desirability of our attempting to put into practice the suggestions of Professor Rugg in his outline of the work done in the field of mathematics. The desirability of greater uniformity and higher standards was emphasized.

The discussion was continued with interest until adjournment was necessitated by the closing of the building under the war department regime.

Secretary Cairns gave a word of greeting from the national Association. A hearty expression of thanks was voted Professor Rugg.

G. N. Armstrong, *Secretary.*

BOOK REVIEW.

Send all communications to W. H. Bussey, University of Minnesota.

Elementary Mathematical Analysis. By John Wesley Young and Frank Millett Morgan. The Macmillan Company, New York, 1917. xii + 548 pages. $2.60.

Instructors in mathematics who are in sympathy with the recent discussions relating to the advisability of unifying the mathematics of the freshman year in college will be pleased to see this new text, for it seems to satisfy the present demand admirably. It places more emphasis "on insight and understanding of fundamental conceptions, less emphasis on algebraic technique and facility of manipulation." It provides for the general cultural aim of mathematical study by arranging the course to "cover as broad a range of mathematical concepts as possible," due consideration being given to "modern mathematical disciplines." The disciplinary value of mathematics is sought "primarily in the domain of thinking, reasoning, reflection, analysis, not in the field of memory, nor of skill in a highly specialized form of activity."

The text is distinctive in a number of respects. It embodies many features which make for economy of time and for increased efficiency. It employs progressive methods and takes advantage of recent developments in teaching freshman mathematics. The various topics of analysis are treated as belonging to a single science, and the emphasis placed upon the notion of function gives the subject a real educational and practical value. The calculus is introduced, wherever convenient, by considering the change ratio $\Delta y/\Delta x$.

The book is divided into five parts, of which the first deals with introductory concepts. The student is made acquainted, in Chapter I, with the fundamental idea of a mathematical function and its representation by analytic, tabular, and graphic methods. This is followed by Chapter II, on the relations of algebraic principles to geometry, which includes an excellent review of algebraic technique.

The second part, the longest of the five, consists of Chapters III–X. It discusses algebraic, trigonometric and exponential functions. The graphic treatment of the algebraic functions somewhat resembles the work usually left to analytic geometry. Interesting applications of slopes and maximum and minimum points are made.

The material in the chapters on trigonometry is well selected and admirably arranged. Time is saved by developing the trigonometric functions from the general angle; and the graphing in polar coördinates, another subject usually deferred to analytic geometry, adds materially to a clearer conception of the variation of the six functions. A section of particular interest at the present time is one which discusses the use of angular measurement in artillery service and gives several problems on indirect fire. We regret that space did not permit more such applications and that simple problems in aviation were not included. There is an excellent chapter on the use of logarithms and another on numerical computation; in these are considered absolute and relative errors, significant figures, and the use of the slide rule. Trigonometric equations, circular measure, and the inverse functions are well treated, and throughout the chapters on trigonometry there are many practical problems and questions.

Part III, Chapters XI–XV, takes up the study of analytic geometry. The curves, which earlier in the course were studied for the purpose of investigating properties of the functions, are here studied from the viewpoint of the equation. The familiar loci, the straight line and circle, are treated at length. The chapter on conic sections introduces only the most essential properties of those curves. Special mention should be made of the excellent chapter on parametric equations.

Topics in college algebra, including the elementary work in the theory of equations, permutations, combinations, probability, the binomial theorem, complex numbers, and the elements of determinants, occupy Part IV, Chapters XVI–XX. The placing of these subjects after the work in trigonometry and analytic geometry is logical, for the student is better prepared to understand their theory.

The last part of the book, Chapters XXI–XXII, takes up a study of the elements of solid analytic geometry. The idea of a function of two variables is emphasized and just enough is given to enable the student to grasp quickly the applications of the calculus to volumes and surfaces of solids of revolution. Four-decimal-place tables constitute the last nine pages.

This text is one of the series of mathematical texts edited by Professor Hedrick, and like the others is an example of excellent workmanship. The cuts and print are clear and attractive and the typographical errors are remarkably few. The text is used with much success in liberal arts and engineering classes in the University of North Dakota. Both instructors and students are enthusiastic over its adoption.

RAYMOND R. HITCHCOCK.

UNIVERSITY OF NORTH DAKOTA.

PROBLEMS FOR SOLUTION.

Send all communications about problems to B. F. Finkel, Springfield, Mo.

2709. Proposed by E. V. HUNTINGTON, Cambridge, Mass.

The following problem was suggested to the proposer by a professor of biology, who has found the result useful in certain problems concerning the equilibrium of chemical reactions.

Starting with $\mu(c_1 + y - x)(c_2 - x)\cdots(c_n - x) = \lambda(b_1 + x)(b_2 + x)\cdots(b_m + x)$, where $\mu c_1 c_2 c_3\cdots c_n = \lambda b_1 b_2 b_3\cdots b_m$ (all the letters being positive), find the limit of x/y as y approaches zero; and show that for small values of y, the value of x/y is always less than this limit.

2710. Proposed by ROGER E. MOORE, The University of Wisconsin.

If $a_k^{(r)}$ denotes the kth term of an arithmetic progression of order r; and c_k denotes the kth binomial coefficient in the expansion of $(a - b)^n$ (n being a positive integer), show that

$$S \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \text{ if } n > r.$$

2711. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the curves (a) $a^8y_1^2 = x^4(a^2 - x^2)^3$, (b) $a^8y_2^2 = x^8(a^2 - x^2)$ bound ten areas, of which two are each $(a^2/4)(\frac{1}{4}\pi - \frac{1}{3})$ and the remaining eight are each $a^2/24$.

2712. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Given the conic $ax^2 + 2hxy + by^2 - 2x = 0$. Find the locus on which lie the four points of intersection of pairs of tangents to the conic from a pair of points on the x-axis equidistant from the origin.

2713. Proposed by G. PAASWELL, New York City.

In the design of gravity retaining walls the following relation exists,

$$\frac{k\cos(\phi' + a + b)}{\tan^2 b\cos b} - \frac{m\cos\phi'\sec a}{\tan^2 b} = 1,$$

where k, m, ϕ' are constants, a and b are the angles formed by the vertical with (1) the diagonal of the section running from the lower left of the section to the upper right, (2) the inside (right) face of the wall. The left face of the wall is vertical. Solve the equation for a, either exactly or by a good approximation. The area of the wall is given by $A = (\tan a + \tan b)/2$, the height of the wall is taken as unity. With the first equation given, determine a value of either a or b which will minimize this area, *i. e.*, what relation must exist between a and b to give the most economical section of wall?

2714. Proposed by H. R. HOWARD, University of St. Francis Xavier's College, Nova Scotia.

A shuffled pack of $2(p + q)$ cards contains $2p$ honors. Show that the chance of securing exactly half the honors in taking half the pack is $[F(p, q)]^2 \div F(2p, 2q)$, where $F(p, q)$ denotes the number of different sets of p cards which can be selected from $(p + q)$ cards.

Show also that if one honor is removed from the pack, the chance is not thereby affected. Is this true for the chance of getting any other assigned number of honors?

2715. Proposed by H. R. KINGSTON, University of Manitoba.

A', B', C' are points on the sides BC, CA, AB, respectively, of the triangle ABC, and AA', BB', CC' are concurrent in O. X, Y, Z are the three collinear points in which, by Desargues' theorem, the corresponding sides of the triangles ABC and $A'B'C'$ intersect. If A'', B'', C'' are the vertices of the triangle formed by the lines AX, BY, CZ, show that AA'', BB'', CC'' are concurrent.

2716. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

To a passenger in a train moving at the rate of 40 miles per hour, the rain appears to be rushing downward and towards him at an angle of 20 degrees with the horizontal. If the rain is

actually falling in a vertical direction, show that the velocity of the raindrops in feet per second is 21.35.

2717. Proposed by ENOS W. WITMER, Sophomore in Franklin and Marshall College.

Determine the integral values of m and n for which the equation $x^4 + mx^2y^2 + ny^4 = z^2$ has non-trivial solutions. [Carmichael's *Diophantine Analysis*, Prob. 13, p. 53.]

The following problems in volumes XX to XXIII still remain unsolved:
Algebra: numbers 406, 430, 461;
Geometry: numbers 446, 470, 472, 478, 494, 501, 510, 519, 523;
Calculus: numbers 339, 340, 342, 348, 353, 360, 385, 411, 415, 425, 429, 432, 434, 436, 440;
Mechanics: numbers 277, 279, 285, 287, 291, 308, 309, 313, 315, 322, 328, 335, 343, 344, 348, 350, 351, 356, 357;
Number Theory: numbers 189, 190, 191, 192, 200, 205, 231, 232, 234, 239, 245, 247, 260, 261, 263, 270, 271, 273, 274, 275.
The editors will be glad to receive solutions of any of these unsolved problems.

SOLUTIONS OF PROBLEMS.

Note. 1. Florence P. Lewis sent in a solution of 486 and Albert Babbitt a solution of 493, Algebra, after selection for publication had been made and sent to the Editor-in-Chief.

2. The following correction should be made: On page 24, of the January, 1918, number of the MONTHLY, 16th line from bottom, for $(n^2 + 1) - 4n$, read $(n + 1)^2 - 4n$. EDITORS.

427 (Calculus). Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, O.

Of all ellipses cirumscribed about a given parallelogram, the minimum (maximum) with regard to area has as conjugate diameters the diagonals of the parallelogram.

II. SOLUTION BY O. D. KELLOGG, University of Missouri.

I venture to add my solution to those given in the January number of the MONTHLY because it illustrates the fruitfulness of the notion "shear," a simple transformation which should doubtless find more use in elementary mathematics.[1] The coördinate axes having any position in the plane, the transformation $x = x'$, $y = y' - ax'$ defines a shear. The following are invariants: area, ellipse, conjugacy, parallelism.

Suppose E' and E'' are two ellipses circumscribed about the parallelogram P, the former having the diagonals of P as conjugate diameters. A shear may be found which carries E' over into a circle C', and consequently P into a square R', since its diagonals are conjugate diameters of a circle. Using the letters to denote areas, we have $P/E' = R'/C'$.

A second shear may be found which will carry E'' into a circle C'', and consequently P into a rectangle R'', and we have $P/E'' = R''/C''$.

But a square has an area whose ratio to that of the circumscribed circle is greater than that of any other rectangle. Hence $R'/C' > R''/C''$, and the above equations yield $E'' > E'$, so that E' is the circumscribed ellipse of *minimum* area.

438 (Calculus). Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the locus of the equation

$$y^6 - 3(a^2 - x^2)y^4 - 2ax^2y^3 + 3(a^2 - x^2)^2y^2 - 6ax^2(a^2 - x^2)y + a^2x^4 - (a^2 - x^2)^3 = 0,$$

first showing that it can be reduced to the form

$$y = kx^n \pm (a^2 - x^2)^m,$$

and finding the points of maximum abscissa, of maximum ordinate, and of inflection.

[1] In fact the idea is used in Young and Morgan's *Elementary Analysis*, pages 132, 288, and 289.

Solution by the Proposer.

Inspection of the two equations shows that $m = \frac{1}{2}$, $n = \frac{2}{3}$, and since the first equation is homogeneous in (a, x, y), $k = a^{1/3}$. The given equation is in fact the result of rationalizing

$$y = a^{1/3}x^{2/3} \pm \sqrt{a^2 - x^2}.$$

Let $x/a = \sin\theta$. Then $y/a = \sin^{2/3}\theta \pm \cos\theta$. Hence, $y'/a = 2/3 \sin^{-1/3}\theta \mp \tan\theta$, since $d\theta/dx = \sec\theta$. Hence, $y''/a = -2/9 \sin^{-4/3}\theta \mp \sec^3\theta$. When $y' = 0$, $\tan\theta \cdot \sin^{1/3}\theta = 2/3$. Solving, $\theta_1 = 38° \; 4.2'$, $x_1/a = 0.6166$, and $y_1/a = .7245 \pm .7873 = 1.5117$ or -0.0628. The first value gives a maximum ordinate. When $y'' = 0$, $\tan^{.8}\theta \sec\theta = (2/9)^{.6}$. Solving, $\theta_2 = 17° \; 1'$, $x_2/a = 0.292\overline{7}$, and $y_2/a = .4408 \pm .9562 = 1.3970$ or -0.5154. $y_2/a' = 1.004\overline{1} \mp .306\overline{1} = 0.6981$ or 1.3102.

The second value gives an inflection.

The curve is readily constructed by adding ordinates of the semi-cubical parabola and the circle.

(a, a) is evidently the point of maximum abscissa.

Also solved by Adele Holtwick.

439 (Calculus). Proposed by Clifford N. Mills, Brookings, S. Dak.

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution by C. C. Yen, Tangshan, North China.

Let $P \equiv (x, y, z)$ be the vertex of the rectangular parallelopiped lying in the first octant of the ellipsoid. Then the volume of the parallelopiped is $V = 8xyz$.

Since P lies on the ellipsoid, the coördinates (x, y, z) satisfy the equation of the ellipsoid, and therefore

$$V = 8xyz = 8cxy(1 - x^2/a^2 - y^2/b^2)^{1/2} = 8c \cdot F(x, y), \tag{1}$$

where $F(x, y) \equiv x \cdot y(1 - x^2/a^2 - y^2/b^2)^{1/2}$ is maximum when and only when V is maximum.

Differentiating, we get

$$\begin{aligned} \frac{\partial F}{\partial x} &= y(1 - 2x^2/a^2 - y^2/b^2) \div (1 - x^2/a^2 - y^2/b^2)^{1/2}, \\ \frac{\partial F}{\partial y} &= x(1 - x^2/a^2 - 2y^2/b^2) \div (1 - x^2/a^2 - y^2/b^2)^{1/2}. \end{aligned} \tag{2}$$

Equating to zero the left-hand members of (2), we have

$$2b^2x^2 + a^2y^2 = a^2b^2, \qquad b^2x^2 + 2a^2y^2 = a^2b^2,$$

which give

$$x^2 = \frac{a^2}{3}, \qquad y^2 = \frac{b^2}{3}; \qquad \text{and, therefore,} \qquad x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}.$$

Differentiating (2), we get

$$\frac{\partial^2 F}{\partial x^2} = -\frac{xy}{a^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-3/2}\left(3 - \frac{2x^2}{a^2} - \frac{3y^2}{b^2}\right), \quad = -\frac{4b}{a\sqrt{3}} \text{ when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}};$$

$$\frac{\partial^2 F}{\partial y^2} = -\frac{xy}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-3/2}\left(3 - \frac{3x^2}{a^2} - \frac{2y^2}{b^2}\right), \quad = -\frac{4a}{b\sqrt{3}} \text{ when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}};$$

and

$$\frac{\partial^2 F}{\partial y \partial x} = \left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right)\left\{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} + \frac{y^2}{b^2}\right\} - \frac{2y^2}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2},$$

$$= -\frac{2}{\sqrt{3}} \quad \text{when } x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}.$$

Therefore, when

$$x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}, \qquad \Delta \equiv \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial y \partial x}\right)^2 = 4;$$

and since

$$\frac{\partial^2 F}{\partial x^2} < 0, \qquad \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0,$$

we have all the conditions for a maximum value of $F(x, y)$ fulfilled at $(a/\sqrt{3}, b/\sqrt{3})$.

Hence, finally, substituting in (1), we have the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid equal to $8abc/(3\sqrt{3})$.

Also solved by L. E. Lunn, L. E. Mensenkamp, O. S. Adams, C. E. Githens, J. L. Riley, Paul Capron, and H. C. Feemster.

441 (Calculus). Proposed by J. L. RILEY, Stephenville, Texas.

Find the minimum value of

$$\int \left\{ \left(\frac{dy}{dx}\right)^2 \sin x + (y + x - \sin x)^2/\sin x \right\} dx.$$

Solution by Elijah Swift, University of Vermont.

This problem is indefinite and no solution is possible, until the conditions that the end points must satisfy are stated. In fact, if we take $y = mx$ and integrate from $\pi + \epsilon$ to $2\pi - \epsilon$, we can make the integral as small as desired by decreasing ϵ.

It is not difficult to find the equation of the extremals. Euler's differential equation is $d(F_{y'})/dx - F_y = 0$. This becomes for our problem $y'' \sin^2 x + y' \sin x \cos x - y = x - \sin x$. A solution of the equation where we make the right-hand member zero is

$$y = C_1 \tan (x/2) + C_2 \operatorname{ctn} (x/2),$$

obtained by taking $2y'$ as an obvious integrating factor. Completing the solution by any one of several methods, there results

$$y = C_1 \tan (x/2) + C_2 \operatorname{ctn} (x/2) - x + \operatorname{ctn} (x/2) \log \sec^2 (x/2).$$

For any further investigation of the problem, however, a knowledge of the boundary conditions is necessary.

Also solved by Alexander Dillingham.

349 (Mechanics). Proposed by S. A. COREY, Albia, Iowa.

A 9-pound weight is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to and supports a smooth pulley P_1 of weight 1 pound, over which passes a second string to one end of which is attached a 3-pound weight, and the other end of which is attached to and supports another smooth pulley P_2 of weight 1 pound. Over the pulley P_2 passes a third string supporting weights, 2 pounds and $3\frac{1}{3}$ pounds.

If the system is acted on by gravity alone show that the accelerations of the 9-pound weight, $3\frac{1}{3}$-pound weight, and pulley P_2 are 0, $\frac{1}{2}g$, and $\frac{1}{8}g$, respectively.

Determine the motion of the weights when pulleys are not smooth, that is, when friction is present.

Solution by the Proposer.

Let x = distance of 9-pound weight from center of fixed pulley, y = distance of center of P_2 from center of P_1, and z = distance of $3\frac{1}{3}$-pound weight from center of P_2.

Then will $\dot{x}$ = velocity of 9-pound weight, $-\dot{x}$ = velocity of P_1, $\dot{y} - \dot{x}$ = velocity of P_2 $-\dot{y} - \dot{x}$ = velocity of 3-pound weight, $\dot{z} + \dot{y} - \dot{x}$ = velocity of $3\frac{1}{3}$-pound weight, and $-\dot{z} + \dot{y} - \dot{x}$ = velocity of 2-pound weight.

If T = kinetic energy of entire moving system, and if m represent mass of 1-pound weight, we have

$$T = \frac{m}{2}\Big[9\dot{x}^2 + \dot{x}^2 + (\dot{y}^2 - 2\dot{x}\dot{y} + \dot{x}^2) + 3(\dot{y}^2 + 2\dot{y}\dot{x} + \dot{x}^2) + \tfrac{10}{3}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2\dot{x}\dot{y}$$

$$+ 2\dot{z}\dot{y} - 2\dot{z}\dot{x}) + 2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2\dot{x}\dot{z} - 2\dot{y}\dot{z} - 2\dot{x}\dot{y})\Big]$$

$$= \frac{m}{3}(29\dot{x}^2 + 14\dot{y}^2 + 8\dot{z}^2 - 10\dot{x}\dot{y} + 4\dot{y}\dot{z} - 4\dot{x}\dot{z});$$

whence,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = \frac{2m}{3}(29\ddot{x} - 5\ddot{y} - 2\ddot{z}) = (9 - 1 - 3 - 1 - 2 - \tfrac{10}{3})mg = -\tfrac{4}{3}mg,$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} = \frac{2m}{3}(14\ddot{y} - 5\ddot{x} + 2\ddot{z}) = (-3 + 2 + 1 + \tfrac{10}{3})mg = \tfrac{10}{3}mg,$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{z}} = \frac{2m}{3}(8\ddot{z} + 2\ddot{y} - 2\ddot{x}) = (-2 + \tfrac{10}{3})mg = \tfrac{4}{3}mg,$$

which give $\ddot{x} = 0$, $\ddot{y} = \frac{1}{3}g$, and $\ddot{z} = \frac{1}{6}g$, and establishes proof sought if we notice that velocity of $3\frac{1}{3}$-pound weight is $\dot{y} + \dot{z}$.

Inasmuch as the 9-pound weight always remains at rest if the system starts from rest, the presence of friction in the fixed pulley would not alter the motion of the system as long as no friction be present in the movable pulleys. But if the fixed pulley is frictionless and friction is present in the movable pulleys, then the diminished accelerations of the weights and movable pulleys on the one side of the fixed pulley must, in part, be offset by an upward acceleration of the 9-pound weight on the other side, and in part by the heat of friction generated as long as the friction is not too great to prevent the revolving of the movable pulleys.

352 (Mechanics). Proposed by C. N. SCHMALL, New York City.

A glass rod is balanced partly in and partly out of a cylindrical tumbler, with lower end resting against the vertical wall of the tumbler. If φ and ψ are the maximum and minimum angles, respectively, which the rod can make with the vertical plane, and θ is the angle of friction, show that

$$\theta = \tfrac{1}{2}\tan^{-1}\frac{\sin^3\varphi - \sin^3\psi}{\sin^2\varphi\cos\varphi + \sin^2\psi\cos\psi}.$$

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let $AB = 2a$ be the rod with its lower end at A; G, its middle point; P, the point of support in the edge of the tumbler; CD, the element of the surface of the tumbler in which A lies; and W = the weight of AB. From P draw a perpendicular to CD cutting it in E, and, similarly, draw GK.

The rod in either extreme position is kept in equilibrium by its weight acting vertically downwards, the two reactions R, S, at A and P respectively, and the friction at A in the direction of the element CD and at P in the direction of the rod.

Resolving vertically and horizontally and taking moments about A,

$$W + S\tan\theta\cdot\cos\varphi = S\sin\varphi - R\tan\theta, \tag{1}$$

$$S\cos\varphi + S\tan\theta\cdot\sin\varphi = R, \tag{2}$$

$$W\cdot GK = S\cdot AP, \tag{3}$$

or

$$W\{(a - x)\sin\varphi + b\} = S\cdot x. \tag{4}$$

From the geometry,

$$x = b/\sin\varphi, \tag{5}$$

and this in (4) gives

$$Wa\sin^2\varphi = b. \tag{6}$$

Substituting R from (2) in (1),

$$W = S\{\sin\varphi(1 - \tan^2\theta) - 2\cos\varphi\tan\theta\}. \tag{7}$$

(6) ÷ (7) gives

$$a\sin^2\varphi = \frac{b/S}{\sin\varphi(1-\tan^2\theta) - 2\cos\varphi\tan\theta}. \tag{8}$$

For the other extreme angle ψ, $\tan\theta$ changes sign, and we have, by analogy,

$$a\sin^2\psi = \frac{b/S}{\sin\psi(1-\tan^2\theta) + 2\cos\psi\tan\theta}. \tag{9}$$

(8) ÷ (9) gives

$$\frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta = \frac{\sin^3\varphi - \sin^3\psi}{\sin^2\varphi\cos\varphi + \sin^2\psi\cos\psi}, \tag{10}$$

giving the required angle.

It may be interesting to note that when the upper edge is smooth, the angle of friction is twice as great as that in (10).

Also solved by A. M. HARDING and G. PAASWELL.

353 (Mechanics). Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A uniform beam of oak, 10 feet long, 15 inches deep and 10 inches wide, sustains, in addition to its own weight, a load of 5,000 lbs. placed at the center. Find the greatest bending moment and the greatest stress in the fibers. Take the specific gravity of oak as 0.934.

SOLUTION BY W. J. THOME, Detroit, Michigan.

Let l, b, d be the length, breadth, and depth, respectively, of the beam; M the greatest bending moment; p, the greatest stress in the fibers; and S, the section-modulus of the beam's cross section. Taking the inch and the pound as units and the weight of a cubic foot of water $= 62.5$ pounds, we have

$$\begin{aligned} M &= \tfrac{1}{4}(5000)l + \tfrac{1}{8}\left(bdl \times 0.934 \times \frac{62.5}{1728}\right) l \\ &= \tfrac{1}{4}(5000)10 \times 12 + \tfrac{1}{8}\left[10 \times 15(10 \times 12) \times 0.934 \times \frac{62.5}{1728}\right] 10 \times 12 \\ &= 159121 \text{ inch-pounds.} \end{aligned}$$

$$p = M/S = M/\tfrac{1}{6}bd^2 = 6M/bd^2 = 6 \times 159121/10 \times 15^2 = 424 \text{ lbs. per sq. in.}$$

Also solved by PAUL CAPRON.

354 (Mechanics). Proposed by G. PAASWELL, New York City.

The acceleration of an electric train is constant and equal to a ft. per sec. per sec. Its braking or deceleration is variable and equal to the square root of the velocity. If the distance between stations is 5,000 ft., show that the acceleration must cease and braking ensue when the train is about 960 ft. from the stopping point; also that the maximum velocity attained for a minimum time run is 88 m.p.h. and the time of run is 54 seconds.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Under a constant acceleration, a final speed of 88 mi./hr., or about 129 ft./sec., is acquired, in a distance of 4,040 ft., at the average rate of $64\frac{1}{2}$ ft./sec., in about 62.6 secs. The corresponding acceleration is 2.06 ft./sec.^{2}. Evidently the numerical values are incorrect. It will appear further that the distance to be run under acceleration, the greatest velocity acquired and the time of run are all dependent on the acceleration, and that the time of run is a minimum (about 39.15 secs.) when the acceleration is infinite, its duration zero, and the greatest speed about $261\frac{1}{4}$ mi./hr.

Let the corresponding values of time in seconds, distance in feet, and speed in feet per second, be t, s, v at any point between stations, and let these letters, with appropriate subscripts, represent

the particular values at P_0, the starting point, P_1, the point where acceleration ceases, and P_2, the stopping point. It is assumed that each of the values t_0, s_0, v_0 and v_2 is zero, and that $s_2 = 5{,}000$; that the acceleration is a constant, a, from P_0 to P_1, and is equal to $-\sqrt{mv}$ from P_1 to P_2, m having the units ft./sec.^{3}.

From P_0 to P_1, $dv/dt = v(dv/ds) = a$, so that, under the initial conditions, $[A]$ $t_1 = v_1/a$. $[B]$ $s_1 = v_1^2/2a$.

From P_1 to P_2, $dv/dt = v(dv/ds) = -\sqrt{mv}$; whence,

$$\int_{P_1}^{P_2} dt = \int_{P_2}^{P_1} \frac{1}{\sqrt{m}} v^{-1/2} dv; \qquad \int_{P_1}^{P_2} ds = \frac{1}{\sqrt{m}} \int_{P_2}^{P_1} v^{1/2} dv,$$

so that, under the final conditions, $t_2 - t_1 = (2/\sqrt{m})v_1^{1/2}$, $s_2 - s_1 = (2/3\sqrt{m})v_1^{3/2}$, or

$$[C] \quad t_2 = \frac{v_1}{a} + \frac{2}{\sqrt{m}} v_1^{1/2}; \qquad [M] \quad s_2 = \frac{v_1^2}{2a} + \frac{2}{3\sqrt{m}} v_1^{3/2}.$$

From $[A]$, $[B]$, $[C]$, $[M]$:

$$[N] \quad \frac{1}{a} = \frac{2s_2}{v_1^2} - \frac{4}{3\sqrt{mv_1}}, \qquad [\beta] \quad s_1 = s_2 - \frac{2v_1^{3/2}}{3\sqrt{m}}.$$

$$[\alpha] \quad t_1 = \frac{2s_2}{v_1} - \frac{4v_1^{1/2}}{3\sqrt{m}}, \qquad [\gamma] \quad t_2 = \frac{2s_2}{v_1} + \frac{2v_1^{1/2}}{3\sqrt{m}}.$$

From $[\gamma]$,

$$dt_2/dv_1 = -\frac{2s_2}{v_1^2} + \frac{1}{3\sqrt{mv_1}} = 0$$

when $v_1 = \infty$, which is impossible by $[M]$, since s_2 is finite, and when $v_1 = \bar{v}_1 = (6\sqrt{m}\, s_2)^{2/3}$. When $v_1 = \bar{v}_1$, moreover, $a = -(6s_2m^2)^{1/3}$, $s_1 = -3s_2$, so that, as a and s_1 must be positive, dt_2/dv_1 vanishes only under impossible conditions.

Differentiating $[N]$, $[\alpha]$, $[\beta]$, $[\gamma]$ and using $[M]$ to eliminate s_2, we find

$$[\mathrm{I}] \quad da = 2\left(\frac{a}{v_1} + \frac{a^2}{\sqrt{mv_1^3}}\right) dv_1, \qquad [\mathrm{II}] \quad dt_2 = -\left(\frac{1}{a} + \frac{1}{\sqrt{mv_1}}\right) dv_1,$$

$$[\mathrm{III}] \quad ds_1 = -\sqrt{\frac{v_1}{m}}\, dv_1, \qquad [\mathrm{IV}] \quad dt_1 = -\left(\frac{1}{a} + \frac{2}{\sqrt{mv_1}}\right) dv_1.$$

Since each of the values a, v_1, and $\sqrt{mv_1}$ is positive, it appears from [I–IV] that as a is increased, v_1 increases and t_1, s_1, t_2 decrease.

From $[M]$,

$$v_1^{3/2} = \frac{3\sqrt{m}}{2}\left[s_2 - \frac{v_1^2}{2a}\right],$$

so that, as $a = \infty$ (s_2 being finite),

$$v_1 \doteq \left(\frac{3}{2} s_2 \sqrt{m}\right)^{2/3},$$

and from $[C]$,

$$t_2 \doteq 2\sqrt{\frac{v_1}{m}} \doteq \left(\frac{12s_2}{m}\right)^{1/3}.$$

From $[A]$ and $[B]$, $t_1 \doteq 0$ and $s_1 \doteq 0$.

Consequently, the greater the acceleration, the shorter will be the time and distance through which it lasts, the greater will the velocity become, and the sooner will the train arrive. For the quickest run, the train should be shot from a gun with a muzzle velocity of about 261¼ mi./hr., braking to start immediately (with an initial intensity of about 1,362 lbs. to the (long) ton).

If $\sqrt{v_1} = x$, we have $x^4 + (4a/3)x^3 = 2as_2 = 10{,}000\,a$, $t_1 = x^2/a$, $t_2 = (x^2/a) + 2x$, $s_1 = (x^4/2a)$, $s_2 - s_1 = \frac{2}{3}x^3$. [$m = 1$ ft./sec.^{3}, $s_2 = 5{,}000$ ft.]. From these the following values may be computed:

a	x	v_1 (ft./sec.)	u_1 (mi./hr.)	t_1 (secs.)	(t_2-t_1) (secs.)	t_2 (secs.)	s_1 (ft.)	s_2-s_1 (ft.)
1	9.6825	93.75	63.92	93.754	19.365	113.12	4,394.9	605.1
2	11.278	127.2	86.72	63.595	22.555	86.15	4,044.5	955.5
3	12.264	150.4	102.55	50.13	24.53	74.66	3,770.3	1,229.7
100	18.735	351.0	239.32	3.51	37.47	40.98	616.0	4,384.0
∞	19.573	383.15	261.24	0	39.149	39.15	0	5,000

355 (Mechanics). Proposed by HORACE OLSON, Chicago, Ill.

A solid spheroid, axes a, a, b, is placed with its axis of revolution vertical. From its highest point a particle is projected horizontally with a speed s. Where will it leave the spheroid, assuming that it slides on the surface without friction?

Solution by William Hoover, Columbus, Ohio.

The entire motion is in a vertical central section of the spheroid and all of such sections are equal ellipses.

Let $a^2y^2 + b^2x^2 = a^2b^2$ (1) be the equation of any one of them.

Resolving tangentially,

$$v\frac{dv}{ds} = g\frac{dy}{ds}. \tag{2}$$

Multiplying by ds and integrating,

$$v^2 = 2gy + C. \tag{3}$$

When $v = v_0$, $y = b$, and $C = v_0^2 - 2gb$, and (3) becomes

$$v^2 = v_0^2 - 2g(b - y). \tag{4}$$

If ρ = the radius of curvature, we have, at the point where the particle leaves the curve

$$\frac{v^2}{\rho} = -g\frac{dx}{ds}. \tag{5}$$

Now from (1),

$$\rho = \{(a^2 - b^2)y^2 + b^4\}^{3/2} \div ab^4 \tag{6}$$

and

$$\frac{dx}{ds} = -ay \div \{(a^2 - b^2)y^2 + b^4\}^{1/2}. \tag{7}$$

Substituting (6) and (7) in (5) and reducing,

$$(a^2 - b^2)gy^3 + 3b^4gy + b^4(v_0^2 - 2gb) = 0, \tag{8}$$

a cubic for y. Now put $a = a/2$, $b = b/2$, $v_0 = s$.

In (8), if $b = a$, and $v_0 = 0$, $y = \frac{2}{3}a$, as is well known for a circle of radius a.

Also solved by Paul Capron.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

NEW QUESTION.

35. Is the theorem given below new or has it previously been published?

Theorem. *If two parallel planes, π and π', cut sections from a cylindrical surface S and two spherical surfaces S_1 and S_2, and if the sum of the sections of S_2 is equal in area to the sum of the sections of S and S_1, then the part of S_2 included*

between π and π' is equal in volume to the sum of the parts of S and S_1 included between π and π'.

Note. In communicating the above question Mr. Elbert O. Brower, of Cicero, Illinois, says (in substance): "The truth of this proposition so easily follows from a consideration of the ordinary formula for getting the volume of a spherical segment, that it is difficult to suppose that it is unknown; yet I am led to wonder how it happens that, if known, it has not been accorded the prominence which it would seem to deserve. In sending it to the MONTHLY I wish to determine whether or not it is original with myself."—U. G. M.

DISCUSSIONS.

I. RELATING TO THE TRANSITION CURVE.

By GEORGE PAASWELL, Civil Engineer, New York City.

The transition curve is a so-called railroad spiral used to ease the approach to a circular curve. It is defined by its intrinsic equation $d\varphi/ds = 2ks$, where s is the distance along the curve measured from the point of zero curvature and k is a constant determined by the special data of the circular curve for which the transition is an easement. The integration of this equation gives $\varphi = ks^2$. Taking the length of the transition as L and the radius of the circular curve as R, the value of k is found from the intrinsic equation to be $1/2RL$.

From the geometry of the infinitesimal triangle $dy = ds \sin\varphi$ and $dx = ds \cos\varphi$, or, substituting from the above the values of ds and s, we have

$$dy = \frac{1}{2\sqrt{k}}\frac{\sin\varphi}{\sqrt{\varphi}}d\varphi; \quad dx = \frac{1}{2\sqrt{k}}\frac{\cos\varphi}{\sqrt{\varphi}}d\varphi; \quad y = \frac{1}{2\sqrt{k}}\int_0^{\phi}\frac{\sin\varphi}{\sqrt{\varphi}}d\varphi\,;$$

$$x = \frac{1}{2\sqrt{k}}\int_0^{\phi}\frac{\cos\varphi}{\sqrt{\varphi}}d\varphi.$$

Expanding the integrals, integrating term by term and replacing k by its value $\sqrt{\varphi}/s$, we get

$$y = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n+1}}{(4n+3)\,\underline{|2n+1}}; \qquad x = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n}}{(4n+1)\,\underline{|2n}}.$$

Defining two functions,

$$\text{tran}\,\varphi = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n+1}}{(4n+3)\,\underline{|2n+1}} \quad\text{and cotran}\,\varphi = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n}}{(4n+1)\,\underline{|2n}},$$

we have $y = s\ \text{tran}\,\varphi$, $x = s\ \text{cotran}\,\varphi$, and, maintaining the same analogy, $\text{tatran}\,\varphi = \text{tran}\,\varphi/\text{cotran}\,\varphi$, so that $y = x\,\text{tatran}\,\varphi$. It may be advantageous to make up tables of these functions, similar to the trigonometric tables, and problems in the transition may be expressed in terms of these new functions. At present there is no rigorous mathematical discussion of this curve.

It may be interesting to establish the coördinates of the terminus of the transition. Since

$$\int_0^\infty \frac{\sin \varphi}{\sqrt{\varphi}} d\varphi = \int_0^\infty \frac{\cos \varphi}{\sqrt{\varphi}} d\varphi = \sqrt{\frac{\pi}{2}},$$

the coördinates become $x_\infty = y_\infty = \frac{1}{2}\sqrt{\pi/2k}$, $x = y = \frac{1}{2}\sqrt{\pi RL}$.

The coördinates of the transition are related to the Bessel functions as follows:

$$J_{1/2}(\varphi) = \frac{1}{\sqrt{\pi}} \frac{\sin \varphi}{\sqrt{\varphi}}\,;\quad J_{-1/2} = \frac{1}{\sqrt{\pi}} \frac{\cos \varphi}{\sqrt{\varphi}}\,;\quad C = \tfrac{1}{2}\int_0^\phi J_{-1/2}(\varphi)d\varphi;\quad S = \tfrac{1}{2}\int_0^\phi J_{1/2}(\varphi)d\varphi;$$

whence $y = \sqrt{\pi RL}\, S(\varphi)$ and $x = \sqrt{\pi RL}\, C(\varphi)$. (Cf. Jahnke and Emde's Tables, p. 23 seq.)

II. Relating to the Graph of a Cubic Equation having Complex Roots.

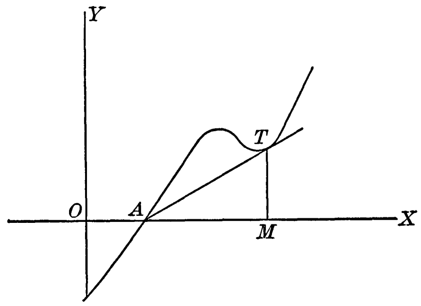
By Edwin S. Crawley, University of Pennsylvania.

The note on "The Graphical Solution of a Cubic Equation having Complex Roots," pp. 70–71 of the Monthly for February, 1918, recalls to my mind something similar which I learned a number of years ago and which might possibly interest some readers of the Monthly.

Every cubic with one real and two imaginary roots is expressible in the form $(x - k)(x^2 - 2px + p^2 + q^2) = 0$, and the graph of

$$y = (x - k)(x^2 - 2px + p^2 + q^2)$$

(*i. e.*, of $y = a_0x^3 + a_1x^2 + a_2x + a_3$) always has a form more or less like the figure. Then it is easy to show that $OM = p$ and $\tan MAT = q^2$, where $p \pm qi$



are the imaginary roots. AT is the tangent to the curve drawn from its real intersection with OX.

For, if $OA = k$ the line $y = \lambda(x - k)$ will be tangent to the curve if

$$x^2 - 2px + p^2 + q^2 - \lambda = 0$$

has equal roots, that is, if $\lambda = q^2$; and $y = q^2(x - k)$ touches the curve at $x = p$.

[Since the foregoing was put in type the editors have called my attention to the fact that the same construction was given by Irwin and Wright in the *Annals of Mathematics*, Vol. 19 (1917), pp. 157–158.]

III. Relating to the Selection of Material for Class Reviews.

By G. R. Clements, United States Naval Academy.

I would like to suggest as a topic for discussion in the Monthly the best use that can be made of the time available for reviews in courses in mathematics.

It seems to me that the best test of a student's mental capacity and of his mastery of a particular course of study in mathematics is his ability to take a pertinent problem and analyze it to see what principles are involved, to select for its solution the most suitable tools from those he has been accumulating during his term's work and to use them intelligently to derive and discuss the results that must follow from his data. And I know of no better way to help a student to correlate his material and get a comprehensive grasp of it than to set him to work at the end of a course on a list of problems carefully selected but *not* closely graded.

When I was teaching at the University of Wisconsin, we secured better results than we had previously obtained in the course in mathematical theory of investments by going rather quickly through the course and then spending a considerable amount of time on just such a list of problems, where the student could have no idea in advance as to whether the solution of a particular problem involved the principles of the first chapter of his text, or of the last chapter, or of both. I have followed a somewhat similar course in analytic geometry, assigning twice each term a list of problems from outside the text (and regarded as somewhat difficult by the students), solutions to be handed in at the end of two weeks, the daily work being somewhat lightened in the meantime.

The increasing list of separate problem texts and the number of texts that have considerable lists of general problems as an appendix would seem to make such a plan rather easy of introduction so far as the mere selection of material is concerned. For example, Miller and Lilly's *Analytic Mechanics* has a very considerable list of miscellaneous problems, and I can think of no better way of reviewing the subject, if that book is being used as a text, than to set the students to mastering a selected portion of this list.

I believe the criticisms that we ourselves and our colleagues in allied departments make of the results of our teaching would be considerably softened if we could conclude each of our courses by going over its subject matter a last time in some such manner as I have suggested.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

The editor earnestly desires that readers should make suggestions to him whereby this department may better serve those for whom it is designed. Very general coöperation in this respect would surely be in the best interests of all concerned.

Every club which has not sent in a complete report of its meetings during 1917–18 is requested to do so as soon as possible.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF CONNECTICUT COLLEGE, New London, Conn.

This newest club at the newest New England college was organized last December through the inspiration of Professor D. D. Leib. Membership is open to those who are taking, or have already taken, a course in mathematics beyond the regular freshman requirement. "The two-fold purpose announced by the club was to foster in the college a proper appreciation of the claims of mathematics upon students, and to acquaint its members, by the presentation of formal papers and by informal discussion, with topics of mathematical interest or value along lines not likely to be included in regular class work. While at least one paper on a definite topic has been presented at each meeting, no attempt has been made to confine the informal discussion to any one topic, and the concluding half hour or more is enlivened by simple 'eats' of some sort." The club had seven members this year.

Officers: President, Ruth F. Avery '19; secretary, Justine McGowan '20; treasurer, Dorothy Peck '19. These officers constitute the committee on program and arrangements. The absence of seniors is due to the fact that the college is just in the third year of its history and the first class will be graduated in 1919. The programs of the meetings have been as follows:

December: "Mathematics and Mathematics Clubs at other Colleges" (particularly Women's Colleges)—a general discussion; articles from the MONTHLY and other sources were summarized.

January: "The Fourth Dimension" by Professor Leib.

February: "Magic Squares" by Margaret Maher '19.

March: "Methods of Multiplication, especially with the Roman Notation" by Ruth F. Avery '19.

April 26: "Zeno's Paradoxes" by Ruth F. Avery '19.

May 10: "The Mathematics of Actuarial Work" by Justine McGowan '20.

THE UNIVERSITY OF SASKATCHEWAN MATHEMATICAL SOCIETY, Saskatoon, Saskatchewan.

This Society was formed on November 16, 1916, with an initial membership of 12. The average attendance now is about 14. All students interested in mathematics are eligible for membership.

Officers, 1917–18: Honorary president, Professor L. L. Dines; president, Rhoda S. Russell '19; secretary-treasurer, Nelson W. Taylor '18; program convenor, John H. Simester '20.

The following meetings of the Society have been held:

November 30, 1916: "Flatland" by Nelson W. Taylor '18.

January 19, 1917: "Non-Euclidean Geometry, Lobachevsky's System" by Roy E. Shuttleworth '18.

February 6: "The Squaring of the Circle" by Edward J. Baldes '18; "A Problem on Refraction of Light" by John H. Simester '20.

February 20: "The Duplication of the Cube and the Trisection of an Angle" by Oscar C. Bridgeman '18.

March 15: "A History of Algebra" by J. H. Simester '20.

November 16, 1917: Organization Meeting.

December 7: "The Construction of a Honey Bee's Cell" by J. H. Simester '20; "A, B, and C—The Human Element in Mathematics," from S. Leacock's *Literary Lapses,* read by Nelson W. Taylor '18.

January 28, 1918: "The Fundamental Theorems for Geometric Constructions by Means of the Compasses alone" by Oscar C. Bridgeman '16.

February 15: "The History and Computation of Logarithms" by J. H. Simester '20.

March 1: "Application of the Compasses alone to certain Geometric Constructions including Regular Polygons" by N. W. Taylor '18.

March 15: "Interesting Relations between Plane and Spherical Geometry" by Gladys Shannon '21; "Parallel Postulates and Riemann's System of Geometry" by Frances Schiltz '14.

Pi Mu Epsilon Fraternity, Syracuse University, Syracuse, N. Y.

The Mathematical Club of Syracuse University was founded in November, 1903. After ten years of successful operation it was reorganized into the Pi Mu Epsilon Fraternity, which aims to promote mathematics and scholarship. On May 25, 1914, this fraternity was incorporated under the laws of the State of New York. Among the powers secured in accordance with the articles of incorporation is that of granting charters to other chapters to be organized elsewhere.

Membership in the chapter is open to "members of the mathematical faculty, former members of the club, any person whose work in the mathematical sciences is distinguished, former students in mathematics, and major and minor students who have taken certain specified courses in the subject and who have attained a certain standard of scholarship set by the chapter." There are 44 members in the university this year, 16 of the faculty and graduates, and 28 undergraduates. The average attendance at meetings was 30.

Officers, 1917–18: Director, Professor Warren G. Bullard; vice-director, Professor John L. Jones; secretary, Helen N. Hale '18; treasurer, Howard A. DoBell

'19; librarian,[1] Cornelia A. Tyler '19. Executive committee: the above officers and Christabel A. Christie '18, Edna R. Howe '18, Charles D. Hurd '18, Rennie B. Smith '18. Scholarship committee: Professors Floyd F. Decker and Louis Lindsey, and Helen N. Hale '18, M. Gladys Medbery '18 and Lawrence J. Blackmar '18.[2]

October 23, 1916: Election of new faculty members. Professor L. Lindsey reported concerning the summer meeting of the Mathematical Association of America. Other members reported on their summer work.

November 13: Report of scholarship committee.[3] A committee was appointed to cast the ballot of Pi Mu Epsilon for officers in the Mathematical Association of America.

December 4: Initiation of new members. Committee on new books appointed. "The Development of the Use of Imaginary Numbers" by Beatrice Reynolds '17.

December 18: Report of the committee on new books. Christmas Party.

February 5, 1917: "Integral Equations" by Professor J. L. Jones.

February: Sleighing Party.

February 26: "Aggregates" by Johanna B. Guelzow '17; "The Brachistochrone" by Harold Hendershot '17.

March 19: "The Sphere in Four-Dimensional Space" by Mildred McKay '17.

March 31: Annual banquet.

April 16: "Non-Euclidean Geometry" by Florence Wilcox '17.

May 7: "A Discussion of Some New Curves" by Leon V. Foster '17.

May: Picnic.

October 8, 1917: Election of new faculty members. Reports of summer experiences by Professors Edward D. Roe and Decker.

October 30: "The Emblems of Pi Mu Epsilon" by H. A. DoBell '18; "The History and Ideals of Pi Mu Epsilon" by Professor Roe.

November 19: "Representation of Complex Loci" by Professor Roe.

December 10: "Some Applications of Mathematics to Modern Warfare" by Lawrence J. Blackmar '18.

February 11, 1918: "Applications of Mathematics to Economics" by Professor J. L. Jones; the scholarship grades of the undergraduate members of the chapter for the first semester were read by the scholarship committee and discussed, and the grades were put on record in the minutes.

March 11: "The Relation of Mathematics to Physics and Theoretical Chemistry" by Edna Howe '18; "The Nature of the Atom" by John J. Hopfield '16.

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[1] A large number of standard mathematical works have been bought by the chapter and are kept in the mathematical seminary. The annual dues are one dollar, yet as much as $30 has been spent for books in a single year.

[2] The students of this committee are those of highest rank in the senior class.

[3] After discussion the minimum scholarship standard for a new student member was set at "cum laude for general work and near to magna cum laude for mathematical work, but in the case of sophomores the latter was put distinctly above magna cum laude."

April 8: "The Impossibility of Squaring the Circle" by M. Gladys Medbery '18; "Geometric Interpretations of Hyperbolic Functions" by Helen H. Hale '18; "Some Ideas of Statistics" by Christabel A. Christie '18; reports of committees on the annual banquet and on new books.

April 29: "Relativity" by Charles D. Hurd '18; "Application of Mathematics to Every Day Life" by Doris A. Bourne '18; Review of Skinner's *Theory of Investments* by Ruth Taylor '18; election of officers for 1918–19.

May 4: Annual banquet.

"It is the duty of every member who presents a paper to copy it in a blank book provided for that purpose. We have three large volumes of such papers in our library—covering the time from 1903 to the present."

The Pentagram, University of Texas, Austin, Texas.

As the result of Professor Albert A. Bennett's initiative, The Pentagram was organized on October 18, 1916, "to assist in promoting the interests of mathematics among the students of the University of Texas." Papers and problems by both the students and staff of instruction constituted the program of biweekly meetings. "The heaviest duty fell of course on Professor Bennett, who suggested most of the problems as well as the subjects of most of the papers. His unwearying enthusiasm was contagious and the interest never flagged. The club counted over 30 members[1] and at the close of the term a banquet was held." [2]

The name of the club was chosen by the charter members for two reasons: "One was because the pentagram was intimately connected with the beginnings and early growth of mathematics; and the other was because of its significance to every Texan, Texas being called the Lone Star State, there being five letters in the word, Texas having fought under five flags and having served under five governments, and the State seal being a five-pointed star artistically wreathed and engraved."

Under its constitution the club regards as eligible to membership: (1) "those students who have completed an equivalent of five terms of work in mathematics with an average grade (in mathematics) of B; (2) all graduate students in mathematics and persons in the city teaching mathematics of high-school grade; and (3) those students who have completed mathematics 3 (first-year calculus) or its equivalent and are continuing the study of mathematics. Other persons may be elected to membership by a two thirds vote of all members present at a regular meeting and with the approval of the council (which consists of the president, vice-president, secretary-treasurer, student member, and faculty adviser). All members of the mathematical faculty teaching staff are invited to join. Other members are elected after an invitation by the council."

The Club uses printed cards of invitation (3×5 inches) with the following inscription: "The Pentagram invites to Membership . . . . This Club

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[1] Average attendance about 20.

[2] "Mathematics Clubs," *The Texas Mathematics Teachers' Bulletin*, April 1, 1917, Vol. 2, p. 33. Professor Bennett is now captain in the coast artillery.

meets twice a month to consider mathematical topics of interest to the undergraduate. Dues, one dollar, payable in the winter term. Address acceptances to . . . . Secretary."

A facsimile of The Pentagram's certificate of membership is given on the opposite page. The original is printed on cardboard.

Officers, 1917–18: President, Charles E. Normand '19; vice-president, Ruth Stocking '18; secretary-treasurer, Dr. Goldie P. Horton, instructor in pure mathematics; faculty member of the council, Professor Thomas McN. Simpson; student member of council, Lloyd Kerr '18.

The following programs have been given:

October 18, 1916: Organization meeting; short talk by Professor Milton B. Porter on "What a Mathematics Club can Accomplish."

November 1: "History and Significance of the Name Pentagram," by Clarence E. Brand '17; "The History of Algebraic Notation" by Henry H. Hammer '18.

November 15: "Intuition in Mathematics" by Professor Porter.

December 6: "Some Propositions in Number Theory" by Amelia K. Benson '17.

January 10, 1917: "Map-making" by Rufus R. Rush '16.

January 24: "Trigonometrical Series" by Hyman J. Ettlinger, instructor [at this time] in applied mathematics.

February 7: "The Computation of Logarithm Tables" by Thomas B. McCarter '16.

February 21: "Point Sets" by Dr. Goldie P. Horton.

March 21: Roll-call program, each member responding to roll-call with something of mathematical interest: an incident, quotation, problem, or the like.

April 4: "Some mathematical Instruments: Planimeter, Slide-Rule, Inversors, Sextant" by Llewellyn Notley '17.

April 19: "Some mathematical Problems: The Pythagorean Relation, Some Number Relations, Some Analysis Situs Relations" by Professor A. A. Bennett.

May 3: "Philosophical and Mathematical Views of Infinity" by Clarence E. Brand '16.

October 17, 1917: "Why We Study Mathematics" by Professor M. B. Porter.

October 31: "Newton, Man and Mathematician" by Professor T. McN. Simpson.

November 14: "The Dimensionality of Space as dependent on the Choice of the Elements" by Lloyd Kerr '18.

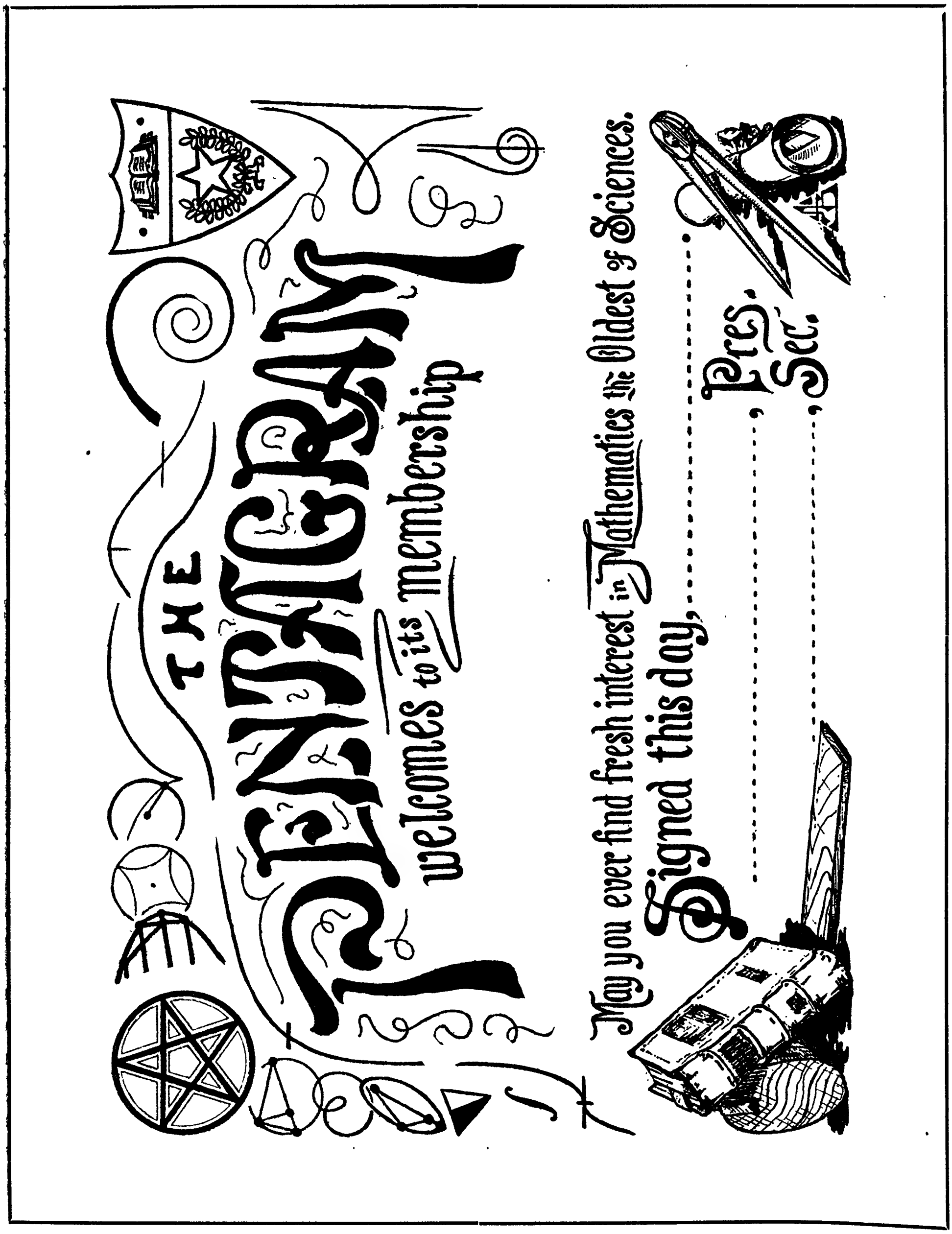
November 28: "The Equation of Life" by Ruth Stocking '18.

December 1: Open meeting, "How we aim the Big Gun"[1] by Professor A. A. Bennett, Captain C. A. R. C.

December 12: "Dialling" by Professor Porter; "The Equation of Time" by Professor Simpson.

January 16, 1918: "The Mathematical Theory of the Chemical Balance" by Charles E. Normand '16.

[1] Cf.. "Remarks on mathematics of artillery," by A. A. Bennett, in *The Texas Mathematics Teachers' Bulletin*, May, 1918, pp. 9–16.


THE
PENTAGRAM
welcomes to its membership
May you ever find fresh interest in Mathematics the Oldest of Sciences.
Signed this day,
Pres.
Sec.

January 30: "The Theory of Flight" by H. J. Ettlinger, adjunct professor of applied mathematics.
February 13: "Continued Fractions" by Bertha Potash '18.
February 27: "Railway Transition Curves" by Marvin Nichols '18.
March 20: "The Logarithm as a Direct Function" by Dr. Paul M. Batchelder, instructor in applied mathematics.
April 4: "The Use of the Mean-Value Theorem in Finding Linear and Curvilinear Asymptotes" by Robert G. Wulff '20; "The Derivative of Surface Area" by Essie Lipscomb '19.
April 18: "Elementary Properties of Groups" by Lula Whitehouse '19. [Miss Whitehouse was absent on account of illness, and the topic was presented by Dr. Batchelder.]
May 2: "Long Division as Taught in the Middle Ages," by Clara Seymour '18; "The Development of Arabic Numerals" by Phillis Henry '16.
May 16: "Fitting Curves to Biometric Data" by Edward L. Dodd, adjunct professor of actuarial mathematics.

TOPICS FOR CLUB PROGRAMS.

11. Euler Integrals and Euler's Spiral—Sometimes Called Fresnel Integrals and the Clothoïde or Cornu's Spiral.[1]

The integrals in question are

$$\int_0^\infty \sin x^2 dx = \tfrac{1}{2}\int_0^\infty \frac{\sin v}{\sqrt{v}}\, dv$$

and

$$\int_0^\infty \cos x^2 dx = \tfrac{1}{2}\int_0^\infty \frac{\cos v}{\sqrt{v}}\, dv;$$

and the equations of the spiral are

$$x = K\int_0^s \cos s^2 ds, \qquad y = K\int_0^s \sin s^2 ds.$$

These were considered by Euler at least as early as 1743 in a problem of his celebrated work on the calculus of variations: *Methodus inveniendi lineas curvas maxime minimive proprietate gaudentes*[2]. . . . The discussion of the problem was somewhat as follows:[3] Consider an elastic spring freely coiled up in the form

[1] For historical sketches and properties of this curve see F. Gomes Teixeira, *Traité des courbes spéciales remarquables planes et gauches*, tome 2, Coïmbre, 1909, pp. 102–107; G. Loria, *Spezielle algebraische und transzendente ebene Kurven* . . . 2. Auflage. Band 2, Leipzig, Teubner, 1911, pp. 70–73.

[2] Lausannæ & Genevæ, MDCCXLIV, pp. 276–7. *Cf. Verzeichnis der Schriften Leonhard Eulers* bearbeitet von G. Eneström. Erste Lieferung, Leipzig, Teubner, 1910, p. 16. See also P. S. Laplace, "Sur la réduction des fonctions en tables," *Journal de l'École Polytechnique*, tome 8, cahier 15, 1809, pp. 250–251.

[3] As Euler's writings to which reference must be made are very scarce, it would seem best to give more details than otherwise would be necessary.

of a spiral. Let us suppose that the interior extremity is fixed and that the spring can be developed into a horizontal position by a weight p suspended at the other extremity. Under these conditions the action of the weight on an element of the spring placed at a distance s from the extremity is ps; and the elasticity of the element preserves it in equilibrium. This elasticity is "the reciprocal of the osculating radius of the spring in its unrestricted state." Setting r equal to this radius with respect to the part s of the spring, taken from its exterior extremity, we have $ps = Ek^2/r$, Ek^2 being a constant depending upon the elasticity of the spring. Let $Ek^2/p = a^2$. For the spring in its natural position we will thus have (1) $rs = a^2$, "quæ est æquatio naturam curvæ . . . complectens." Whence, introducing rectangular coördinates,

$$\text{“}x = \int ds \sin \cdot \frac{ss}{2aa}, \qquad \& \qquad y = \int ds \cos \cdot \frac{ss}{2aa}.\text{”}$$

Euler then remarks: "From the fact that the osculating radius steadily decreases the longer the arc taken it is evident that the curve is not produced to infinity. The curve therefore will be in the nature of a spiral so that when the spiral is completed it is rolled up, as it were, in a certain point which may be called the center. The point seems to be very difficult to discover by this construction. Therefore we must admit that analysis will make no small gain should anyone find a method whereby, approximately at least, the value of this integral would be determined in the case that s is taken to be infinite. This problem does not seem to be unworthy the best strength of geometers."

Euler then expresses x and y as converging series in s for approximating to these values but adds that if s be made infinite the values of x and y cannot be determined in this way. He sets, finally, $s^2/2a^2 = v$, obtaining

$$(2) \qquad x = \frac{a}{\sqrt{2}}\int \frac{\sin vdv}{\sqrt{v}}, \qquad y = \frac{a}{\sqrt{2}}\int \frac{\cos vdv}{\sqrt{v}},$$

and shows that approximate values of x and y could be found by considering the intervals $v = 0$ to $v = \pi$, $v = \pi$ to $v = 2\pi$, $v = 2\pi$ to $v = 3\pi$, $\cdots$, and certain converging series "requiring long operations and very tedious calculations to evaluate them."

Thirty-eight years later, however, Euler had solved the problem completely. This solution is to be found in one of the last papers which he wrote (he died in 1785). It is entitled "De valoribus integralium a terminus variabilis $x = 0$ usque ad $x = \infty$ extensorum" and was presented to the Academy at Petrograd on April 30, 1781.[1] Again he considers the curve the radius of curvature at each point of which is inversely proportional to the arc of the curve and is led, as before, to the equation $rs = a^2$ from which, Euler says, it would not be difficult to discuss the form of such a curve. He refers to the infinite number of whorls

[1] *Leonhardi Euleri Institutionum calculi integralis*, Vol. 4, Petropoli, 1794, pp. 337–345; editio tertia, Petropoli, 1845, pp. 337–345. German edition: *Vollständige Anleitung zur Integralrechnung* . . . übersetzt von J. Salomon, Wien, 1830, pp. 321–328.

(*infinitas spiras*) about a fixed point "which may be called the pole of this curve." He then proceeds to determine the coördinates of this pole. Introducing the angle of contingence (v) he is led to the form (3)[1] $s^2 = 2a^2v$, from which he finds readily the equations (2). Concerning the evaluation of these integrals for the coördinates of the pole he remarks that he had "recently found by a happy chance and in an exceedingly peculiar manner" that

$$x = \frac{a}{\sqrt{2}}\,\frac{\sqrt{\pi}}{2}, \qquad y = \frac{a}{\sqrt{2}}\,\frac{\sqrt{\pi}}{2}.$$

Euler's method of evaluation is based upon that of $\int_0^\infty x^{n-1}e^{-x}dx = \Gamma(n)$ which, in turn, leads to

$$\int_0^\infty x^{n-1}e^{-px}\cdot\cos qxdx = \frac{\Gamma(n)\cos n\alpha}{r^n}, \qquad \int_0^\infty x^{n-1}e^{-px}\sin qxdx = \frac{\Gamma(n)\sin n\alpha}{r^n},\text{[2]}$$

where $p = r\cos\alpha$, $q = r\sin\alpha$. Euler then sets $q = 1$, $p = 0$, $n = \frac{1}{2}$ and finds the required result. He notes also:

$$\int_0^\infty \frac{e^{-px}\cos qxdx}{\sqrt{x}} = \sqrt{\frac{\pi}{r}}\cos\frac{\alpha}{2},\text{[3]} \qquad \int_0^\infty e^{-px}\sin\frac{qx}{\sqrt{x}}dx = \sqrt{\frac{\pi}{r}}\sin\frac{\alpha}{2},$$

[1] A curve with an equation of this same form was named by K. C. F. Krause "parabola originaris longitudinaris" (*Nova theoria linearum curvarum*, Monachii, anno MDCCCXXXV, p. 79). The reason for the name is clear. None of Krause's discussion of the curve is worth referring to; Loria's mention of it seems misleading in part.

[2] These were the integrals investigated by Poisson in his "Mémoire sur les intégrales définies," *Journal de l'École Polytechnique*, Paris, tome 9, cahier 16, pp. 215–246, 1813, especially pp. 215–219. See also Poisson, *Nouveau Bulletin des Sciences* par la société philomathique de Paris, 3 année, 1811, tome 2, p. 251; Lacroix, *Traité du calc. diff. et integ.*, tome 3, 2e éd., Paris, 1819, pp. 486–490; Grunert, Crelle's *Journal*, Band 8, 1832, pp. 146–151; J. Plana, "Sur trois intégrales définies," *Acad. Sci. Mém.*, Bruxelles, Vol. 10, 1837; A. De Morgan, *Differential and Integral Calculus*, London, 1842, p. 630; Schlömilch, "Ueber einige Integrale welche goniometrische Funktionen involvieren," *Arch. d. Math. u. Phys.*, 1845, Band 6, pp. 200 ff.; E. F. A. Minding, "Ueber $\int_0^\infty \sin x^m\cdot x^{-n}dx$ wo $m \geqq n > 0$," *Arch. d. Math. u. Phys.*, Band 30, 1858, pp. 171–183; W. Walton, "On a Pair of Definite Integrals," *Quarterly Journal of Mathematics*, 1871, Vol. 11, pp. 373–375; J. W. L. Glaisher, "On certain Definite Integrals," *Report of the British Association for the Advancement of Science*, 1871, London, 1872, Report, pp. 10–12.

[3] For $p = 1$ and $q = 0$ we find, on substituting t^2 for x, that $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2$, an integral (of great importance in many parts of applied mathematics) definitely evaluated by Laplace in a memoir published in 1781 (*Mém. Acad. Paris*, 1778; *Œuvres*, Paris, Vol. 9, "Mémoire sur les probabilités"). The Euler integrals, and spiral in connection with the elastic spring, of these notes were also discussed by Laplace in "Sur la réduction des fonctions en tables," *Journal de l'École Polytechnique*, tome 8, cahier 15, pp. 229–265, 1809. A slip made by Mascheroni is here corrected; in his *Adnotationes ad calculum integralen Euleri* ec. (Ticini, M.DCCXC; [also L. Euler, *Opera Omnia*, series 1, Vol. 12, Leipzig, Teubner, 1914]) Mascheroni has a note on Cap. V, Sect. I, Vol. 1, entitled: "De integratione Formularum $x^ndx \sin x$, $x^ndx \cos x$," pp. 38–57 [pp. 454–471]. In the special case $n = -\frac{1}{2}$ he gives (p. 53) $\sqrt{2\pi}$, instead of $\sqrt{\pi/2}$.

and the well-known result

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2},^{1}$$

the evaluation of which "up to the present has defeated all known artifices of calculation."

While Euler was not the first to discuss some of the problems mentioned above, he was the first to publish any results of importance in connection with them. In 1694 (at the close of his memoir "Curvata Laminæ Elasticæ" [2]) James Bernoulli (1654–1705) mentions among problems which might be worked out: To find the curvature a lamina should have in order to be straightened out horizontally by a weight suspended at one end. In the posthumous edition of his works, in 1744, there is a reference from this passage to a fragment entitled "Invenire Curvarum, cujus curvedo in singulis punctis est proportionalis longitudini arcus; id est, quæ ab appenso pondere flectitur in rectam." [3] Here there is the equation $rs = a^2$ and a construction for points on the curve; but there is not the slightest indication that Bernoulli had any conception, such as Euler had, of the real form of the curve.

In the nineteenth century the Euler integrals and spiral became of special interest through discoveries of Fresnel (1788–1827) in connection with the diffraction of light. By making certain assumptions and approximations Fresnel deduced (in 1818)[4] for the intensity of the illumination at any point of a diffraction pattern

$$I = [\int \cos \tfrac{1}{2}\pi v^2 dv]^2 + [\int \sin \tfrac{1}{2}\pi v^2 dv]^2.$$

For this reason the integrals which here occur are often called Fresnel's integrals. From what has been indicated above the value of each, for the limits $v = 0$ to $v = \infty$, is $\frac{1}{2}$.

In his Note of 1818, Fresnel gave a table of the values of $A = \int_0^v \cos \frac{1}{2}\pi v^2 dv$ and $B = \int_0^v \sin \frac{1}{2}\pi v^2 dv$ for values of v (differing by 0.1) from 0.1 to 5.1 (later extended to 5.5), to 4 places of decimals. This table is reproduced by E. Verdet in his *Leçons d'Optique physique* (1869).[5] More detailed tables (to the nearest hundredth) were calculated by Abria.[6] Modifications of Fresnel's method of evaluating A and B, and criticisms and corrections of his results were given by

[1] See G. H. Hardy's discussion of eleven proofs of this in *Mathematical Gazette*, July, 1916, Vol. 8, pp. 301–303; see also Vol. 5, pp. 98–103, 1909, and Vol. 6, pp. 223–4, 1912.

[2] *Acta Eruditorum*, 1694, p. 276; *Opera*, Genevæ, M.DCCXLIV, tome 1, p. 600.

[3] J. Bernoulli, *Opera*, tome 2, pp. 1084–1086.

[4] *Œuvres complètes d'Augustin Fresnel*, tome 1, Paris, MDCCCLXVI, pp. 176–181; see also pp. 198–9, 315–352.

[5] Publiées par A. Levistal, tome 1, Paris, 1869; *cf.* pp. 343 ff. German edition by K. Exner, Band 1, Braunschweig, Vieweg, 1881, pp. 236–309; the table is given on p. 240 and on p. 241 A and B are graphed as oscillating curves.

[6] "Sur la diffraction de la lumière," *Journal de mathématique pures et appliquées*, 1839, tome 4, pp. 248–260.

Knochenhauer,[1] Cauchy,[2] and Gilbert.[3] Knochenhauer's method was good for small values and Cauchy's for large values of v. Peters gives (*l. c.*, page 48) a table similar to that of Fresnel but slightly more extensive, in that intervals of 0.01 are considered from $v = 0.01$ to $v = 0.10$, and intervals of 0.05 from $v = 0.10$ to $v = 1.00$. This is the table most frequently quoted;[4] but it should be remembered that, except for some corrections and slight additions, it is identical with Fresnel's given some forty years earlier.

The tables of Ignatowsky[5] give (among other things), from $v = 0.0$ to $v = 8.5$ (for intervals 0.1), the values of A and B to four places of decimals, and of $\log A$ and $\log B$ to six places. Lommel published[6] a table for

$$A' = \int_0^z \frac{\cos z}{\sqrt{z}}\, dz, \qquad B' = \int_0^z \frac{\sin z}{\sqrt{z}}\, dz$$

from $z = 0$ to $z = 50$ at unit intervals. From $z = 0.0$ to $z = 50.0$ at intervals of 0.1, and to four places, it is printed in Jahnke and Emde's tables.[4]

In 1874 Cornu plotted Euler's spiral accurately[7] by means of Peters's table. (Euler had already given half the spiral.) In a sketch of Cornu, Poincaré has written as follows:[8] "Aussi, quand il aborda l'étude de la diffraction, il eut bientôt fait de remplacer cette multitude rebarbative de formule herissées d'intégrales par une figure unique et harmonieuse, que l'oeil suit avec plaisir et où l'esprit se dirige sans effort. Tout le monde aujourd'hui pour prévoir l'effet d'un écrau quelconque sur un faisceau lumineux, se sert de la *spirale de Cornu.*" The expression "Cornu's spiral" was used by Preston,[9] Wood,[10] and others before Poincaré[11] employed it in the sketch quoted; but the term is evidently highly inappropriate in the light of Euler's discoveries set forth above.

Besides the works on physics to which reference has been made already in

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[1] K. W. Knochenhauer (1) "Ueber die Oerter der Maxima und Minima des gebeugten Lichtes nach den Fresnel'schen Beobachtungen," *Annalen der Physik und Chemie*, Leipzig, 1837, Band 41, pp. 103–110; (2) "Ueber eine besondere Klasse von Beugungserscheinungen," *idem*, 1838, Band 43, pp. 286–292; (3) *Die Undulationstheorie des Lichtes*, Berlin, 1839, p. 36f.

[2] A. Cauchy, *Comptes Rendus*, Paris, 1842, tome 15, "Note sur la diffraction de la lumière," pp. 554–6; "Addition à la Note sur la diffraction de la lumière," pp. 573–578.

[3] Ph. Gilbert, "Recherches analytiques sur la diffraction de la lumière" (mémoire présenté le 3 août, 1861). *Mémoires couronnés . . . acad. roy. d. sc. . . . de Belgique*, 1863, tome 31, pp. 1–52.

[4] For example: E. Jahnke und F. Emde, *Funktionentafeln mit Formeln und Kurven*, Leipzig, 1909, pp. 23–26; R. W. Wood, *Physical Optics*, New York, Macmillan, 1905, p. 198.

[5] W. v. Ignatowsky, *Annalen der Physik*, 1907, Band 328, pp. 895–898.

[6] E. Lommel, *Abh. Münch. Ak.*, Band 15, 2. Abtheilung, 1880, p. 230.

[7] A. Cornu, (1) "Méthode nouvelle pour la discussion des problèmes de diffraction dans le cas d'une onde cylindrique," *Journal de physique théorique et appliquée*, Paris, 1874, pp. 5–15; (2) "Études sur la diffraction; méthode géométrique pour la discussion des problèmes de diffraction," *Comptes Rendus*, tome 78, 1874, pp. 113–117.

[8] H. Poincaré, *Savants et écrivains*, Paris, Flammarion, [1910], p. 106.

[9] T. Preston, *The Theory of Light*, 2d edition, London, Macmillan, 1895, p. 274 [4th ed., 1912, p. 291].

[10] R. W. Wood, *Physical Optics*, New York, Macmillan, 1905, p. 158 and on the plate at the end of the volume.

[11] Loria seems to be in error here (*l. c.*, p. 71).

connection with our topic we may note those by Drude,[1] Pockels,[2] and Chwolson.[3]

Lommel seems to have been the first[4] to observe the connection between A', B' and Bessel's functions:

$$\int \frac{\cos z}{\sqrt{z}} dz = \sqrt{\frac{\pi}{2}} \int J_{-1/2}(z) dz; \qquad \int \frac{\sin z}{\sqrt{z}} dz = \sqrt{\frac{\pi}{2}} \int J_{1/2}(z) dz.$$

Amongst the many methods for evaluating A and B reference may be given: (1) to the method of Godefroy[5] who, by a slight modification of Laurent's discussion,[6] starts with $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ and avoids all use of imaginaries; (2) to a paper by Cayley[7] discussing several interesting points which later occupied the attention of Glaisher,[8] Jamet,[9] and Humbert;[10] (3) to other methods illustrated by Pierpont[11] and d'Adhémar;[12] and (4) to Noumoff's "Interprétation géométriques des intégrales de Fresnel" [13] in which the projection of helices generated by a certain parabola rolling on a right circular cylinder are curves the sum of the areas under which give the required values. The latter part of the article contains a description of a mechanical integrator for calculating the integrals A and B.

In recent times Cesàro has given to Euler's Spiral the name Clothoïde[14] and exhibited a number of remarkable properties of the curve.[15] Among these the

[1] P. Drude, *Lehrbuch der Optik*, Leipzig, Hirzel, 1900, pp. 174–187; English translation by C. R. Mann and R. A. Millikan, London, Longmans, 1902, pp. 188–202.

[2] F. Pockels, pp. 1051–1064 of *Handbuch der Physik*, Zweite Auflage herausgegeben von A. Winkelmann, Band 6: Optik, Leipzig, Barth, 1906.

[3] O. D. Chowlson, *Traité de physique*, ouvrage traduit sur les éditions russe et allemande, tome 2, Paris, Hermann, 1909, pp. 652–656.

[4] E. Lommel, "Ueber die Anwendung der Bessel'schen Funktionen in der Theorie der Beugung," *Zeitschrift für Mathematik und Physik*, Leipzig, 1870, pp. 141–169. *Cf.* A. Gray and G. B. Mathews, *Treatise on Bessel Functions*, London, 1895, p. 41.

[5] A. Godefroy, "Sur les intégrales de Fresnel," *Nouvelles annales de mathématiques*, 1898 (3), Vol. 17, pp. 205–206.

[6] H. Laurent, *Traité d'analyse*, tome 3, Paris, 1888, p. 137.

[7] A. Cayley, "Note on the Integrals $\int_0^\infty \cos x^2\, dx$ and $\int_0^\infty \sin x^2\, dx$," *Quarterly Journal of Mathematics*, Vol. 12, 1873, pp. 118–126; also *Collected Mathematical Papers*, Cambridge, Vol. 9, 1896, pp. 56–63.

[8] J. W. L. Glaisher, "On the Integrals $\int_0^\infty \sin x^2 dx$ and $\int_0^\infty \cos x^2 dx$, *Quarterly Journal*, 1875, Vol. 13, pp. 343–349.

[9] V. Jamet, "Sur les Intégrales de Fresnel," *Nouvelles annales de mathématiques*, 1896 (3), tome 15, pp. 372–376.

[10] G. Humbert, *Cours d'analyse*, Paris, Gauthier-Villars, tome 1, 1903, pp. 307–08.

[11] J. Pierpont, *Lectures on the Theory of Functions of Real Variables*, Boston, Ginn, Vol. 1, 1905, pp. 499–500.

[12] R. d'Adhémar, *Exercises et Leçons d'Analyse*, Paris, Gauthier-Villars, 1908, pp. 23–25.

[13] *Journal de physique théorique et appliquée*, Paris, 1847 (3), tome 6, pp. 281–289.

[14] From the Greek word meaning to twist by spinning—since the curve spins or turns about its asymptotic points.

[15] E. Cesàro, (1) "Les lignes barycentriques," *Nouvelles annales de mathématiques*, 1886 (3), tome 5, pp. 511–520; (2) "Sur la courbe représentative des phenomènes de diffraction," *Comptes Rendus*, 1890, tome 110, pp. 1119–1122; (3) *Nouvelles annales de mathématiques*, 1905, 4e série, tome 5, pp. 570–573. See also *L'Intermédiaire des mathématiciens*, 1916, tome 23, pp. 187–189.

following may be mentioned: (*a*) The clothoïde is the only curve enjoying the property that the center of gravity of any arc is a center of similitude of the circles osculating the extremities of the arc; (*b*) when a clothoïde rolls on a straight line, the locus of the center of curvature corresponding to the point of contact is an equilateral hyperbola asymptotic to the line considered.

Wieleitner discussed "Die Parallelkurve der Klothoide."[1] For different values of m the intrinsic equation $rs^m = a^2$ represents a clothoïde, a logarithmic spiral, a circle, the involute of a circle, and a straight line.

NOTES AND NEWS.

Edited by D. A. Rothrock, Indiana University, Bloomington, Indiana.

Assistant Professor J. L. Coolidge, of Harvard University, has been promoted to an associate professorship of mathematics.

Dr. Anna J. Pell, of Mount Holyoke College, will fill the vacancy at Bryn Mawr College caused by the resignation of Dr. Olive C. Hazlett.

At Leland Stanford Junior University Dr. H. C. Moreno has been promoted from an assistant to an associate professorship of applied mathematics.

Dr. Flora E. Le Stourgeon, of the Liggett School, Detroit, Mich., has accepted an instructorship in mathematics at Mount Holyoke College.

At the University of Illinois Mr. R. F. Borden, a graduate student, has been appointed instructor in mathematics, and Mr. Joseph Rosenbach, of the University of New Mexico, has been appointed assistant in mathematics.

Professor R. E. Moritz, of the University of Washington, has in *School and Society*, April 27, 1918, a Reply to Ernest C. Moore's paper "Does the study of mathematics train the mind specifically or universally?", which was published in the same periodical on October 27, 1917.

A fund of 150,000 crowns has been donated by Mr. C. Hennevig as a memorial to N. H. Abel, the income from which is to be used to encourage mathematical research in Norway.

Professor E. V. Huntington, President of the Mathematical Association of America, has taken leave of absence from Harvard University and with the rank of major in the national army is assigned to statistical study under the chief of staff with residence in Washington.

Professor W. D. Cairns, of Oberlin College, Secretary of the Mathematical Association of America, is giving courses in the fundamental concepts of algebra and geometry and in the teaching of mathematics, and a graduate course in the calculus of variations in the summer session of Ohio State University.

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[1] *Archiv der Mathematik und Physik*, 1907 (3) Band 11, pp. 373–375.

Three men received the doctorate in mathematics at the University of Chicago on June 11, 1918, W. G. SIMON, E. P. LANE, and I. A. BARNETT. Announcement was made last month of the appointments for next year of Mr. SIMON and Mr. LANE. It may now be added that Mr. BARNETT has been appointed to an instructorship in mathematics at Washington University, St. Louis.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY. Summer Session, beginning June 18, 1918.—By Professor C. L. E. MOORE: Solid analytic geometry; Calculus.—By Professor L. M. PASSANO: Integral calculus.—By Professor H. B. PHILLIPS: Analytic geometry; Elementary calculus.—By Professor N. R. GEORGE: Solid geometry; Elementary algebra.

The Summer Session of the UNIVERSITY OF MICHIGAN will extend from July 1 to August 23, 1918. The following mathematical courses will be offered: Elementary and advanced calculus, Differential equations, Advanced algebra, Projective geometry, History of mathematics, Mathematical finance and insurance, and Theory of potential. The instructors will be Professors BEMAN, FORD, KARPINSKI and BRADSHAW, and Dr. ALLEN, Dr. NELSON, Mr. ROUSE and Mr. COE.

INDIANA UNIVERSITY. The mathematical courses announced for the year 1918–1919 are as follows: By Professor S. C. DAVISSON: Differential equations, three hours; Theory of investment, three hours; Non-Euclidean geometry, two hours.—By Professor D. A. ROTHROCK: Advanced analytic geometry, three hours; Theory of equations and determinants, two hours; Advanced calculus, three hours; History of mathematics, two hours.—By Professor U. S. HANNA: Analytic mechanics, three hours; Higher algebra, two hours.

UNIVERSITY OF ILLINOIS.—The following courses in mathematics are announced for the year 1918–1919, all courses being three hours during the year except as indicated.—By Professor E. J. TOWNSEND: Functions of a complex variable; Differential equations; Advanced calculus.—By Professor G. A. MILLER: Continuous and finite groups; Theory of equations and determinants (first semester).—By Professor J. STEBBINS: Least squares (two hours, first semester).—By Professor J. B. SHAW: Fundamental functions (first semester); Functional transformations (second semester).—By Professor C. H. SISAM: Invariants and higher plane curves; Solid analytic geometry (second semester).—By Professor R. D. CARMICHAEL: Elliptic functions.—By Professor A. EMCH: Projective geometry.—By Professor A. R. CRATHORNE: Actuarial theory.—By Dr. E. B. LYTLE: History of mathematics (two hours, second semester).—By Dr. G. E. WAHLIN: Theory of numbers.

The April *Bulletin of the American Mathematical Society* reports the following university and college teachers of mathematics as having entered the military service: Mr. J. E. DAVIS, of Pennsylvania State College, Mr. J. J. TANZOLA, of the U. S. Naval Academy, and Mr. A. L. WECHSLER, of Columbia University,

have joined the national army; Mr. H. M. TERRILL, of Columbia University, has been made ensign in the naval reserve; Professor V. H. WELLS, of the University of Pittsburgh, has been commissioned lieutenant in the science and research division of the signal reserve corps; and Mr. FREDRICK WOOD, of the University of Wisconsin, has been appointed lieutenant in the field artillery.

Assistant Professor E. S. SMITH, of the University of Cincinnati, has been appointed acting commandant in addition to his duties in the department of mathematics.

At Wellesley College, Miss ROXANA H. VIVIAN will be professor in the department of mathematics; she has also been appointed director of the department of hygiene. Miss MARY F. CURTIS, Ph.D., of Radcliffe, and this year instructor at Woman's College of Western Reserve University, has been appointed instructor in mathematics at Wellesley. For the year 1918–1919, Professor CLARA E. SMITH will exchange professorships with Associate, Professor FLORENCE P. LEWIS, of Goucher College.

The eighth regular meeting of the Association of Mathematics Teachers of New Jersey was held at Asbury Park on May 18, under the presidency of Professor R. H. RIVENBURG, of Peddie Institute. In addition to the addresses of welcome the program as announced was as follows: "Application of efficiency principles to a recitation in algebra," by Dr. F. DURELL, of Lawrenceville School; "The centers of circles tangent to three out of four given lines," by Dr. C. R. MACINNES, of Princeton University; "Report of the committee on first year mathematics," by Mr. A. W. BELCHER, of Newark High School. The meeting closed in the afternoon with the president's address on "In darkest algebra, analysis of problems, and locus of a point," followed by the annual business meeting and election of officers.

The thirtieth educational conference of the academies and high schools in relations with the University of Chicago was held at the University on Thursday and Friday, May 9 and 10, 1918. At the Thursday Conference for principals and superintendents, Professor H. O. RUGG, of the School of Education, spoke on "Scientific method in the reconstruction of high school subjects," taking for his chief illustration the courses in mathematics, while emphasizing the applicability to all courses in the high school curricula. In the afternoon the curriculum of the junior high school was discussed by several speakers, including Mr. J. R. CLARK of the Parker High School, Chicago, who presented the case for mathematics. On Friday afternoon, thirteen departmental conferences were held, all of which were attended by large numbers of teachers. In the mathematical conference, Professor Rugg spoke in greater detail as to proposed reorganization of mathematical work in the high school, with special reference to the first year. His topic was "Ninth grade mathematics on trial," and he set forth in convincing manner the plan and method recently published by the University of Chicago Press in a Monograph by H. O. RUGG and J. R. CLARK, entitled "Scientific Method in the Reconstruction of Ninth-Grade Mathematics."

The *Bollettino di Bibliografia e Storia delle Scienze Mathematiche,* which has for some years been edited by Professor GINO LORIA, of the University of Genoa, is about to begin a new series. This series will be published by the scientific publishing house of D. CAPOZZI, of Palermo. The journal has filled a decided need on the part of the mathematicians interested in the history and bibliography of their subject. The change of publisher will enable Professor Loria to give even more assistance to students than he has been able to give in the old series.

The United States Bureau of Education has recently issued a Union List of Mathematical Periodicals prepared by Professor DAVID EUGENE SMITH and Dr. CAROLINE EUSTIS SEELY. This list contains the leading mathematical periodicals needed by research students and to be found in a number of the larger libraries in various parts of the country. Copies may be secured by addressing the United States Commissioner of Education, Washington, D. C.

The second number of Volume 3 of the Texas Mathematics Teachers' Bulletin is before us. The Bulletin is edited by Associate Professor CALHOUN and Adjunct Professor ETTLINGER, of the University of Texas, and is open to the teachers of mathematics in Texas for expression of their views, the editors, however, assuming no responsibility for statements or opinions not written by themselves. The content of the present number is intended to present elementary matter of interest to the teachers of mathematics in the high schools of the state. Among the topics discussed are "Mathematics in everyday life," by T. M. SIMPSON; "High School Algebra," by M. B. PORTER; "The Length of the Circle," by J. W. CALHOUN; "A new course in mathematics at the University of Texas (Introduction to statistics)," by E. L. DODD; "The factoring of polynomials," by P. M. BATCHELDER; and a reprint from *School and Society* of Professor D. E. SMITH'S "A glimpse at early colonial algebra." The mathematical department of the University of Texas has provided, during the past two summers, a special four-day Mathematics Teachers' Conference held at the opening of the Summer Session of the University. Owing to the disturbed conditions of the country it has not been definitely decided to hold the Conference during the coming summer term.

Professor R. C. ARCHIBALD of Providence, R. I., will be glad to learn if any library in America contains B. E. Cousinery's *Géométrie élémentaire du compas* (Paris, 1851), or where the volume may be purchased.

THIRD SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The third summer meeting of the Association will be held by invitation of Dartmouth College at Hanover, New Hampshire, on Friday and Saturday, September 6–7, 1918, in conjunction with, and following, the summer meeting of the American Mathematical Society. A joint dinner will be arranged for Thursday evening, September 5, and at a joint session on Friday morning the

subject of the mathematics of warfare is to be treated by men now actively engaged in government service.

During the sessions of the Association on Friday and Saturday, Professor FLORIAN CAJORI, of the University of California, will deliver his address, as retiring president, on "Plans for a History of Mathematics in the Nineteenth Century"; Professor W. F. OSGOOD, of Harvard University, will speak "On the Mathematical Formulation of Physical Concepts and Laws as treated in the Calculus"; and Professor F. L. KENNEDY, of Harvard University, will give a paper on "Some Experiments in the Teaching of Descriptive Geometry," the discussion being led by Dean O. E. RANDALL, of Brown University. Other features of the Association's program will be announced later.

For a session on Friday members are invited to submit papers on topics of their own choosing. Abstracts of such papers in a form suitable for publication in the Secretary's report of the meeting should be sent to Professor R. C. ARCHIBALD, Brown University, Chairman of the Program Committee, not later than August first, in order to be approved by the committee in time for publication in the printed program; authors will please state the time necessary for reading their papers. No other announcement will be made until the program is mailed to members about the middle of August.

The Committee on Arrangements, Professor J. W. YOUNG, Chairman, announces that Dartmouth College will open one of its dormitories for the accommodation of attending members. A separate entrance, or at least a separate floor, will be provided for ladies or married couples. Meals will be furnished under the auspices of the college at reasonable rates. The rates for the occupancy of dormitory rooms will probably be one dollar per day per person. Persons desiring to stay over Sunday and Monday for the purpose of making excursions into the neighboring hills and mountains can probably be accommodated.

Hanover is on the Passumpsic division of the Boston and Maine Railroad, five miles north of White River Junction, Vt. The railroad station is known as "Hanover, N. H.–Norwich, Vt." Junction points for trains from the west are White River Junction, Vt. (Grand Trunk), Greenfield, Mass., and Springfield, Mass. The White Mountain Express is a daily through train from New York, which passes through Hanover. An announcement of details regarding train connections will be made in connection with the program; it should be said here that the trains from the east, south and west reach Hanover between 2 and 5 in the afternoon, no morning trains arriving in Hanover in time for a morning session.

It is very important that members keep the Secretary informed as to their summer addresses, so that the program may be received promptly, as it will be mailed under second-class postage.

VOLUME XXV SEPTEMBER, 1918 NUMBER 7

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DEFINITIONS OF THE DISCRIMINANT OF A RATIONAL INTEGRAL FUNCTION OF ONE VARIABLE.

By G. A. MILLER, University of Illinois.

Many students meet the term *discriminant* for the first time in the discussion of the solutions of the quadratic equation $ax^2 + bx + c = 0$. It is here commonly defined as the expression $b^2 - 4ac$ and it is usually noted that the vanishing of this expression is a necessary and sufficient condition that the given quadratic equation has equal roots. As the vanishing of this expression implies the vanishing of many other expressions and vice versa, it is clear that the given expression is by no means the only one whose vanishing is a necessary and sufficient condition that the quadratic equation under consideration has equal roots. Most of these other expressions might be said to be less simple but they include $4ac - b^2$ which appears to be just as simple as the expression defined above as the discriminant.

We emphasize these evident facts because the mathematical student is apt to meet in his more advanced work definitions of the term discriminant which are either vague or contradictory with his earliest use of the term. As an instance of the latter type of definitions we may cite the one given on page 96 of the *Theory of Equations* by F. Cajori, 1912. The discriminant of $f(x)$ is here defined as the resultant of $f(x)$ and $f'(x)$. If we find this resultant, when

$$f(x) = ax^2 + bx + c,$$

in accord with the illustration of the term resultant found on the same page, we obtain the following determinant:

$$\begin{vmatrix} a & b & c \\ 0 & 2a & b \\ 2a & b & 0 \end{vmatrix} = ab^2 - 4a^2c.$$

which is a times the discriminant of this quadratic as defined in the preceding paragraph.

In order to agree with the special case under consideration the discriminant of $f(x)$ might therefore be defined as the resultant of $f(x)$ and $f'(x)$ divided by the coefficient of the highest power of x in the rational integral function of x denoted by $f(x)$, provided we would adopt the definition of resultant implied in the illustration noted above. This definition, as well as the one given on page 196 of Bôcher's *Introduction to Higher Algebra*, 1907, is, however, not in accord with that found in some other standard works. In particular, these definitions do not coincide with the one found in the *Encyclopédie des Sciences Mathématiques*, tome 1, volume 2, page 75, but the resultants obtained according to these various definitions may differ only with respect to sign. This difference actually presents itself in the special case considered in the preceding paragraph, as can easily be verified.

We thus see that the reader may be considerably perplexed when he tries to harmonize the definitions of the term discriminant found in more advanced works with the earliest example of a discriminant with which he became acquainted, unless he is forewarned that the current definitions relating to this term are not in perfect agreement. The definition given on page 250 of Bôcher's *Introduction to Higher Algebra*, 1907, does not clear up the matter since it applies only to the case when the coefficient of the highest power of the unknown is unity. It may perhaps be taken for granted by many readers that the definition found in the large mathematical encyclopedias[1] will eventually be universally adopted. If this will be done we shall have to say that the discriminant of $ax^2 + bx + c$ is $4ac - b^2$ instead of $b^2 - 4ac$.

The question arises whether this change is desirable in view of the fact that $b^2 - 4ac$ appears under the radical sign in the common solution of the equation $ax^2 + bx + c = 0$ and hence it seems convenient to call it the discriminant of the quadratic equation rather than to say that this expression multiplied by -1 is the discriminant of the given equation. On the other hand, uniformity in definitions is so desirable that this slight convenience might well be sacrificed for the sake of avoiding the confusion caused by contradictory definitions of mathematical terms.

Another solution of this difficulty would be secured if mathematicians could agree on not following the large mathematical encyclopedias in this particular. In fact, such a well-known authority as the second edition of Pascal's *Repertorium der höheren Mathematik* has thrown its weight in favor of this movement by adopting a definition according to which the discriminant of $f(x)$ is the resultant of $f(x)$ and $f'(x)$ multiplied by

$$\frac{1}{a_0}(-1)^{n(n-1)/2},$$

[1] *Encyklopädie der Mathematischen Wissenschaften*, Vol. 1, p. 251; *Encyclopédie des Sciences Mathématiques*, tome 1, Vol. 2, pp. 97, 100.

where a_0 is the coefficient of the highest power of x in $f(x)$ and n is the degree of this function.[1] If this definition and the usual definition of resultant are adopted the discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$, which agrees with common usage in our textbooks.

When J. J. Sylvester first used the term discriminant as a technical mathematical term, *Philosophical Magazine*, 1851, page 406, he added the following explanatory remark: "Discriminant because it affords the discrimen or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally, of the existence or otherwise of multiple points in the locus represented or characterized by an algebraical function." Several authors have used somewhat similar explanatory words as a definition. Thus we find in the *Theory of Equations* by F. Cajori, 1912, page 96, "The discriminant of an equation $f(x) = 0$ may be otherwise defined as the simplest function of the coefficients, or of the roots, whose vanishing signifies that the equation has equal roots." Such a definition is clearly too vague to determine a definite expression as the discriminant of a given function. This is also true of such definitions as the following: "The discriminant of an equation involving a single unknown is the simplest function of the coefficients in a rational integral form, whose vanishing expresses the condition for equal roots," *Theory of Equations* by Burnside and Panton, volume 2, 1904, page 83; cf. *Theory of Equations* by F. Cajori, 1912, page 97.

These observations may suffice to forewarn the young mathematician in regard to the different meanings assigned to common mathematical terms by various authors. Although the term discriminant is sixty-seven years old and has found a place in many elementary textbooks it is still being defined differently by eminent authorities, so that it would even now be difficult to determine by the process of comparing authorities what expression should be called the discriminant of the general quadratic equation in one unknown. In addition to the noted difficulties in the way of such a determination we may add that on page 315 of the first volume of Borel's *Die Elemente der Mathematik* translated by P. Stäckel, 1908, it is stated that

$$\frac{b^2 - 4ac}{4}$$

is sometimes called the discriminant of the equation $ax^2 + bx + c = 0$.

It is a very singular fact that the definition of the discriminant of $f(x)$ contained in what are now universally regarded as our foremost works of reference, viz., the large mathematical encyclopedias in course of publication, is not in accord with the common textbook use of this term in the special case when $f(x) = ax^2 + bx + c$. The main objects of this note are to direct more general attention to this fact and to save the beginner from the perplexities into which

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[1] Pascal's *Repertorium der höheren Mathematik*, Vol. 1 (1910), p. 274. In the 1900 edition of this work, p. 88, the definition of the discriminant of $f(x)$ agrees with that found in the large mathematical encyclopedias.

he is naturally led by definitions found in some of our reliable American textbooks. In a previous note published in this MONTHLY, volume 24, 1917, page 106, we directed attention to desirable changes in the definitions of the term discriminant found in two of our standard dictionaries and in the *International Encyclopedia.*

In view of the fact that such eminent authorities give different definitions of the term discriminant of $f(x)$ it may appear almost presumptuous on the part of the present writer to express an opinion in regard to what appears to him as the most desirable definition to adopt. The advantages of uniformity along this line impel him, however, to say that he would adopt the definition found in the second edition of Pascal's *Repertorium* noted above, and hence he would not follow the large mathematical encyclopedias in this particular case. He would, on the other hand, follow these encyclopedias as regards the definition of the resultant of two functions of a single variable, and hence he would digress also in this instance from the definition adopted in Bôcher's *Algebra* cited above. These conclusions are based mainly on what appears to the writer as the most feasible steps towards securing uniformity.

Dr. A. J. Kempner recently directed my attention to the following definition: "By the *resultant* of two equations $f(x) = 0$ and $\phi(x) = 0$ is meant that integral function of the coefficients of $f(x)$ and $\phi(x)$ whose vanishing is the necessary and sufficient condition that $f(x) = 0$ and $\phi(x) = 0$ have a common root"; *College Algebra* by H. B. Fine, 1904, page 512. The fact that this definition appears in one of our most advanced American algebras may justify a reference to it here, especially since the definitions of the terms discriminant and resultant are so closely related. It is difficult to see how such a vague definition can fail to embarrass the serious student.

In this connection it may be of interest to refer to another term (*real curve*) which is used in college mathematics with a meaning differing from that assigned to it in some of our best reference books. To illustrate this fact we may note that in the *Encyclopédie des Sciences Mathématiques,* tome 3, volume 3, page 261, it is stated that "a real curve need not contain any real point; as happens, for instance, in the case of the real conic $x^2 + y^2 + 1 = 0$." On the contrary, it is customary to call such a conic in our textbooks an *imaginary* curve notwithstanding the fact that all the coefficients of its equation are real, and this usage seems to be also in accord with the terminology employed on page 71 of the volume just mentioned. The use of the term *real curve* for an imaginary ellipse in analytic geometry does not appear desirable, and hence it is the more questionable whether this use should be sanctioned in more advanced subjects.

WHAT IS THE ORIGIN OF THE NAME "ROLLE'S CURVE"?

By FLORIAN CAJORI, University of California.

It is a curious circumstance that the two topics of mathematical interest for which Michael Rolle is now best known—"Rolle's theorem" and "Rolle's curve" —are the ones which modern readers have been experiencing difficulty in finding in Rolle's published books or articles and in legitimately associating with his name. As late as 1903 A. v. Braunmühl[1] declared that "Rolle's theorem"—the theorem according to which $f'(x) = 0$ has at least one real root lying between two successive real roots of $f(x) = 0$—had not been found in Rolle's works and was therefore wrongly attributed to him. The present writer[2] was finally able to show that the theorem is rightly attributed to Rolle, it being found in an out-of-the-way publication of Rolle bearing the title *Dêmonstration d'une Methode pour rêsoudre les Egalitez de tous les degrez,* Paris, 1691.

As regards "Rolle's curve," historians have been thus far less fortunate. Some years ago Gino Loria[3] frankly admitted that he had not been able to ascertain the reason why the cubic curve $xy^2 = a(y + x)^2$ or $xy^2 = a(y - mx)^2$ should be named after Rolle. F. G. Teixeira[4] speaks of "Rolle's curve" and fully describes its properties, but gives no information as to Rolle's connection with the curve. Nor does Felix Müller,[5] who simply mentions its name. The earliest appearance that we have seen of this name in print is 1896, when Elgé[6] wrote an article "Sur la courbe de Rolle. Sa construction par points et par tangents."

In 1898 Gino Loria[7] published a note of inquiry regarding the origin of the name, and said that de Longchamps believed he had seen the name earlier in the *Educational Times.* As this note elicited no response, it was republished ten years later[8] with an annotation by H. Brocard who states that the discussion carried on between Saurin and Rolle on the "problème général des tangentes" in the *Journal des Savans* in the years 1702 and 1703 seemed to be a promising place for search, but that it is not given there; Brocard admits that he was not able to find any justification for attributing the curve to Rolle. No greater success rewarded the efforts of the present writer who examined the booklet quoted above, which contains "Rolle's theorem," also Rolle's *Traitê d'Algebre,* Paris, 1690, and the different volumes of the *Histoire de l'Acadêmie royale des*

[1] A. v. Braunmühl in *Bibliotheca Mathematica,* 3. F., Vol. 4, p. 399.

[2] F. Cajori in *Bibliotheca Mathematica,* 3. F., Vol. 11, 1910–1911, pp. 300–313.

[3] G. Loria, *Spezielle algebraische und transcendente Ebene Kurven.* Deutsch v. F. Schütte, Leipzig, 1902, p. 75; 2d ed., 1910, p.78.

[4] F. G. Teixeira, *Traitê des courbes spéciales remarquables planes et gauches,* in *Obras,* Vol. 4, Coimbra, 1908, pp. 115, 134.

[5] F. Müller, *Mathematisches Vokabularium,* Leipzig, 1900, "Curve." See also F. Müller in *Bibliotheca Mathematica,* 3. F., Vol. 2, 1901, p. 302.

[6] Elgé in *Journal de mathêmatiques spéciales,* 4 S., Vol. 5, 1896, pp. 32–34. See also p. 55.

[7] *L'Intermédiaire des mathêmaticiens,* Vol. 5, 1898, p. 76. No. 1257.

[8] *L'Intermédiaire des mathêmaticiens,* Vol. 16, 1908, p. 244.

sciences at Paris. The curve and its graph are given without name in Cramer's *Introduction à l'analyse des lignes courbes,* 1750, pp. 188, 478, and in O. Gherli's *Elementi teorico-pratici delle matematiche pure,* Vol. IV, Modena, 1773, pp. 343, 345.

The question still remains to be solved, who first used the name "Rolle's curve" for $xy^2 = a(y + x)^2$ and what connection, if any, Rolle had with it.[1] Who can throw further light on this?

THE MATHEMATICS OF AËRODYNAMICS.[2]

By EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

1. In the development of a branch of engineering there are several stages, sometimes long drawn out and poorly differentiated, sometimes brief and sharply defined.

There is the stage in which the bold inventors, trying one way and another, often with personal danger or even disaster, accomplish the first crude beginnings. In ship construction this epoch is prehistoric; in steam engineering it starts, at least in story, with Watts; in electrical engineering it lies within the memory of men still living; in aëronautics it stretches over a long period admirably described by A. F. Zahm in his book *Aërial Navigation.* The first essays in aëronautics, if we overlook the legendary Icarus, were the balloon and the balloon-type machine; progress with the heavier-than-air type may well be dated from Langley and the Wrights, though experiments with gliders are known for some time previous to them.

There is another stage—that in which the properly equipped mathematician and physicist construct a theory to correspond to the branch of engineering and by the indications of that theory aid materially in hastening the development of improved types of machines, or at least, if they arrive too late to help in the major development, they assist in codifying the fundamental scientific principles which underlie the subject and make possible systematic training of the youth to carry on the advance in details. The development of mathematics and physics was sufficient before the advent of electrical engineering to enable major advances to be made by William Thomson (later Lord Kelvin), by Oliver Heaviside, and by M. I. Pupin, to mention only three. In the case of the simpler parts of mechanical engineering and of naval architecture, the pressing necessities of prehistoric and early historic man were such that the art far outstripped the

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[1] As far as I now know, the only curve which prior to 1896 may be said to have been associated with the name of Rolle is mentioned by J. Bernoulli in a letter to Leibniz, dated May 7, 1701. Bernoulli there refers to the "ultima Rolii curva a Varignonio delineata

$$y = 2 + \sqrt{(4 + 2x)} + \sqrt{(4x)},"$$

Got. Gul. Leibnitii et Johan. Bernoullii Commercium Philosophicum et Mathematicum, Tomus secundus, Lausannæ & Genevæ, MDCCXLV, pp. 41, 42, 44.

[2] Address read before the Mathematical Association of America at its annual meeting, at Columbia University, New York, Saturday, December 30, 1916.

science; indeed the limitations of our mathematical and physical knowledge of the behavior of liquids are still so serious that when we wish a cup-defender we have to rely almost entirely upon the fine sense of a genius like Herreshoff instead of upon the theories and calculations of engineers like Professor C. H. Peabody and Rear-Admiral Taylor.

In the development of the airplane, however, since the early experimental and inventive successes of the Wrights and their competitors and followers in France, much has been accomplished in a single decade by the mathematician and physicist, in particular by G. H. Bryan, whose book *Stability in Aviation* was epochal, and by L. Bairstow, of the National Physical Laboratory, Teddington, England, whose experimental and theoretical work has done so much toward rendering Bryan's ideas directly and surely applicable to actual flying machines. I have been fortunate enough under commission from our government to add my mite to the theoretical advance (*Motion of an Airplane in Gusts*), and we in this country should all do much homage to the pioneer work which Lieut. J. C. Hunsaker, U. S. N., performed at the Massachusetts Institute of Technology.

There is a third stage, a relatively late stage in this uprearing of a branch of engineering. Here we find the scientific and technical knowledge so codified in text-books and manuals that it is not particularly hard for a school of technology, properly equipped with laboratories and teaching staff, to take in as raw material average young fellows and turn out as manufactured product tolerably competent engineers, able to design and make minor improvements in standard machines. This stage is not yet quite attained in aëronautics, but with the obviously great technical advances which have come about with such perplexing rapidity during the past two or three years under the pressure of war necessities in Europe and in this country, and with the presumably great theoretical advances which have accompanied them, it certainly will not take long after the war shall have terminated and after the present belligerent governments shall have ventured to divulge the ways and means and ends in aëronautics which they now guard so sedulously—it certainly will not then take long to work down the science and art of aëronautical engineering to the point where it takes its place in this third stage with mechanical, civil, and electrical engineering. Already a preliminary codification may be found in the course prepared by Messrs. Klemin and Huff, on the basis of the course given by their former teacher, Dr. Hunsaker, which is now appearing serially in the recently established, interesting and important journal *Aviation and Aëronautical Engineering.*

My subject as announced by your program committee is the "Mathematics of Aërodynamics," and if I have been long in my preamble it is merely because I feel that before we examine one special phase of aëronautics, the phase that interests us here most nearly, we should first take a broad view of the whole field to acquire the perspective necessary for a just appraisal of the value and of the limitations of mathematics in engineering. Mathematicians as investigators are concerned chiefly with but a single stage of the work. They cannot aid much in the first hardy adventures; they cannot help greatly in the final codifica-

tion—except as they strive as teachers better to prepare the freshman and the sophomore in his mathematical courses for the technical work to which he looks forward; they can help materially in the intermediate stage, provided they have schooled themselves in that style of mathematics which is suited to the treatment of engineering problems, that is in mechanics and mathematical physics.

2. In discussing aëronautical machines mathematically, or otherwise, a sharp distinction must be made between the lighter-than-air and the heavier-than-air types—the balloon and the plane types, as we may briefly designate them.

For the balloon type the first mathematical work will naturally be very elementary, dealing with buoyancy, with the expansion and contraction of gases owing to rise and fall in the level of flight and to changes of temperature. Arithmetic and algebra alone or the simplest portions of the calculus will suffice. If the coefficients of resistance to motion are known or assumed for different shapes of balloon, simple calculations will determine the power necessary for uniform rectilinear motion. As resistances are supposed to vary with some power, generally the square, of the velocity, easy problems in the calculus may be made regarding stopping and starting with no further use of differential equations than the "variables separable" type.

In discussing the airplane some small use of trigonometry is necessary. The pressure P in a plane surface of area S, which is taken as a first approximation to a wing, is assumed to be normal to the plane (the viscous tangential drag being neglected), and of the form $P = kSv^2 \sin i$, where i is the angle between the wing and the direction of motion. This is resolved vertically and horizontally to give the "lift" L and "drift" D as

$$L = kSv^2 \sin i \cos i, \qquad D = kSv^2 \sin^2 i.$$

In uniform horizontal flight $L = W$, the weight of the machine, and $D = F$, the thrust of the propeller. (To a first approximation the resistance of all surfaces other than the main wing of the monoplane or the pair of wings of the biplane are neglected.) The power necessary to drive the wing through the air is then nearly

$$\frac{Dv}{550} = \frac{W^2}{550kSv}.$$

This shows that for a given machine, the power is inversely as the velocity. The apparent anomaly is explicable by virtue of the fact that at higher velocities the machine flies at smaller angles of incidence i. A pretty exercise may be had by assuming that the resistance other than that due to the wings may be lumped together in a term Cv^2 added to the drift D so that the power becomes

$$\frac{W^2}{550kSv} + \frac{Cv^3}{550};$$

the minimum power may then be determined with its corresponding speed; and the further result that for the speed which minimizes the power, the weight per horsepower varies inversely as the speed.

There are a number of such simple applications of trigonometry and calculus which may be introduced in many a course in mathematics for the purpose of enlivening the work. One particularly interesting calculation is found by contrasting the expression for the power which has been obtained from the assumption (known to Euler) that the pressure varies with the sine of i with the expression which would be obtained from the erroneous assumption (generally attributed to Newton) that the pressure varies with the square of the sine of i. Those of you who are interested in the application of elementary mathematics to the airplane will find a great deal of suggestion in Painlevé and Borel's *L'Aviation*.

I wish next to mention those parts of advanced mathematics which are used by writers on aërodynamics. Beginning with Kirchhoff and Lord Rayleigh and reaching its culmination in recent work by Sir George Greenhill, the theory of functions of a complex variable has been indispensable in deriving theoretical formulas for the pressure P of a stream of air on a wing. Problems in hydrodynamics which take account of eddy (or vortical) motion or of internal friction (viscosity) are so difficult to solve that one may with reasonable accuracy say that for theoretical work in aërodynamics, eddies and friction must be disregarded; and for similar reasons the fluid motion is ordinarily restricted to plane motion. There are to be sure "end-effects" when a wing moves through the air, but owing to the length of span an approximation of value may be had by neglecting the "end-effects" and assuming that the motion of the air in all planes perpendicular to the edge of the wing is identical. Now by the method of "conformal representation" of the theory of functions the pressure exerted on a plane wing of various shapes, by the motion of the air, may in some cases be calculated, and the center of pressure may also be found.

These theoretical determinations of pressure and center of pressure are not precisely verified when experimentally measured in the wind tunnel and it is now customary to use the experimental values in place of those theoretically calculated. Nevertheless no student of theoretical aërodynamics can afford to be ignorant of the elements of hydrodynamics, including the theory of the reactions of streams on given contours, and this means that he must have some knowledge of the theory of functions of a complex variable, or rather let me say, some knowledge of the means of applying the principle of conformal mapping to the setting up of the functions which gives the map of one boundary and enclosed region upon another boundary and enclosed region. It is unhappily true that many extended and beautiful courses on the theory of functions leave the student wholly unable to carry through the applications necessary to the solution of these practical, or at least semi-practical, problems in hydromechanics. Might I suggest to all who teach the complex variable and the conformal map the possibility of drawing on uniplanar fluid motion as a source of instructive exercises?

3. Owing partly to the divergence between theory and experiment in fluid motion in simple cases and partly to the impossibility of obtaining a theoretical hydrodynamic solution for the motion of so complicated an object as an airship

or airplane, it is necessary in the main to treat the ship or plane as a rigid body to which are applied certain forces (resultant fluid pressures and moments of pressures) determined by wind tunnel experiments upon models. The problem of the motion of the machine then becomes one in the dynamics of a rigid body free to move in space (*i. e.*, with six degrees of freedom) subject to known forces. It is customary to refer the motion to axes fixed in the machine and moving with it. Now unfortunately moving axes are regarded as belonging to really advanced rigid dynamics. Routh in his two volumes *Elementary and Advanced Rigid Mechanics* leaves moving axes to the beginning of the Advanced Part. Single volume treatises such as Loney's *Dynamics of a Particle and of Rigid Bodies* are apt to stop before reaching moving axes. It is not at all unlikely that the present interest in the dynamics of flight will force us so to alter our texts and our courses as to include the theory of moving axes; for we cannot expect students to prolong their studies of mechanics unduly before entering upon the motion of aërial machines.

Whether referred to moving or fixed axes the equations of motion of an airplane or airship, with the forces represented by empirical equations experimentally determined, are too difficult to integrate without approximation. The first case treated is that of motion in a line nearly straight and inclined to the horizontal at a nearly constant angle. By considering the motion as made up of small departures from uniform flight in straight lines, the differential equations of the motion become linear equations with constant coefficients, and their solution is therefore a matter of well-known, though sometimes very tedious, routine. And right here allow me to intercalate the remark that we must not ourselves disdain arithmetic nor allow our students to be discouraged by it; there is current all too much of a feeling that only the beautiful general theories of mathematics are worthy the attention of real mathematicians and their students; the art and the science of getting a sufficiently accurate numerical solution of a numerical problem are of equal importance and dignity with general theory in all applications of mathematics.

The utility of the solution of the nearly uniform motion of the machine is found in the discussion of dynamical stability. The approximate equations are solved, as all linear equations with constant coefficients are solved, in terms of exponential functions with or without combination with trigonometric terms. If a machine dynamically stable is slightly disturbed from uniform motion the small departures from that standard state will die out as time goes on, *i. e.*, the exponential functions must have negative exponents like e^{-at}. It is a pretty and none too easy algebraic problem, solved in Routh's *Rigid Dynamics* but not in many books on algebra, to determine the conditions under which an algebraic equation has its real roots, and the real parts of its imaginary roots, negative. The practical importance of dynamical stability in a machine is that under normal atmospheric conditions and for short periods of time the machine may safely be flown "hands off" and the pilot has therefore some freedom to attend to other matters than the guiding of his craft.

I have been told that the theory of Bryan, recast by Bairstow, and supplemented by data of wind tunnel experiments, has enabled a machine to be designed and constructed which actually has been flown "hands off." If this be true it marks a great triumph for mathematical physics; whether it is true or not, we can confidently assert that to the English training in mathematical physics is in no small degree due the great and sudden advance in airplane design and the great success in aircraft warfare which have been realized in England in relatively few months.

4. I have shown you that the mathematics of aërodynamics leads from elementary algebra and arithmetic to the theory of functions of a complex variable and to the solution of linear differential equations with constant coefficients. I have shown how the theoretical work has come in that stage of the development of aëronautical engineering where it has been of real help in rapidly advancing the art. There are always with us branches of engineering and physics in which the right kind of mathematics is of great value for the rapid advance of those branches. This right kind of mathematics is the good old traditional Cambridge, England, type, the mathematics of Newton, of Green, of Maxwell, of Kelvin, of Rayleigh, a type of mathematics which in this country, owing perhaps to our preponderance of German-trained mathematicians, has all too little prestige. I sometimes wonder whether we do right to aim so exclusively at the continental type. It is worthy of note that our two great native mathematicians, George W. Hill and J. Willard Gibbs, were concerned with the applied side. May it not be that we in this country have such a natural bent toward the practical that a diligent cultivation of the British sort of mathematics would find a readier response among our students?

We are here not as research mathematicians but as teachers of collegiate mathematics. Our country has great industrial problems of peace and war to solve, and every one of us must help as he may. As we bend the mathematical twig, so will the tree incline. Let us without prejudice consider our curricula and with open mind introduce any necessary changes to make sure that the type of mathematics which we place before our students is that which will contribute most to the victory of our country in time of stress and to her prosperity in times of peace.

SPRING MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section of the Mathematical Association of America was held at Carleton College on May 4, 1918. Seventy-eight people attended, including Edla G. Berger of College of St. Catherine, Father W. E. Etzel of St. Thomas College, C. H. Gingrich of Carleton College, chairman of the section, Jessie G. Quigley of College of St. Teresa, G. N. Bauer, W. H. Bussey, R. R. Shumway, H. L. Slobin, R. M. Barton of the University of Minnesota, members of the Association. All in attendance at the meeting were, as the guests of Carleton College, most hospitably entertained through the day and at luncheon.

The program was arranged to give place to a paper from a subject closely allied to mathematics and it was the very good fortune of the section to be able to have Professor H. C. Wilson of Carleton College set before them with lantern slides the principal matter of his paper in the May number of *Popular Astronomy* upon the eclipse of the sun of June 8.

Professor Slobin presented a paper upon integration. He traced the development of the theory of integration corresponding to the development of the function concept, chronologically, from the work of Leibnitz and Newton through that of Cauchy, Dirichlet, Fourier, Riemann and Lebesgue, and established the relation of the Lebesgue integral to the preceding notions of integration.

Professor G. N. Bauer gave a talk upon the progress of the campaign for war savings stamps. He explained the meaning of the war savings stamp, its worth to the Government and to the purchaser and outlined the plan of the campaign, giving the amounts to be raised in the state and the method of reaching all people of the state.

It appeared to the program committee that it would be well to have some discussion of work which while not included in the regular text might be introduced in the class room at times. To this end Father Etzel was to discuss the quadratic involving a single parameter. Due to the fact that the early papers of the program had taken so much time it became necessary that Father Etzel simply present the subject of discussion and postpone the full discussion to the next meeting.

The Section wishes to coöperate in all possible ways with the Association and to this end arranged a discussion of the advisability of introducing the study of Descriptive Geometry into the college course in mathematics. This subject was discussed at the annual meeting of the Association with the general idea of having material on the program of direct interest in secondary school mathematics; it had the same purpose in the meeting of the section. This discussion was introduced by Professor W. H. Kirchner of the University of Minnesota. Professor Kirchner gave a historical outline of the study of the subject and the published text books; he showed how the subject could be a definite help to our present courses in mathematics and how the study would serve to clear up much of our present work which is difficult for the student to see, by giving him the power to visualize. He believes that if the subject were introduced into the present course in college mathematics, not as a distinct subject by itself but as related to other matter customarily treated, there would be a decided gain in clearness of understanding of mathematics and that the increased amount of time required for this work would not be appreciable. Professor Kirchner gave several illustrations of his discussion. The subject was discussed at considerable length by the section.

R. M. Barton, *Secretary.*

SPRING MEETING OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION.

The spring meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Catholic University, Washington, D. C., on May 4, 1918. Among the thirty-two persons in attendance were the following members of the Association: O. S. Adams, U. S. Coast and Geodetic Survey; Clara L. Bacon, Goucher College; C. C. Bramble, U. S. Naval Academy; J. A. Bullard, U. S. Naval Academy; Paul Capron, U. S. Naval Academy; G. R. Clements, U. S. Naval Academy; A. Cohen, Johns Hopkins University; Alexander Dillingham, U. S. Naval Academy; Angelo Hall, U. S. Naval Academy; W. M. Hamilton, U. S. Nautical Almanac Office; Wm. E. Heal, U. S. Bureau of Plant Industries; L. S. Hulburt, Johns Hopkins University; W. W. Johnson, U. S. Naval Academy; A. E. Landry, Catholic University; Florence P. Lewis, Goucher College; Frank Morley, Johns Hopkins University; O. J. Ramler, Catholic University; H. M. Roeser, U. S. Bureau of Standards; R. E. Root, U. S. Naval Academy; W. F. Shenton, U. S. Naval Academy; C. E. Van Orstrand, U. S. Geological Survey and George Washington University.

The annual election of officers of the Section resulted as follows: Chairman, Professor A. E. Landry, Catholic University; Secretary-Treasurer, Professor R. E. Root, U. S. Naval Academy; Third member of the Executive Committee, Professor L. S. Hulburt, Johns Hopkins University.

The president, Professor A. Cohen, presided at both sessions of the meeting. The program was opened by a cordial and inspiring address of welcome by Rev. Dr. Edward A. Pace, of Catholic University. In the interval between the two sessions those attending were the guests of Catholic University at a most enjoyable luncheon, served in Graduates Hall.

Titles and authors of papers read were as follows:

"On the problem of elimination." Professor FRANK MORLEY, Johns Hopkins University.

"The doubly periodic functions connected with the curve $x^3 + y^3 = 1$." Mr. O. S. ADAMS, U. S. Coast and Geodetic Survey.

"Some results relating to the in- and circumscribed triangle of the rational quartic." Professor A. E. LANDRY, Catholic University.

"On Duhamel's theorem." Professor L. S. HULBURT, Johns Hopkins University.

"On the Missouri system of grading students." Professor FLORENCE P. LEWIS, Goucher College.

"A college training course in secondary mathematics." Dr. H. C. GOSSARD, U. S. Naval Academy.

"The use of polar line-coördinates." Mr. PAUL CAPRON, U. S. Naval Academy.

"A general system of approximate integration formulas." Mr. M. SASULY, Bureau of Standards (introduced by Mr. Harry M. Roeser).

"Matrices connected with the invariants of the equation of the second degree." Dr. J. A. BULLARD, U. S. Naval Academy.

RALPH E. ROOT, *Secretary.*

MEETING OF THE KENTUCKY SECTION.

The Tenth Annual Meeting of the Mathematics Section of the Kentucky Colleges and the Second Annual Meeting of the Kentucky Section of the Mathematical Association of America, was held at Georgetown College, Georgetown, Ky., May 11, 1918, in the Physics Lecture Room, Physics Building. The chairman, Prof. A. L. Rhoton, in a few words welcomed the members and visitors. The program with brief abstracts follows.

"Illustrated Lecture on Snowflakes." D. W. Martin, Professor of Physics, Georgetown College, assisted by Mr. C. V. Mullins.

Prof. Martin gave a brief account of the earliest work in snowflake photography by J. G. Greenough of Jericho, Vermont. Many slides were shown and attention was especially called to the pronounced hexagonal shape of the flakes, there being only one exception to this.

"Photogrammetry." Mr. V. G. Grove, University of Ky.

Photographic principles were first enunciated by Beautemps-Beaupré in 1791–1793. Following the invention of the sensitized plate, Colonel A. Laussedat in 1864 published the first work on photographic surveying. In 1865 A. Meydenbauer published applications to architecture. Guido Hauch then (1884) gave a graphical construction of a third perspective from two given ones. S. Finsterwalder laid the foundation of photogrammetry. Besides these, practical surveyors used photogrammetric methods to aid them in mapping mountainous regions. The two cases of one and two perspectives were discussed; the perspective being either vertical or inclined. The problems of orientation, graphical construction, reconstruction of an auxiliary figure, were given. An analytical expression for the coördinates of the space point in terms of the coördinates of the images was obtained. G. Hauch's construction of a third perspective was then discussed and it was shown that this subject has a very wide application.

"The Decipherment of Military Code Messages." H. R. Phalen, Berea College, Berea, Ky.

The speaker showed by some dozen large printed sheets the various methods of enciphering military code messages. All messages are divided into two great classes: transposition and substitution. In the first case the letters are simply rearranged according to some predetermined scheme, but each letter represents itself; that is, "a" means "a" and "k" means "k" wherever they are found. Consequently the vowel and consonant frequencies will be the same in this type of message as they are in any ordinary page of reading matter. Of this transposition type messages were presented in the simple vertical, diagonal, spiral, keyword and route cipher methods involving English, German, French and Spanish texts. The other great class is the substitution class where letters are

interchanged in their alphabetical position and where the vowel and consonant frequency tables are useless. Under this type the most interesting point was the construction of a military message by the method of the famous Playfair Cipher of the British army.

"A Generalization of the Mean Value Theorem." H. H. Downing, University of Ky.

The conditions of $0 < \theta < 1$ and $0 < h$ were removed and several functions were examined to find a relation between θ and h. For the function $ax^3 + bx^2 + cx + d$ we set $\varphi = \theta h$ and substitute the function in

$$\frac{f(x+h) - f(x)}{h} = f'(x + \varphi),$$

and obtain the equation

$$ah^2 - 3a\varphi^2 + (3ax + b)h - 2(3ax + b)\varphi = 0.$$

This is seen to be an ordinary hyperbola with transverse axis vertical and the hyperbola passing through the origin. (φ-axis vertical, H-axis horizontal.) The straight lines $\varphi = 0$, $\varphi = 1/3\ h$, $\varphi = 2/3\ h$, $\varphi = h$, $\varphi = 4/3\ h$, are the loci, respectively, of a focus, vertex, center, vertex and focus.

Prof. E. L. Rees was elected chairman for the year 1918–1919 and the present secretary was reëlected. The secretary's report was then given. Regret was expressed at the absence of Prof. W. H. Garnett, Wesleyan College, Winchester, Ky., and Prof. Henry Lloyd, Transylvania College, Lexington, Ky., both caused by illness. Regret was also expressed that Prof. H. R. Phalen was leaving the state.

Those present during the meetings were: A. L. Rhoton, David W. Martin, Georgetown College; G. C. Crooks, Center College; H. R. Phalen, Berea College, J. M. Maxey, Asbury College; P. P. Boyd, J. M. Davis, E. L. Rees, V. G. Grove, H. H. Downing, University of Kentucky.

H. H. Downing, *Secretary.*

BOOK NOTICES.

Send all communications about books to W. H. Bussey, University of Minnesota.

The number of books in the English language on the theory of functions of a complex variable continues to grow. One of the latest is "Functions of a Complex Variable," by Thomas M. Macrobert, Lecturer in Mathematics in the University of Glasgow. It is published by Macmillan and Co., London. The book is designed for beginners who have acquired a good working knowledge of the calculus. In order to make the subject not too difficult for beginners, the author has abstained from the use of strictly arithmetical methods and, while endeavoring to make his proofs sufficiently rigorous, he has based them mainly on geometrical conceptions.

"Has mathematics a ealm apart from human life, fitting daily experiences in

places closely enough to be of use, but still not at all identical with it; or is it, indeed, the very same as the realm of human life? Is the differential equation only a refinement upon the real law of physics, the irrational only an approximation to the actual number of nature? Is the universe stable or will it some day disappear, wind its way back into chaos, leaving nothing but the truths of mathematics still standing? Is it true that chance does not exist really but only in seeming, or is everything purely chance, and are the laws of the universe merely the curves which we have drawn through a random few of an infinitely compact set of points? The consideration of these problems is what we mean by the phi osophy of mathematics." A new book on this subject has recently been published by the Open Court Publishing Company, 122 South Michigan Ave., Chicago, Illinois. It is entitled "Lectures on the Philosophy of Mathematics" and is by James Byrnie Shaw of the Department of Mathematics of the University of Illinois.

The number of trigonometry text books on the market is so very large that the author of every new one feels called upon to explain in his preface just why he has added to the list. The preface to the new "Plane and spherical trigonometry" by Professor Leonard M. Passano of the Massachusetts Institute of Technology says that the chief aim of the text are brevity, clarity and simplicity. The text "aims to present the trigonometry in such a way as to make it interesting to students approaching some maturity, and so as to connect the subject, not only with the mathematics which the student has already had, but also with the mathematics which, in many cases at least, is to follow." The book a ms to avoid the tendency to amplification which the author says trigonometry texts of late years have shown. The book is published by the Macmillan Company.

One of the most interesting chapters in the first course in calculus is the chapter on maxima and minima. The many practical problems that can be solved by the simple criteria given in the ordinary calculus text book interest the student exceedingly. They make him feel that the calculus has power. Mathematicians have always been interested in problems of maxima and minima. Those of the present day who are especially interested in the subject and who want a somewhat full treatment rather than the brief inadequate treatment given in the current text books will welcome the 193-page book on "The Theory of Maxima and Minima," by Professor Harris Hancock, of the University of Cincinnati, published by Ginn and Company.

PROBLEMS FOR SOLUTION.

Send all communications about problems to B. F. Finkel, Springfield, Mo.

2719. Proposed by R. P. BAKER, University of Iowa.

Show that, $2x(\log x)^2 - x(x-1)(x+3)\log x + (x-1)^2(3x-1)$ is negative for $1 < x < \infty$.

2720. Proposed by CAPT. A. A. BENNETT, C.A.R.C., Galveston, Texas.

Given three points, A, B, C, in a plane, draw from an arbitrary fourth point D the segments AD, BD, CD. Also draw rays AA', BB', CC', making equal (small) angles respectively with

segments AD, BD, CD. The triangle determined by the three rays does or does not contain the point D according as the original triangle ABC does or does not contain D.

Prove the theorem, considering also the case in which A, B, C, D, are concyclic.

2721. Proposed by G. PAASWELL, New York City.

A, B are the termini of a horizontal line of length w. At a point C in this line, at a distance k from A, is applied a force P making an angle φ with the vertical. Determine the family of curves extending from A to B (and below AB) such that an equal normal distribution of loading p per unit length of curve around the periphery of the curve, will hold in equilibrium the force P. The parameter of the family so determined will be p.

2722. Proposed by FRANK IRWIN, University of California.

The number of terms in the general polynomial of the nth degree in m variables and in that of the mth degree in n variables is the same. It would be interesting to devise schemes which, without assuming this result, should exhibit the terms of these polynomials in one-to-one correspondence with each other.

2723. Proposed by GEORGE Y. SOSNOW, Newark, N. J.

The feet of the perpendiculars from the intersection of the diagonals on the sides of a cyclic quadrilateral M are joined to form a second quadrilateral N. Prove that N is a quadrilateral of minimum perimeter inscribed in M.

2724. Proposed by FRANK IRWIN, University of California.

Show that there is a unique set of real values, $x_1, x_2, x_3, \cdots, x_n$, that satisfy the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_1x_2 - x_2x_3 - x_3x_4 - \cdots - x_{n-1}x_n - x_n + \frac{n}{2(n+1)} = 0.$$

2725. Proposed by S. A. COREY, Albia, Iowa.

Establish the identity,

$$(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8)(a_1a_2 + c_1a_3a_4 + c_2a_5a_6 + c_1c_2a_7a_8)$$
$$\equiv (r_9r_{10}r_{11}r_{12})^{1/2} + c_1(r_{13}r_{14}r_{15}r_{16})^{1/2} + c_2(r_{17}r_{18}r_{19}r_{20})^{1/2} + c_1c_2(r_{21}r_{22}r_{23}r_{24})^{1/2}$$

where $r_9 = a_1r_1 - c_1a_3r_3 - c_2a_5r_5 + c_1c_2a_7r_7$, $r_{10} = a_1r_2 - c_1a_3r_4 - c_2a_5r_6 + c_1c_2a_7r_8$, $r_{11} = a_2r_1 - c_1a_4r_3 - c_2a_6r_5 + c_1c_2a_8r_7$, $r_{12} = a_2r_2 - c_1a_4r_4 - c_2a_6r_6 + c_1c_2a_8r_8$, $r_{13} = a_3r_1 + a_1r_3 - c_2a_7r_5 - c_2a_5r_7$, $r_{14} = a_3r_2 + a_1r_4 - c_2a_7r_6 - c_2a_5r_8$, $r_{15} = a_4r_1 + a_2r_3 - c_2a_8r_5 - c_2a_6r_7$, $r_{16} = a_4r_2 + a_2r_4 - c_2a_8r_6 - c_2a_6r_8$, $r_{17} = a_5r_1 + c_1a_7r_3 + a_1r_5 + c_1a_3r_7$, $r_{18} = a_5r_2 + c_1a_7r_4 + a_1r_6 + c_1a_3r_8$, $r_{19} = a_6r_1 + c_1a_8r_2 + a_2r_5 + c_1a_4r_7$, $r_{20} = a_6r_2 + c_1a_8r_4 + a_2r_6 + c_1a_4r_8$, $r_{21} = -a_7r_1 + a_5r_3 - a_3r_5 + a_1r_7$, $r_{22} = -a_7r_3 + a_5r_4 - a_3r_6 + a_1r_8$, $r_{23} = -a_8r_1 + a_6r_3 - a_4r_5 + a_2r_7$, and $r_{24} = -a_8r_2 + a_6r_4 - a_4r_6 + a_2r_8$.

2726. Proposed by E. H. MOORE, The University of Chicago.

Let $\alpha_1(x, y)$, $\alpha_2(x, y)$, $\alpha_3(x, y)$, $\alpha_4(x, y)$, $\kappa_1(x, y)$, $\kappa_2(x, y)$ be six real-valued continuous functions of (x, y) over the unit-square S: $(0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1)$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be symmetric functions and of positive type, i. e., for every real-valued continuous function $\xi(x)(0 \leqq x \leqq 1)$ $\int_0^1 \int_0^1 \xi(x)\alpha_g(x, y)\xi(y)dxdy \geqq 0$. Prove the inequality:

$$(J_{15}'J_{26}''J_{37}'''J_{48}^{\text{iv}} - J_{16}'J_{28}''J_{35}'''J_{47}^{\text{iv}} - J_{25}'J_{46}''J_{17}'''J_{38}^{\text{iv}} + J_{26}'J_{48}''J_{15}'''J_{37}^{\text{iv}})\kappa_1\kappa_2\kappa_1\kappa_2 \geqq 0.$$

Here on the left there are four terms of which the first is

$$\int_0^1 \cdots \int_0^1 \alpha_1(u_1u_5)\alpha_2(u_2u_6)\alpha_3(u_3u_7)\alpha_4(u_4u_8)\kappa_1(u_1u_2)\kappa_2(u_3u_4)\kappa_1(u_5u_6)\kappa_2(u_7u_8)du_1 \cdots du_8;$$

thus the eight variables of integration $u_1 \cdots u_8$ are in order the eight arguments of the four functions $\kappa_1\kappa_2\kappa_1\kappa_2$ while the "integration" symbols J indicate how the variables are to be supplied as arguments to the four functions $\alpha_1\alpha_2\alpha_3\alpha_4$, e. g., J_{26}'' indicates that α_2 (the superscript'' determining the subscript 2) has the arguments u_2u_6.—Indicate another inequality of this type and determine the number of such inequalities.

SOLUTIONS OF PROBLEMS.

489 (Algebra). Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove the following:

$$\begin{vmatrix} -x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ -v & -u & y & -x \end{vmatrix}^2 + a\begin{vmatrix} x & -x & -bu & abv \\ y & y & -bv & -bu \\ u & u & x & ay \\ v & -v & y & -x \end{vmatrix}^2$$

$$+ b\begin{vmatrix} x & -ay & -x & abv \\ y & x & y & -bu \\ u & av & u & ay \\ v & -u & -v & -x \end{vmatrix}^2 + ab\begin{vmatrix} x & -ay & -bu & -x \\ y & x & -bv & y \\ u & av & x & u \\ v & -u & y & -v \end{vmatrix}^2 = \begin{vmatrix} x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ v & -u & y & -x \end{vmatrix}^2.$$

II. Solution by A. M. Harding, University of Arkansas.

The quantities X, Y, U, V, W, defined by the equations

$$\begin{aligned} xX - ayY - buU + abvV &= -xW, \\ yX + xY - bvU - buV &= yW, \\ uX + avY + xU + ayV &= uW, \\ vX - uY + yU - xV &= -vW, \end{aligned}$$

are proportional to the five determinants taken in order.

It can be easily shown that the last determinant in the left member of the proposed equation is equal to zero for all values of x, y, u, v, a, b. Hence the above equations may be written in the form

$$\begin{aligned} (X + W)x - aYy - bUu &= 0, \\ Yx + (X - W)y - bUv &= 0, \\ Ux + (X - W)u + aYv &= 0, \\ Uy - Yu + (X + W)v &= 0. \end{aligned}$$

The quantities x, y, u, v, can satisfy this system of homogeneous linear equations if, and only if, the determinant

$$\begin{vmatrix} X+W & -aY & -bU & 0 \\ Y & X-W & 0 & -bU \\ U & 0 & X-W & aY \\ 0 & U & -Y & X+W \end{vmatrix}$$

is equal to zero. It can be shown that the value of this determinant is k^2, where

$$k = (X^2 - W^2) + aY^2 + bU^2.$$

Hence $X^2 + aY^2 + bU^2 = W^2$, or, since $V = 0$, $X^2 + aY^2 + bU^2 + abV^2 = W^2$. That is, the relation stated in the problem holds for all values of x, y, u, v, a, b.

Note. We are publishing a second solution of this problem for two reasons: First, because the solution above is essentially different from the one published in the March number; and second, because the conclusion drawn in that solution that the identity does not always exist is incorrect. Editors.

271 (Number Theory). Proposed by HORACE OLSON, Chicago, Illinois.

Prove that if x, y, z, u, v, and w are integers such that $x^2 + y^2 = u^2$, $x^2 + z^2 = v^2$, $y^2 + z^2 = w^2$, then the product $xyzuvw$ is divisible by 518400.

Solution by the Proposer.

I shall first prove some lemmas.

1) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then either x or y is divisible by 3. This follows from the fact that any perfect square is congruent, modulo 3, to either 0 or 1.

2) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then either x or y is divisible by 4; for any perfect square is congruent, modulo 16, to 0, 1, 4, or 9.

3) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then at least one of the numbers x, y, u is divisible by 5; for any perfect square is congruent, modulo 5, to 0, 1, or 4.

Hence, from the hypotheses, at least two of the numbers x, y, z are divisible by 3. If the notation be so chosen that these are x and y, the first equation shows that u also is divisible by 3. We can then divide both terms of this equation by 9; it then follows that either $x/3$ or $y/3$ is divisible by 3. Thus the product $xyzuvw$ is divisible by 3^4. In the same way it can be proved that the product is divisible by 4^4.

By lemma 3, at least one of the numbers x, y, u is divisible by 5. Similarly, at least one of each of the sets x, z, v and y, z, w is divisible by 5. Thus at least two of the numbers x, y, z, u, v, w are divisible by 5.

Therefore the product $xyzuvw$ is divisible by $3^4 \cdot 4^4 \cdot 5^2$, or 518400.

Also solved by H. C. Feemster and J. L. Riley.

2661. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find a parallelepipedon whose edges, and the diagonals of its faces, are all rational whole numbers.

Solution by the Proposer.

Denote the edges by x, y, and z; then

$$x^2 + y^2 = \square, \quad x^2 + z^2 = \square, \quad y^2 + z^2 = \square. \tag{1, 2, 3}$$

Assume $x = 2pq$, $y = p^2 - q^2$; then by substitution,

$$x^2 + y^2 = (2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2,$$

a square;

$$x^2 + z^2 = (2pq)^2 + z^2, \quad y^2 + z^2 = (p^2 - q^2)^2 + z^2, \tag{4, 5}$$

which must be made squares.

Put

$$(2pq)^2 + z^2 = \left(z - \frac{2pqr}{s}\right)^2,$$

which gives

$$z = \frac{pq(r^2 - s^2)}{rs}.$$

Substituting in (5) the value of z found above,

$$(p^2 - q^2)^2 + \frac{p^2q^2(r^2 - s^2)^2}{r^2s^2} = \square,$$

or

$$r^2s^2(p^2 - q^2)^2 + p^2q^2(r^2 - s^2)^2 = \square. \tag{6}$$

Expanding (6) it may be written

$$(p^2r^2 + q^2s^2)(q^2r^2 + p^2s^2) - 4p^2q^2r^2s^2 = \square,$$

which will be the case when

$$q^2r^2 + p^2s^2 = 4q^2s^2. \tag{7}$$

By transposition, (7) becomes

$$q^2r^2 = 4q^2s^2 - p^2s^2 = s^2(4q^2 - p^2) = s^2\left(\frac{mp}{n} - 2q\right)^2,$$

say; and we get

$$\frac{p}{q} = \frac{4mn}{m^2 + n^2}, \qquad \frac{r}{s} = \frac{2(m^2 - n^2)}{m^2 + n^2};$$

$$x = 2pq = 8mn(m^2 + n^2), \qquad y = p^2 - q^2 = (4mn)^2 - (m^2 + n^2)^2,$$

$$z = \frac{pq(r^2 - s^2)}{rs} = \frac{2mn[4(m^2 - n^2)^2 - (m^2 + n^2)^2]}{m^2 - n^2}.$$

Reducing the foregoing values of x, y, z to a common deominator $m^2 - n^2$ and then discarding it, we finally have

$$x = 8mn(m^2 - n^2)(m^2 + n^2),$$

$$y = (m^2 - n^2)[(4mn)^2 - (m^2 + n^2)^2],$$

$$z = 2mn[4(m^2 - n^2)^2 - (m^2 + n^2)^2].$$

Take $m = 2$, $n = 1$ and we get $x = 240$, $y = 117$, $z = 44$, the smallest values known. The diagonals of the faces are 267, 244, 125.

Also solved by C. B. HALDEMAN and L. E. LUNN.

2664. Proposed by J. W. NICHOLSON, Baton Rouge, La.

Find the sum of the series, $\frac{1}{3} - \frac{2}{15} + \frac{3}{35} - \cdots + (-1)^{n+1}\dfrac{n}{(2n-1)(2n+1)}$.

SOLUTION BY S. W. REAVES, University of Oklahoma.

The exponent of -1 in the general term should obviously be $n + 1$. Making this correction, the general term may be written,

$$(-1)^{n+1}\frac{n}{(2n-1)(2n+1)} = \tfrac{1}{2}(-1)^{n+1}\left(\frac{n}{2n-1} - \frac{n}{2n+1}\right).$$

Let S denote the required sum. Then

$$S = \frac{1}{2}\left[\sum_{i=1}^{n}(-1)^{i+1}\frac{i}{2i-1} + \sum_{i=1}^{n}(-1)^{i+2}\frac{i}{2i+1}\right]$$

$$= \frac{1}{2}\left[\left(1 - \tfrac{2}{3} + \tfrac{3}{5} - \cdots + (-1)^{n+1}\frac{n}{2n-1}\right) + \left(\tfrac{1}{3} + \tfrac{2}{5} - \cdots + (-1)^{n+2}\frac{n}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - 1 + 1 - 1 + \cdots + (-1)^{n+2}\frac{n}{2n+1}\right].$$

If n be even, $S = \dfrac{n}{4n+2} = \frac{1}{4}$ when n is infinite.

If n be odd, $S = \dfrac{n+1}{4n+2} = \frac{1}{4}$ when n is infinite.

Also solved by BANCROFT H. BROWN, E. H. WORTHINGTON, E. B. ESCOTT, GEORGE F. WILDER, PAUL CAPRON, HORACE OLSON, ELIJAH SWIFT, and ARNOLD DRESDEN.

2665. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A telegraph wire, which weighs 1/10 lb. per yard, is stretched between poles on a level ground so that the greatest dip of the wire is 3 feet. Find, approximately, the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

SOLUTION BY G. PAASWELL, New York City.

MacMahon's *Hyperbolic Functions*, pages 47 to 51 discusses the two cases here, (*a*) when the wire is taken as inextensible, and (*b*) as extensible. Under (*a*) $y/c = \sec\varphi$; $x/c = gd^{-1}\varphi = \log_e \tan(\pi/4 + \varphi/2)$. The origin is the point midway between the poles and a distance c below the lowest point of the wire. Use the yard as the unit of length. $c = H/w = 140/0.1 = 1400$. $y = 1 + c = 1401$. Whence, $\varphi = 2° \; 9' \; 54''$. Thence, $x = 52.913$ and since $L = 2x$, the re-

quired length is 105.826 yards. Note here that if the weight is taken as distributed along the horizontal, i. e., along the x-axis instead of along the wire there is obtained by simple moments $c\,dy = x\,dx$ and with the origin as before except that the low point is taken instead of a point c below, there is obtained the parabolic relation $x = \sqrt{2cy}$, with $y = 1$. Hence, $x = \sqrt{2800} = 52.915$ and $L = 105.83$ yds., which is sufficiently close for this problem. As a matter of interest, it is seen that since $y/c = \cosh x/c$, the origin a distance c below the low point, by expanding the cosh to two terms, $1 + x^2/2c^2$, and replacing y by $y - c$ the parabolic relation above is obtained, which is a measure of the approximation of this method of solving the catenary.

(b) Here $y/c = \sec\varphi + \frac{1}{2}m\tan^2\varphi$; $x/c = gd^{-1}\varphi + m\tan\varphi$. $m = 1H$, where 1 is the unit extension of the wire. Since the wire weighs 0.1 pound per yd., taking steel as weighing 10.2 pounds per sq. in. per yd., and the modulus of elasticity as 30,000,000 pounds per sq. in., we have since $1 = H/AE$, where a is the area of the section of the wire and E the modulus, $m = 6664 \times 10^{-5}$. Expressing the y equation as a quadratic in $\sec\varphi$ and solving for φ, $\varphi = 2° 5' 54''$. Substituting this value in the equation for x and again replacing the gudermanian by its logarithmic equivalent we get $x = 54.74$ and $L = 109.58$ yds. Do not mistake the difference between this value and the above as the extension of the wire. Its meaning is that when allowance is made for extension it is necessary to spread the supports above the difference apart in order to maintain the deflection of one yard. The parabolic expression for this case is more complicated than the exact.

Also solved by J. V. Balch, Horace Olson, A. R. Nauer, Paul Capron, and Louis G. Pooler.

2666. Proposed by W. WOOLSEY JOHNSON, Annapolis, Md.

Ten equations between five quantities, x_1, x_2, x_3, x_4, x_5 being written as follows: $x_1 = 1 - x_3x_4$ and four others formed by the cyclic interchange of the suffixes; also $x_5x_1x_2 = x_5 + x_2 - 1$ and four others formed by the cyclic interchange; prove that only three of these equations are independent. In other words, the values of x_1 and x_2 being assumed at pleasure, x_3, x_4, and x_5 can be so determined as to satisfy all ten equations.

I. Solution by Arnold Dresden, University of Wisconsin.

Denote by E_i the function $x_i + x_{i+2}x_{i+3} - 1$ and by $\bar{E}_i$ the function $x_ix_{i+1}x_{i+2} - x_i - x_{i+2} + 1$. Then the proposed equations may be written in the form $E_i = 0$, $\bar{E}_i = 0$, $(i = 1, \cdots, 5)$, the subscripts being reduced modulo 5.

We have the following identities:

$$(1) \qquad \begin{cases} x_{i+1}E_i - E_{i+3} \equiv \bar{E}_{i+1} \\ x_{i-1}E_i - E_{i+2} \equiv \bar{E}_{i+2}, \end{cases}$$

so that the equations $\bar{E}_i = 0$ are dependent upon the equations $E_i = 0$.

But from the relation (1), we derive, moreover, the following

$$(5) \qquad E_{i-1} + x_{i-1}E_i \equiv x_{i+2}E_{i+1} + E_{i+2}.$$

Hence from any three of the equations, $E_i = 0$, the remaining two must follow.

II. Solution by Horace L. Olson, Heidelberg University, Tiffin, Ohio.

The ten equations mentioned are

$$\begin{array}{llll} (1) & x_1 = 1 - x_3x_4 & (6) & x_5x_1x_2 = x_5 + x_2 - 1 \\ (2) & x_2 = 1 - x_4x_5 & (7) & x_1x_2x_3 = x_1 + x_3 - 1 \\ (3) & x_3 = 1 - x_5x_1 & (8) & x_2x_3x_4 = x_2 + x_4 - 1 \\ (4) & x_4 = 1 - x_1x_2 & (9) & x_3x_4x_5 = x_3 + x_5 - 1 \\ (5) & x_5 = 1 - x_2x_3 & (10) & x_4x_5x_1 = x_4 + x_1 - 1. \end{array}$$

Let us assume, first, that $x_1x_2 \neq 1$. Then from equations (4), (2), and (3), respectively, we find that,

$$x_4 = 1 - x_1x_2, \qquad x_5 = (1 - x_2)/(1 - x_1x_2), \qquad x_3 = (1 - x_1)/(1 - x_1x_2).$$

To show that the remaining equations are dependent upon these three, we note that equations (3), (4), (2) lead respectively to

$$1 - x_2x_3 = 1 - x_2 + x_1x_2x_5, \qquad 1 - x_2x_3 = 1 - x_2 + x_5(1 - x_4), \qquad 1 - x_2x_3 = x_5.$$

This last equation is equation (5).
Similarly we have

$$1 - x_3x_4 = 1 - x_4 + x_4x_5x_1, \qquad 1 - x_3x_4 = 1 - x_4 + x_1 - x_1x_2, \qquad 1 - x_3x_4 = x_1,$$

This last equation is (1).

Multiplying (4) by x_5, we have $x_5x_1x_2 = x_5 - x_5x_4$ and then $x_5x_1x_2 = x_5 + x_2 - 1$ from (2).

This last equation is (6). Since, then, equations (2) to (5) and (7) to (10) are obtained from (1) and (6) by cyclic permutation of the suffixes, it follows that upon equations (2), (3), and (4) all the others depend. This, we note, is true whether or not $x_1x_2 = 1$. If $x_1x_2 = 1$, we find from (4), (1), and (2), respectively, $x_4 = 0$, $x_1 = 1$, $x_2 = 1$. Equations (3) and (5) then both become $x_3 + x_5 = 1$. Under these conditions equations (6) to (10) are satisfied identically, and x_3 and x_5 are not determined.

Also solved by Elijah Swift, Pauline Sperry, C. C. Yen, and Herbert N. Carleton.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas.

DISCUSSIONS.

I. Concerning Direction Cosines and Hesse's Normal Form.

By Maxime Bôcher, Harvard University.

For many years I have felt that Hesse's form for the equation of a line is out of place in a course on elementary analytic geometry[1] and have omitted it from my own course. I am glad to find the same view expressed by two writers in recent numbers of the Monthly (p. 476, 1917 and p. 181, 1918) who give proofs of the formula for the distance from a point to a line which do not involve the use of Hesse's form. Perhaps I may be permitted to call attention to a proof of this kind which I have published elsewhere[2] and which seems to me to have two advantages over those recently given in the Monthly, namely, *first,* that it requires no reference to the figure with the consequent necessity (if we wish to be complete, that is, really to give a *proof*) of examining exhaustively the various forms the figure may take on; and *secondly,* that it admits of immediate extension to the corresponding problem in solid analytic geometry.

It is perhaps not quite so clear that Hesse's form for the equation of a plane in solid analytic geometry should be removed from the pages of elementary text books or relegated to small-type sections, since this form is commonly used not merely in connection with the formula for the distance from a point to a plane but also as the primary form of the equation of a plane from which the other forms are deduced. I will indicate how I am accustomed to dispense with it altogether, and at the same time to relegate the too pervasive direction cosines

[1] It has its definite, though very limited, place in higher work.

[2] Bôcher, *Plane Analytic Geometry*, Henry Holt & Co., 1915, p. 38.

of the ordinary treatment to the subordinate position in which I think they belong.

The direction of a line in space is determined by the ratios $l : m : n$ of the projections of any segment of this line on the coördinate axes. These ratios are called the *direction ratios* of the line. They are the natural generalization of the slope of a line in plane analytic geometry. These direction ratios may, of course, be regarded as the ratios of the direction cosines,[1] and, conversely, the direction cosines may be expressed in terms of the direction ratios by means of the formulæ

$$(1) \quad \cos\alpha = \frac{l}{\sqrt{l^2+m^2+n^2}}, \quad \cos\beta = \frac{m}{\sqrt{l^2+m^2+n^2}}, \quad \cos\gamma = \frac{n}{\sqrt{l^2+m^2+n^2}}.$$

If two lines have the direction ratios $(l_1 : m_1 : n_1)$ and $(l_2 : m_2 : n_2)$ respectively, the condition for parallelism is

$$l_1 : m_1 : n_1 = l_2 : m_2 : n_2,$$

and the condition for perpendicularity is

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

The line connecting the points (x_1, y_1, z_1) and (x_2, y_2, z_2) evidently has as its direction ratios $(x_2 - x_1 : y_2 - y_1 : z_2 - z_1)$.

By the direction ratios of a plane we understand the direction ratios of its normals.

It now easily follows that the equations of the line through (x_1, y_1, z_1) with direction ratios $(l : m : n)$ are

$$x - x_1 : y - y_1 : z - z_1 = l : m : n,$$

which, if none of the quantities l, m, n are zero may be written

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

On the other hand, if we have a plane through (x_1, y_1, z_1) with direction ratios $(l : m : n)$, the point (x, y, z) lies in this plane when and only when the line connecting it with (x_1, y_1, z_1) is perpendicular to the normals to the plane. Consequently the equation of the plane is

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0.$$

From this it is easy to prove that every equation of the first degree

$$Ax + By + Cz + D = 0$$

represents a plane whose direction ratios are $(A : B : C)$. The further formulæ concerning lines and planes are now easily obtained.

[1] Except in the case of isotropic lines which have no direction cosines but do have direction ratios.

The method of presentation here sketched not only dispenses with Hesse's normal form entirely, but is to my mind simpler and more appropriate than the one ordinarily given. Direction cosines can be introduced by means of (1) whenever they are wanted, but there are very few occasions where there is any advantage in using them, the formula

$$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2$$

for the cosine of the angle between two lines being one of the few formulæ which are a little simpler when direction cosines are used than when written in terms of direction ratios.

As an introduction to more advanced work the method here suggested has the advantage of being applicable to isotropic lines and planes as well as to others.

II. Concerning a Method of Constructing the Graph of an Equation in Which the Variables May Be Separated.

By E. L. Rees, University of Kentucky.

If an equation can be put in the form $\varphi(x) = \psi(y)$, *i. e.*, if the variables can be separated, the graph may be constructed in the following simple manner: Construct the graphs of the equations $y' = \varphi(x)$, $x' = \psi(y)$. Draw an ordinate of the first curve and an abscissa of the second curve equal to the ordinate of the first. The intersection of this ordinate and abscissa is a point on the required curve. In this manner any desired number of points may be rapidly constructed.

The following theorems[1] giving the properties of the graph as determined from those of the auxiliary curves are easily deduced. Indeed, the truth of most of them may be inferred from geometric intuition.

1. If x' and y' are both increasing or both decreasing functions in corresponding intervals the curve will rise in those intervals.

2. If one function increases and the other decreases the curve will fall in the corresponding intervals.

3. A vertical line of symmetry of the first or a horizontal line of symmetry of the second auxiliary curve is a line of symmetry of the required curve.

4. A turning point (or a multiple turning point) on either auxiliary curve in general corresponds to a turning point (or multiple turning point) on the required curve.

5. Two corresponding turning points give a node if both are maximum or both minimum, or a conjugate point if one is maximum and the other minimum.

6. A double turning po nt and a corresponding turning point give a cusp.

7. A triple turning point and a corresponding turning point if both are maximum or both minimum give a point of osculation.

[1] In the statement of these theorems it is to be understood that turning points and multiple turning points on the second curve are those points at which the tangent is vertical, while these terms as applied to the first curve have their usual meaning.

Other theorems of a similar nature could be added. These theorems may be illustrated by the following simple equations, the numbers in the parentheses indicating the theorems which the equations are intended to illustrate. $y^2 - 1 = x^3 - 3x$, $y^2 - 1 = x^3$(1, 2, 3, 4); $y^2 - 2 = x^3 - 3x$, $y^2 + 2 = x^3 - 3x$ (5); $y^2 = x^3$ (6); $y^2 = x^4 - x^6$ (7).

It is, of course, true that the advantages of this method of constructing graphs are greatest for the more complicated forms of equations.

This principle may also be applied in the construction of graphs of parametric equations. For, if the parametric equations are $x = \varphi(t)$, $y = \psi(t)$ we may use as our auxiliary equations $x = \varphi(y')$, $y = \psi(x')$ and proceed as before.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

In so far as information is obtainable the December issue of this MONTHLY will contain a directory of undergraduate mathematics clubs which have been in operation during at least a part of 1917–18. Certain statistics, and general discussion of club interests, will be introduced also. It is hoped that those clubs which have either failed to discover themselves to the editor or neglected to reply to his repeated solicitations for information, will report to him as soon as possible and, in any case, not later than October 6.

CLUB ACTIVITIES.

MATHEMATICS CLUB OF IOWA STATE TEACHERS COLLEGE, Cedar Falls, Iowa.

This club was organized in December, 1909, "to further the work in mathematics in the college and in the state." The officers for 1917–18 were: President, Ira S. Condit, head of the department of ma hematics; secretary, Irma Hemphill '18; executive committee, Professor Peter Luteyn and Grace Hillier '18, who prepare programs and decide on dates of meetings The programs between June, 1916, and March 1918, were as follows:

June 28, 1916: "Teaching factoring" by Professor Robert D. Dougherty.

July 12: "Standard tests" by Professor E. E. Watson, of Parsons College.

August 2: "The problem and the process of analysis in arithmetic" by Professor Condit.

August 16: "Ratio and proportion," by Professor Charles W. Wester of Iowa State College and Rodney W. Babcock, instructor in mathematics at the University of Wisconsin.

October 18: "The place of mathematics in education" by Professor Wester.

November 15: Report of Committee on Elimination, of Iowa State Teachers Association, by Ruth Smith '18 and Tracey Hodson '18.

April 24, 1917: "Methods of statistical study" by Professor Wester.

May 16: Conservation in mathematics—"Conservation of time and energy by the development of algebraic symbolism" by Irma Hemphill '18; "The

metric system" by Alice O'Connor '18; "Short cuts in arithmetic" by Mae Howell '18; "Short cuts in high school mathematics" by Doris de Bar '18; "Conservation in geometry" by Ruth Smith '18.
July 11: "Elimination in mathematics" by Professor Emma L. Lambert.
July 25: "Elimination in mathematics" by Professor Luteyn.
March 6, 1918: "Contents of a course in arithmetic" by Professor Wester.

JUNIOR MATHEMATICS CLUB, University of Minnesota, Minneapolis, Minn.

The first meeting of this club was held on December 9, 1914. All students who have completed the course in integral calculus, and seniors who are majoring in mathematics, are eligible for membership. The staff of instruction in the department of mathematics and astronomy is also invited to attend the meetings of the club. The number of its members was 20 last year and the average attendance at meetings was 15.

Officers, 1917–18: President, Jemima Olson '18; secretary, Elizabeth Carlson Gr. These officers constitute the program committee also.
November 8, 1917: "The drawing of a circle having five times the area of a given circle" by Jemima Olson '18.
November 22: "How to draw a straight line without the use of a ruler" by Adelina Anderson '18.
December 6: "The Russian peasant method of multiplication" by Elma F. Hario '18.
December 13: Joint meeting with the University Mathematical Club[1]—"A generalized form of mathematical induction" by Professor William H. Bussey; "A note on the eight queens problem"[2] by Professor Bussey.
January 17, 1918: "The reduction of a quadratic form to the sum of squares" by Vera Wright Gr.
March 7: "Magic squares" by Hildegarde Swenson '18.
March 28: "Wallace lines" by Roger A. Johnson, Professor of Mathematics at Hamline University, St. Paul, Minn.
April 10: "Calculating machines" by Ralph M. Barton, instructor in mathematics, and Professor Bussey.
April 18: Annual banquet.
April 24: "The educational value of the history of mathematics" by William O. Beal, assistant astronomer to the university.

MATHEMATICS CLUB OF MOUNT HOLYOKE COLLEGE, South Hadley, Mass.

This club, which has just completed the tenth year of its existence, was founded for the purpose of "presenting to its members a broader view of mathematics." Last year it had 38 active and 15 associate members, the average attendance at

[1] This club is composed of faculty members and graduate students. Its programs are usually of too advanced a character for members of the Junior Mathematics Club.

[2] *Cf.* W. W. R. Ball, *Mathematical Recreations and Essays*, fifth edition, London, Macmillan, 1911, pp. 113–118.

meetings being 20. Junior and senior students who are doing major work in mathematics are eligible for active membership and such students become members by signing the constitution. Any sophomore who has elected mathematics as one of her major subjects may become an associate member by a majority vote of the members present at any meeting. Any member of the staff of the mathematics department and any graduate student in mathematics is eligible for either active or associate membership. Only active members may vote or hold office. Active members are charged a tax of fifty cents a year.

Officers 1917–18: President, Alice Weeks '18; vice-president, Evelyn Clift '19; secretary-treasurer, Jeannette Bickford '18; executive board: Ruth Carpenter '18, and Professor Emilie N. Martin.

October 21, 1916: Social meeting; the club was entertained by guessing various mathematical puzzles.

November 17: "Mathematical theory of probabilities and its application to games of chance" by Ruth Carpenter '18; "Magic squares" by Alice Weeks '18.

December 16: "Poincaré" by Helen Hughes '17; "Fermat, Fermat's Theorem, and Painlevé" by Florence Allen '17; an article on Sonja Kovalevski read by Ethel Anderson '19.

January 16, 1917: A talk on the models of surfaces owned by the college, and an exhibition of photographs of snow crystals by Professor Martin.

February 17: "Linkages" by Margaret Wilcox '19 and Evelyn Clift '19 (illustrated by models which they had constructed for the occasion); selections from Cassius Keyser's "Essays and Addresses"[1] read by Ruth Hemenway.

March 24: "Continued fractions" by Jeannette Bickford '18.

April 21: "Photogrammetry and its use in the present war" by Charles L. Bouton, associate professor of mathematics at Harvard University.

May 22: Election of officers for 1917–18.

In 1917–18 the number of meetings was reduced to five on account of war-work in which the members were engaged.

September, 1917: Social meeting to receive new members.

November: "Training in mathematics in a Russian artillery school" by Mr. Alexander Pell of South Hadley.

February 16, 1918: "Mathematics in the present war" by Captain Peter F. Field, professor at the University of Michigan.

March 9: "Applied mathematics in the laboratory work of a great electric company" by Agnes Eastman '13.

May 25: "Mathematics of insurance" by Helen Hughes '17; election of officers for 1918–19.

THE MATHEMATICAL CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Neb.

This club was organized in 1915 "to stimulate the mathematical interests of its members by presenting illuminating and varied phases of mathematics in

[1] The selections were taken from two essays: "The human worth of rigorous thinking" and "The human significance of mathematics."

pure and in applied form." According to the preliminary organization circular: "Many absorbing topics that can hardly be dealt with in the class room will find a natural place in this club. It is hoped that the members will gain a new and valuable insight into the beauty and wealth of the mathematical realm and a greater knowledge of the many ways in which mathematics touches life."

The charter members were recommended by the faculty. Thereafter members have been elected by the club on the recommendation of its executive committee which, as a matter of established custom, first consults the faculty. "As a rule the faculty recommends to membership those students who have shown distinction in their first year mathematics courses (through analytic geometry) "

Officers, 1917–18: President, Julia L. Torrence '18; vice-president, Eimo G. Funke '19; secretary-treasurer, Frances Botkin '19; faculty-adviser, Professor Henry Blumberg. These officers constituted the executive committee.

The club meets once a month, 7.30–9 00 p.m.; the number of members in attendance has varied from 20 to 60, with an average of about 35. All members are expected to contribute to the topics of the evening. Interesting problems are posted on the club's bulletin board and these problems are later discussed at the meetings. During 1916–17 three prizes were awarded by the club, mainly for activity in connection with the bulletin board. The programs for 1915–18 have been as follows:

October 14, 1915: Organization meeting; talks by different members of the faculty.

November 11: "Various definitions of π" by James H. Taylor '16; "The game of nim" by Professor Blumberg.

December 9: "Some problems from analysis situs" by Edward M. Kadlack '16; "Selected facts from the theory of numbers" by Professor William C. Brenke.

January 13, 1916: Social meeting, log fire, refreshments, everybody expected to bring an anecdote about a famous mathematician.

February 10: "Cantor's famous contribution to the study of the infinite" by Professor Blumberg; "The cycloid" by Herbert Grumman '15.

March 9: "Euclid's fifth postulate" by Olive Bayles '16; "Soap bubbles and mathematics" by Professor Oliver Gish.

April 13: "Lightning calculators" by Ezra Andreson '18; "The method of Archimedes" by Professor Albert L. Candy.

May 11: "Some unsolved problems in mathematics" by Professor Ellery W. Davis; social hour.

October 12: Election of officers; social hour.

November 9: "Summation of certain series" by Walter F. Weiland '18; "History of perspective drawing" by Professor Blumberg.

December 14: "Certain problems of interpolation" by Professor W. C. Brenke; bulletin board problems.

January 11, 1917: Social meeting; mathematical games; everybody expected to bring a geometric, algebraic or logical paradox.

February 15: "History of logarithms" by William F. Joachim '18.

March 15: "Selected topics from the theory of probability" by Professor Davis; bulletin board problems.

April 15: "The roots of unity" by Alva L. Sikes '19; "History of the Arabic numerals" by Professor Candy.

October: Election of officers; social hour.

November 9: "Selected topics from projective geometry" by Frances Botkin '19; "Mathematical curricula in different countries" by Professor Candy.

December 13: "Magic squares" by Josefa Seely '18; "Graphical methods" by Professor Lulu Runge.

March 14, 1918: Bulletin board problems; social hour with mathematical games.

April 11: "Selected facts from the theory of numbers" by Professor Blumberg; bulletin board problems; mathematical games.

May 9: "Development of mathematical symbols" by Bernice Downing '18; bulletin board problems; social hour with mathematical games and mathematical conundrums.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OKLAHOMA, Norman, Okla.

At a meeting on October 26, 1916, a group of students, who had decided to organize a mathematics club, appointed a committee, with Professor Samuel W. Reaves, head of the department, as chairman, to draw up a constitution. This constitution, adopted at the following meeting, was as follows:

PREAMBLE.

We, the undersigned, appreciating the advantages to be derived from an association which shall give opportunity for the presentation and discussion of mathematical subjects of interest, do hereby organize ourselves into a mathematical club, and we agree to be governed by the following Constitution.

ARTICLE I. *Name.*

Section 1. This association shall be called The Mathematics Club of the University of Oklahoma.

ARTICLE II. *Members.*

Section 1. Membership in this Club is limited to such students and teachers in the University of Oklahoma as are interested in the subject of mathematics.

Section 2. Proposals for membership shall be in writing and may be submitted at any regular meeting.

Section 3. Voting upon members shall be by ballot, a majority vote being necessary for election.

ARTICLE III. *Officers.*

Section 1. The officers of this club shall be a President, a Vice-President, and a Secretary-Treasurer, which officers shall be elected from among the students majoring in mathematics.

Section 2. Officers of the club shall be elected by ballot at the first regular meeting of each semester.

Section 3. Each officer shall serve until his successor is duly elected. Vacancies may be filled temporarily by presidential appointment until the next regular meeting of the club.

Section 4. The three officers mentioned in Section 1, together with one faculty member selected by the faculty members of the club, shall constitute the Program Committee.

ARTICLE IV. *Meetings.*

Section 1. Regular meetings of the club shall be held on the second and fourth Thursdays of each month.

ARTICLE V. *Miscellaneous.*

Section 1. This club shall have the power to make rules for its meetings, levy assessments upon its members, and perform other acts not inconsistent with this constitution.

Section 2. One-third of all the members of the club shall constitute a quorum.

Section 3. Amendments to this constitution shall be offered in writing, shall lie upon the table for two weeks, and shall require for adoption a two-thirds vote of all members present.

Section 4. In all cases not otherwise provided for this club shall be governed by "Roberts's *Rules of Order*" as parliamentary guide.

During 1916–17 the following meetings (with an average attendance of 15 members) were held:

November 9, 1916: "How we have learned to count" by Dr. Nathan Altshiller, instructor in mathematics.

December 14: "Finger counting" by Harold Gimeno '18; "Finger calculation" by Enoch B. Ferrell '18.

January 16, 1917: "Mathematical wrinkles"[1] by Thomas L. Sorey '18 and Hugh S. Lieber '18.

February 13: "Some properties of triangles" by Professor Reaves.

March 13: "Paper folding" by Ella Mansfield '18.

March 27: "Life of Descartes" by Earl Bonham '17.

April 10: "Invention of logarithms" by Professor Edmund P. R. Duval; "If at the beginning of our era one cent had been placed at four per cent compound interest, what would be the radius of the gold sphere equivalent to the capital accumulated to date?" by Margaret Coleman '17.

April 17: "Mathematics and astronomy" by Professor Harry C. Gossard.

"Owing to the fact that most of the advanced students in mathematics, including both presidents of the club (Thomas L. Sorey '18 and Enoch B. Ferrell '18), entered military service, while the others left college, to fill the vacancies created by the military draft, it was decided to suspend the activities of the club for the year 1917–18."

TOPICS FOR CLUB PROGRAMS.

12. GEOMETRY OF FOUR DIMENSIONS.

By HENRY P. MANNING, Brown University.

If but a single meeting is to be devoted to this subject the most interesting way to take it up is to give a simple statement of theorems and proofs after the manner in which plane and solid geometry is studied in the schools. Most of those who belong to mathematics clubs have heard tales of strange things that are to be found in hyperspace, and what they would like to know is the geometry itself and the mysterious paths by which these things are found. Probably the best topics to take up would be perpendicularity and various kinds of angles, and then, if there is time, some of the theorems about parallel lines, planes and hyperplanes.[2]

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[1] A book entitled *Mathematical wrinkles for teachers and private learners* was published by Samuel I. Jones at Gunter, Texas, in 1912.

[2] A list of theorems as I have taken them up on two or three occasions is given in *The Mathematics Teacher*, Vol. 7, Dec., 1914, pp. 49–58.

Such a treatment ought to be the central or main part of any study of this subject, but if there is time to look at it from other points of view the following topics (arranged under 7 headings) may be considered.[1]

1. The history of the idea of a fourth dimension. It will be convenient to think of three periods: down to 1827, from 1827 to 1870, from 1870 to the present day.

In the writings of Aristotle (384–322 B.C.), Pappus (end of the third century), and, according to Simplicius (sixth century A.D.), of Ptolemy (about 150 A.D.) there are speculations about the number of dimensions of space and the possibility or impossibility of there being more than three. The same matter is referred to by the English philosopher Henry Moore (1614–1687), and by Kant (1724–1804).[2]

In the development of algebra, which at first ascribed a geometrical meaning to all of its terms, there came about a reluctant extension of these terms beyond those which describe figures of two and three dimensions.[3]

Finally, there is a suggestion by d'Alembert (1717–1783), and after him by Lagrange (1736–1813), that time might be regarded as the fourth dimension.[4]

In the second period certain leading mathematicians took up particular parts of hypergeometry and proved theorems or developed formulæ.[5] These developments were mostly by analytical methods, but sometimes synthetic methods were used, and in them all the geometrical conceptions were made quite clear. There was also, however, much work done in pure analysis with equations and algebraic forms in n variables, where the language of geometry is sometimes used, but the conceptions of geometry are almost entirely absent.[6]

[1] Most of the material for this work will be found in the author's *Geometry of Four Dimensions*, Macmillan, New York, 1914, and in a book of fourth dimension essays, *The Fourth Dimension Simply Explained*, Munn, New York, 1910. These will be referred to as *Geometry* and *Essays*.

[2] Pappus, after speaking of the three-line and four-line locus (see Heath's *Apollonius*, Cambridge, 1896, Introduction, chap. v) says that we cannot say of more than six lines, "the ratio of the content of four to that which is contained by the rest, since there is nothing contained by more than three distances. Men a little before our time have allowed themselves to interpret such things, signifying nothing at all comprehensible, speaking of the product of that which is contained by such lines into the square of this or the content of those. These things might, however, be stated and shown generally by means of compound ratios"—Hultsch ed., Vol. II, Berlin, 1877, p. 680. For other references see *Geometry*, pp. 1–3.

[3] *Geometry*, pp. 2–3.

[4] *Geometry*, p. 4; R. C. Archibald, "Time as a Fourth Dimension," *Bulletin of the American Mathematical Society*, Vol. 20, 1914, p. 409.

[5] Möbius, *Der barycentrische Calcul*, Leipzig, 1827, § 140, p. 184; Cayley, "Sur quelques théorèmes de la géométrie de position," *Crelle's Journal*, Vol. 31, pp. 213–226, in particular, pp. 217–218; Sylvester, "On the Centre of Gravity of a Truncated Triangular Pyramid," *Philosophical Magazine*, fourth series, Vol. 26, Sept., 1863, pp. 167–183 or *Mathematical Papers*, Vol. II, No. 65; Clifford, *Educational Times*, Jan., 1866, or *Mathematical Reprints*, Vol. 6, pp. 83–87, or *Mathematical Papers*, London, 1882, p. 601.

[6] Green, *Mathematical Papers*, edited by N. M. Ferrers, London, 1871, p. 188; C. G. J. Jacobi, "De binis quibuslibet functionibus homogeneis," *Crelle's Journal*, Vol. 12, 1834, p. 1; Cayley, two papers published in the *Cambridge Mathematical Journal*, Vol. 3, 1841, or *Mathematical Papers*, Vol. I, Nos. 2 and 3; Schläfli, (1) "Ueber das Minimum des Integrals," etc., *Crelle's Journal*, Vol. 43, 1852, pp. 23–36; (2) "On the Integral," etc., *Quarterly Journal*, Vols. 2 and 3, 1858–1860.

For further details see *Geometry*, pp. 4–8.

Since 1870 we have witnessed a very rapid growth of the subject. References are given in the *Geometry* to Veronese, Halphen, Jordan, Stringham, Hathaway, Cole (p. 142), Keyser, Craig, and others. Nearly every meeting of a mathematical society has some paper, and nearly every volume of a mathematical journal contains some article bearing on this subject.

2. The many curious phenomena in space of four dimensions make a description of this subject especially interesting, even to those who do not care for mathematics. An account of these phenomena after the manner of the numerous "non-mathematical" articles that have been written would furnish an entertaining paper for a mathematics club. It is easy to write such a paper because almost everything in this geometry is analogous to things of plane and solid geometry. In fact, the mathematician in developing and establishing its theorems is all the time consciously or unconsciously using these analogies. Even the proofs are often analogous to the proofs of the lower geometries, although the analogy itself is never a proof.[1]

3. The analogy with two-dimensional beings has been used very freely to show our relation to four-dimensional space.[2] In particular, this idea has been developed in the form of a romance by E. A. Abbott, in *Flatland,* and by C. H. Hinton in *An Episode of Flatland.* A report on these books is sometimes made the subject in a meeting of a mathematics club. The former has also been dramatized.[3]

The question whether anyone is able, or can acquire the ability, to form a conception of space of four dimensions may depend partly on what we mean by the word conception. It is certainly possible by practice to acquire a facility in describing figures of such a space and in stating and proving the theorems of this geometry.[4]

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[1] The following are some of the phenomena referred to. They are explained many times in the *Essays* and in all articles of a similar nature. The new perpendicular direction, *Essays,* pp. 45, 71, 82, 122. Passing out of a closed room, *Essays,* pp. 116, 123, 138, 149, 180, 188. Making symmetrical figures coincide by turning one of them over through hyperspace, *Essays,* pp. 48, 76, 95, 139, 158, 213. Turning a flexible hollow shell inside out, *Essays,* pp. 49, 170, 249. Knots, links of a chain, *Essays,* pp. 50, 145. Planes meeting only in a point, going around a plane, *Essays,* pp. 23, 24, 49. The hypercube, *Essays,* pp. 46, 72, 88, 92, 113, 147. Cutting apart a polyhedroid and spreading it out in a hyperplane, *Essays,* p. 74, *Geometry,* pp. 68, 238, 240, 248. Some of these subjects are mentioned by Professor Newcomb, "Philosophy of Hyperspace" (presidential address), *Bulletin of the American Mathematical Society,* Vol. 4, Feb., 1898, pp. 187–195; "Fairy land of Geometry," *Harper's Magazine,* Vol. 104, Jan., 1902, pp. 249–252, reprinted in *Sidelights on Astronomy,* New York, Harper and Brothers, 1906, pp. 155–164.

There is a story by Charles Johnstone, "Jones and the Fourth Dimension," *Harper's Magazine,* Vol. 129, June, 1914, pp. 117–122, that is rather amusing, though not quite accurate in its representation of a lack of distance in space of four dimensions.

[2] Essays, pp. 78, 81, 96, 115, 127, 155.

[3] Abbott, *Flatland,* London, 1884, Little, Brown and Company, Boston, 1907. Hinton, *An Episode of Flatland,* London, Swan, Sonnenschein and Co., 1907. In the *Mathematical Gazette,* Vol. 7, Jan., 1914, pp. 228–231, is an account of a dramatic performance of *Flatland* by a girls' school in Acton, England. This is reprinted in *School Science and Mathematics,* Vol. 14, Oct., 1914, pp. 583–587.

[4] See discussion of this question by C. J. Keyser, "Mathematical Emancipations," *Monist,* Vol. 16, 1906, pp. 65–83, particularly, pp. 81–82; reprinted in *The Human Worth of Rigorous Thinking,* New York, Columbia University Press, 1916, pp. 101–121. See also *Geometry,* pp. 15–16 and the last footnote on p. 9.

4. If the various topics suggested are taken up and several meetings are devoted to the geometry of four dimensions, the treatment of the geometry itself by synthetic methods as explained at the beginning of this article should come at this point.

5. The analytical geometry of four dimensions with formulæ for distances and angles of various kinds, the equations of hyperplanes, planes and lines, and a brief account of the hypersurfaces of the second degree, the hyperconicoids.[1]

6. Some particularly puzzling subjects may be suggested for further study.

Kinematics and mechanics of four dimensions promises to be very interesting. The cross section of a rope would be a sphere, not a circle. It would have to be fastened to a spherical ring, not to a circular ring. A chain could not be made of links, but alternate links and hollow spheres could be put together to form a chain. Instead of a two-dimensional wheel and a one-dimensional axle we must have a three-dimensional wheel, or a two-dimensional axis.[2]

If we lived on the three-dimensional boundary of a hypersphere, just as we actually live on the earth, held to it by an attraction towards the center, we could move in three directions. We might study the problem of changing pressure into motion (like that of the locomotive), or the problem of constructing a mill that will saw a hypersolid block into two pieces, just as we saw a log into boards.

Surfaces in hyperspace is another subject that needs to be studied a great deal. A curve in any kind of space extends only forwards and backwards and seems very much the same as in a plane as long as we confine our attention to the curve itself. It is only when we consider the space about the curve that we find in space of three dimensions the phenomena of knots and of torsion. So surfaces, considered by themselves, are very much the same in any space. As long as we confine our attention to the surface itself we find it two-dimensional and we move about on it very much as we do on a simple plane. There is no difficulty in understanding at least a small portion of an ordinary surface in any kind of space. But when we investigate the space about the surface, if the space is of more than three dimensions, we begin to be mystified. Rotation on an axis-plane and the

[1] This treatment is taken up by Jouffret, *Géométrie à quatre dimensions*, Paris, 1903, and (for the case of any number of dimensions) by Schoute, *Mehrdimensionale Geometrie*, *Sammlung Schubert*, XXXV and XXXVI, Leipzig, 1902 and 1905, I, § 6, pp. 125–180. Certain parts are worked out also in the following articles: F. N. Cole, "On Rotations in Space of Four Dimensions," *American Journal of Mathematics*, Vol. 12, 1890, pp. 191–210; T. Craig, "Displacements Depending on One, Two and Three Parameters in a Space of Four Dimensions," *American Journal of Mathematics*, Vol. 20, 1898, pp. 135–156; J. G. Hardy, "Curves of Triple Curvature," *American Journal of Mathematics*, Vol. 24, 1902, pp. 13–38; C. J. Keyser, "Concerning the Angles and the Angular Determination of Planes in 4-Space," *Bulletin of the American Mathematical Society*, Vol. 8, 1902, pp. 324–329. Two articles in the AMERICAN MATHEMATICAL MONTHLY may be mentioned here: B. H. Brown, "Centres of Similitude and their N-Dimensional Analogies," Vol. 23, May, 1916, pp. 155–159, and M. H. Sznyter, "Some Metrical Properties of the Pentahedroid in a Space of Four Dimensions," Vol. 24, Mar., 1917, pp. 113–119.

[2] See *Essays*, p. 31, Hinton, *The Fourth Dimension*, Swan, Sonnenschein and Co., London, 1904, pp. 31 and 71–73. The nature of rotations is taken up briefly at the end of an article by Beltrami, "Formules fundamentales de cinematique dans les espaces de courbure constante," *Bulletin des sciences mathématiques*, Vol. 11, 1876, pp. 233–240, *Opere*, Milan, Vol. III, 1911, pp. 23–29, and very fully by Cole, *loc. cit.*

fact that we can go around a surface just as in ordinary space we go around a line make one of its mysteries. The consideration of an entire surface that does not lie in any one three-dimensional space also involves peculiarities of hyperspace. An example is the surface of double revolution, a surface on which the ordinary plane geometry of Euclid holds true if we take for lines certain systems of circles. The study of surfaces is important in what is called analysis situs.[1]

7. The object and value of the study of this subject may be considered from three points of view: any direct applications that it may have, its relation to the science of geometry and to mathematics in general, and any philosophical discussions that it may involve.

We can use it to give simple proofs of certain theorems of ordinary geometry, such as the equivalence of symmetrical figures,[2] and theorems about certain figures that are sections or projections of simpler figures of hyperspace.[3] New figures have also been discovered in this way. Thus application has been made to architecture.[4] In mechanics it has been found useful to represent time as a fourth dimension. In this way the theory of relativity finds its simplest form of expression.[5] It can be used sometimes to give us graphical solutions of problems[6]. It helps us to visualize certain complicated groups of objects, such as determinants of more than three dimensions.[7] There are many interpretations of this geometry in which some other figure than the point is taken as element. These interpretations have been put forward to help us understand the geometry, but the relation can be reversed and the geometry applied to the study of these figures.[8] Its most important direct application lies in its furnishing names and concrete terms for certain complicated ideas and expressions of analysis.

In the next place we get a broader view of the nature of geometry by a study of geometry of four dimensions, and, as far as we can carry it, of higher dimensions. There are many theories in geometry that we hardly notice, if at all, in studying only plane and solid geometry. We understand better the foundations of geometry and we get an invaluable training in mathematical reasoning.[9]

Finally, the study of this geometry throws light on certain questions of philosophy, as, for example, the nature of space itself. The fact that we are able to develop other kinds of geometry besides the plane and solid geometry of our experience is put forward as disproving the theories of Kant on the subjective nature of our geometrical intuitions.

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[1] See *Geometry*, p. 219 and the reference there to Poincaré.

[2] *Geometry*, p. 149.

[3] See *Geometry*, p. 5, and the references there to Cayley and Veronese.

[4] Claude Bragdon, *Projective Ornament*, Manas Press, Rochester, N. Y., 1915.

[5] *Geometry*, p. 11.

[6] An example is Clifford's solution of a problem in probability referred to in *Geometry*, p. 5, and in footnote above.

[7] See abstract of paper by L. H. Rice, "Determinants of Many Dimensions," *Bulletin of the American Mathematical Society*, Vol. 23, 1916, p. 69.

[8] *Geometry*, p. 10.

[9] *Geometry*, p. 14.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

The chief aim of this department will be to gather and publish information concerning mathematical courses in preparation for war service. The editor invites the contribution of suitable articles and notes, and hopes, since timeliness will be important, that appropriate material will be sent in as soon as it becomes available. If an article is such that the Editors of the MONTHLY deem its early publication sufficiently pressing, preprints of it will be prepared. The editor of this department will also undertake to furnish to the extent of his means material and information desired by individual readers. Address Dr. Henry Blumberg, University of Illinois, Urbana, Ill.

COURSES IN COLLEGE IN PREPARATION FOR THE NAVY.[1]

By R. G. D. RICHARDSON, Brown University.

The Need. To the colleges no call of the country in its war program is more urgent than that of the navy for commissioned officers. Besides the need of line officers for the new ships of the navy proper, the department has announced that it needs 22,000 commissioned officers to man the new mercantile marine which the navy will operate. Very few enlisted men have the education necessary to qualify for anything beyond the rank of petty officer. The Naval Academy can graduate at most a few hundreds yearly. The many auxiliary schools are filled beyond capacity but are turning out only a fraction of the number required. While a minimum of three to six months' sea experience has been a prerequisite, a large part of the theoretical training for candidates for the ensign's commission has in many instances been given in academic institutions. Since all candidates for commission as ensign must have the preliminary education required for admission to college, it is natural that the navy should look to us in increasing measure to supplement the efforts of the other agencies by giving courses which will aid in preparing men for this examination.

All college men who are physically fit must look forward to being called by their country to some arm of the service, and every college recognizes the imperative necessity of giving its students some technical preparation. Many men will prefer the navy to other branches of the service, and it devolves upon each school to ascertain how it can give the elements of the necessary training by adapting its present courses or adding new ones. This will in part anticipate any action which the government may deem it wise to take in directing the instruction in mathematics in those colleges whose students are furloughed for instruction.

What is Already Being Done. The experiences of other institutions furnish

[1] This brief sketch has been prepared very hastily and is subject to charges of inadequateness and inaccuracy. Its only excuse is that it may serve to call attention to the needs before definite programs for the academic year are made up. R. G. D. Richardson, Chicago, August 29 1918.

a basis for judgment as to what may be accomplished. Various types of work are already under way of which some examples may be cited.

Several institutions such as Harvard and Pennsylvania have loaned to the government their facilities for instruction and housing. Enlisted men who have the necessary basis of education and are deemed to possess the requisite personal qualifications are given training by officers of the navy to fit them for an ensign's commission or for the radio service.

In another group are institutions such as Yale, Princeton, Brown and the Universities of Michigan, California, and Washington which have regularly organized Naval Units. The formation of such Units is encouraged by the Navy Department but in most instances no official recognition has as yet been given. The units differ widely in the work they are attempting. While some students in a few engineering schools are being trained for the engineering branch, the greater number are headed for commissions in the line. In some cases a retired officer is assigned to service as head of the Unit and petty officers detailed to take charge of the drills. The greater part of the theoretical instruction, however, is given by members of the regular faculty. Candidates for commissions in the line take courses in Navigation, Seamanship, Navy Regulations, Ordnance, and Gunnery. Among the drills are day and night signalling of all kinds, handling of boats under oars and sail, School of the Company, Manual of Arms, School of the Section and Battery in Artillery, and Battleship Drills. Men who have enrolled in the Naval Reserve are given opportunity for practical work by means of week-end cruises or summer service on battleships or converted yachts.

Besides these two types of instruction, many institutions are adapting present courses or are introducing new ones with a view to giving preliminary training for future officers. Others are coöperating with outside agencies by furnishing mathematical instruction. The University of California supplements courses given to its undergraduates by courses in extension covering part of the same field as the naval units. During the present summer members of the Faculty of the University of Chicago and Northwestern University have been giving courses for those men, enlisted in the mercantile branch of the service, who have not yet been called to the school at Municipal Pier, Chicago. The Commandant of the station has detailed petty officers to supplement by drills the theoretical instruction in mathematics and astronomy.

Navigation. There seems to be no reason why a course in Navigation should not constitute a part of the mathematical program of every institution. Not only is the present demand very urgent but the new merchant marine will continue to call for officers in large numbers after the present war is won.

Prerequisite to such work is a course in Trigonometry with special emphasis on the derivation of those formulas used in computation and on the computation itself. The first topic to be discussed is the determination of position by Dead Reckoning including the methods of Plane, Mid-latitude and Mercator's Sailings. Double interpolation must be taught in connection with the Traverse Tables to be used in these computations. As a preliminary to

the study of the nautical astronomy involved, the student should possess a knowledge of the elements of spherical geometry and spherical trigonometry.[1] Of the latter the essentials include the formulas for right and quadrantal triangles and the Laws of Sines and Cosines together with the modifications of the latter used in calculating the haversine of a side or the haversine of an angle. [$\text{hav } x \equiv \frac{1}{2} \text{ versin } x \equiv \frac{1}{2}(1 - \cos x)$.] An introduction to some of the fundamental notions of astronomy is also necessary at this point.

Since from the standpoint of the student the most difficult ideas are those grouped about the notion of time considerable emphasis should be laid on this topic. One of the most important problems is the complete determination of the position at sea from altitude observations of celestial bodies. This is best accomplished by finding the intersection of two Sumner lines. A Sumner line or line of position is a small circle on the earth from any point of which the observed body will have the same altitude at the same instant. The center of the circle will be the point which has the observed body in its zenith. Perhaps the simplest application of nautical astronomy is the determination of latitude from a meridian altitude by means of the various corrections. For a meridian altitude the Sumner line will then be tangent to the parallel of latitude, and for a prime vertical observation, it will be tangent to the meridian. The most favored scheme for the reduction of the Sumner line observations is that of St. Hilaire. Out of such a computation comes as an almost immediate by-product the azimuth of the observed body. This gives at once the compass error which is another important problem of nautical astronomy.

The standard texts for this course will be cited in the specific instance below. Many books on navigation have been published within the past four years but most of them give rule of thumb methods rather than a mathematical treatment. A standard British text is by Gill (Longmans, Green & Co.) and from the standpoint of college instruction the last edition is very useful. Among other books to be recommended for the instructor are Martin's Navigation (Longmans, Green & Co.), Hall's Navigation (W. B. Clive, London), Muir's Navigation and Compass Deviations (U. S. Naval Institute, Annapolis, Md., $4.20).

The use of the compass, pelorus, vernier and chronometer and of the sextant with the real or an artificial horizon is necessary and to these may be added other instruments used at sea. It is essential that the student become familiar with the nature of charts of various kinds. Mercator Charts, Great Circle Charts and Position Plotting Sheets can be bought at cost from the Hydrographic Office (a catalog of all the publications of this office may be obtained for 30 c.).

If time is limited, fifty hours in Navigation with twice as much outside preparation on the part of the student will serve as a valuable introduction. In that time the central features can be covered. But it would be preferable to extend the course to ninety hours in order to take up the greater part of the topics covered by Bowditch, to acquire accuracy and facility in computation and

[1] Spherical Trigonometry is now a prerequisite for the naval school at Pelham Bay, L. I.

in handling the instruments, and to learn the use of the Kelvin-Aquino method for solving oblique-angled triangles by tables. The nature of the standard maintained by the Navy Department can be ascertained in this, as in other subjects, by an examination of the papers set for the ensign commission. Copies may be obtained on application to the Bureau of Navigation, Washington, D. C.

Other Courses. Since for the navy the target is generally in view, the problems of Ordnance and Gunnery are quite different from those of the army. From such books as Curtin's Naval Ordnance, Alger's Ground Work of Practical Naval Gunnery (U. S. Naval Institute, Annapolis, $4.85, $4.50) and the Bureau of Ordnance pamphlets the instructor could work up a few lectures on this topic. The elements consist largely in practical applications of arithmetic and trigonometry, and considerable practice in examples is necessary to insure correct understanding of the problems involved and to acquire facility.

The larger part of the training for officers is, of course, non-mathematical. While instructors competent to give extended courses in seamanship are not in general available and while many institutions will not find it practicable to obtain petty officers for the drills, some text-book work could in many cases be given to advantage. Instruction in the reading of charts may be supplemented by Buoy and Light Lists and the Coast Pilot. Charts on large and small scale obtainable from the Coast and Geodetic Survey are useful in laying out courses and in the study of the coast and harbors of some particular locality. Actual inspection of light-houses, buoys, and ships in process of construction would be a valuable auxiliary to instruction from the text. Elementary instruction in practical seamanship may be illustrated by means of models and by practise with small boats under oars and sail. Lectures on marine meteorology are valuable. Beside the Blue Jackets Manual (U. S. Naval Institute, 75 c.), which the men should know from cover to cover, students might be given parts of the U. S. Navy Regulations, Watch Officers Manual (U. S. Naval Institute, $1.10) and an elementary text in piloting, buoys, rules of the road, such as Chapman's Department Notes on Seamanship (U. S. Naval Institute, $1.00) contains valuable supplementary matter. Such work might extend to ninety hours. For more extensive instruction Knight's Seamanship is the standard text. Since company drill in the army is not essentially different from that in the navy (Landing Force Manual, U. S. Navy, 1918, $1.00) it may be substituted when no naval drill is possible. Swimming and naval calisthenics are possible substitutes for the regular gymnasium work.

A Specific Instance. As a specific instance of what is being done, the courses to be given to the Brown Naval Unit during 1918–19 may be cited. An elementary year course in Seamanship (Naval Science A, B, C) based on Chapman's Practical Motor Boat Handling (Motor Boating, New York City, $1.00) and Knight's Seamanship (Van Nostrand, $3.00) is required for the new men. For those who have had this elementary work, a year course (Naval Science D, E, F) is given including further topics in Seamanship, lectures and problems in Ordnance and Gunnery, and lectures by Rear Admiral Edwards, the Com-

mandant of the Unit, on engineering features. Men in this course will serve as officers in the drills. Material for the drills such as cutters, rifles, machine guns and artillery has been supplied by the government. The time devoted to class work together with the outside preparation and the drills (which are required of all) will in each of these courses exceed ten hours per week and three hours credit will be given toward the degree. All men are required to wear a uniform and if eighteen years of age will in all cases be enrolled in the Naval Reserve. Students thus enrolled will be taken for week-end cruises on naval vessels of various types.

A third course given for the Unit three hours throughout the year is Navigation A, B, C. This is open to those who have taken the freshman course in mathematics and by special permission to others who have had Plane Trigonometry. Bowditch's American Practical Navigator (U. S. Hydrographic Office, 1918, Washington, D. C., \$2.25 +) is the text and the student must have the American Nautical Almanac (Nautical Almanac Office, Naval Observatory, Washington, D. C., 15 c. +) and the Altitude, Azimuth and Line of Position Tables (published by Hydrographic Office No. 200, 60 c. +). The first term's work consists of a study of the instruments involved, Dead Reckoning, Spherical Trigonometry and an introduction to Astronomy. In the second term determination of position at sea by observations on celestial bodies is studied, while in the third the remaining parts of Bowditch are covered and the Kelvin-Aquino method learned.

It is probable that for those men who can remain but one or two terms an abbreviated course in Navigation will be given. In that case only the main topics will be covered and the student will not have the opportunity to acquire the desired accuracy and facility.

All the distinctive text-books used at the Naval Academy and others of a similar character are in a special library open to the student.[1] Through the courtesy of the Yale authorities, men from the Brown Unit were invited to attend the Summer Nautical Training School which was held for the Yale Unit at Madison, Conn. In this school the students were under rigid naval discipline and received theoretical and practical instruction in Navigation, Seamanship, Naval Regulations, and Ordnance and Gunnery during the months of July and August.

[1] The names of the text-books used at Annapolis are given in the Annual Register U. S. Naval Academy (Government Printing Office, Washington, D. C.) a copy of which may be obtained on application to the Academy. For the Departments of Mathematics (beyond the calculus), Navigation, Seamanship, Ordnance and Gunnery the following are the texts: H. E. Smith's Analytic Mechanics, Alger's Hydromechanics, Smith's Strength of Materials, Bowditch's American Practical Navigator, White's Astronomy, Nautical Almanac, Azimuth Tables, Muir's Navigation and Compass Deviation, Logan's Marine Surveying, Practical Manual of the Compass, Stockton's Manual of International Law, Knight's Seamanship, The Deck and Boat Book, Signal Book U. S. Navy, Forms Procedure U. S. Navy, U. S. Naval Regulations, Grant's School of the Ship, Alger's Exterior Ballistics, Curtin and Johnson's Naval Ordnance 1915, Range and Ballistic Tables 1915, Gunnery Instructions 1913, Bureau of Ordnance Pamphlets, Official Publications.

The list of books published for camp libraries by the American Library Association Library War Service (Washington, D. C.) contains many titles in navigation and naval engineering, but it is not discriminating.

MATHEMATICAL INSTRUCTION AT THE GREAT LAKES NAVAL STATION.

By I. A. Barnett, Washington University, St. Louis.[1]

The work in mathematics at the Great Lakes Naval Station is at present under the auspices of the Y. M. C. A. as one of the phases of the Association's general educational program. Among the 40,000 enlisted men there were, at the time the writer left the station (August 15, 1918), about 800 studying mathematics. Their previous training ranged from a grammar-school to a college education. Attendance in the Y. M. C. A. classes was entirely voluntary. About half of those who attended were preparing themselves for the competitive entrance examinations of the Ensign School with a view, after four months' stay in this school, to an Ensign's commission. The subjects of examination are Geography, English, American history (with special emphasis on naval history), arithmetic, algebra (through quadratics), plane geometry and plane trigonometry. The remaining half consisted of men in the aviation school who wanted to learn some mathematics as an aid in their required studies, and to a smaller extent, of those who desired chiefly to advance their education.

There are now about 35 Y. M. C. A. classes in arithmetic, algebra and trigonometry. The topics discussed are as follows. Arithmetic (chiefly review, 6 hours including outside preparation): multiplication, division, g.c.d., l.c.m., fractions, decimals, mensuration, foreign money. Algebra (partly review, 15–20 hours): general nature, fundamental operations, linear equations and applications to problems, factoring, fractions, simultaneous linear equations, extraction of square roots. Trigonometry (mainly new, 10–12 hours): the right triangle and applications, logarithms, the general angle; no proofs. Outlines of these courses were mimeographed and distributed.

There are numerous difficulties in connection with the work. The teachers must be selected from the enlisted men, are usually inexperienced, and are not provided with extra time for the preparation of lessons. The classes are heterogeneous. No special rooms are provided for instruction, which is given in the barracks, the Y. M. C. A. huts or the regimental headquarters, the men often sitting on the floor. Usually there is no time for study except between 6:30 P.M. and 8:00 P.M., and almost all the work must be concentrated during these hours. The men work hard during the day, and only those especially ambitious and energetic attend the classes.

The most obvious and effective way of overcoming these difficulties would be to secure official recognition, by the naval authorities, of the Y. M. C. A. instruction. The writer has been recently informed that a start in this direction has already been made.

THE NAVAL UNIT AT THE UNIVERSITY OF CALIFORNIA.

For the proposed naval unit at the University of California,[2] the following outline of courses preparatory for naval service has been provisionally made.

[1] [Dr. Barnett was engaged for a period of 7 weeks in the organization and supervision of the work here described. Editors.]

[2] We are indebted to Prof. T. M. Putnam for the information about the war courses at the University of California.

(*a*) *For Deck Officer:* Plane and Spherical Trigonometry and Elementary Theory of Map Construction, Introduction to Plane and Solid Analytic Geometry, Navigation and Nautical Astronomy, Naval History, Oceanography and Marine Meteorology, Seamanship, Naval Regulations, and Physical Education. With the exception of Physical Education, each course is given 3 hours weekly, $\frac{1}{2}$ year. There are also drills and practical work at sea, arranged with the coöperation of naval officers. (*b*) *For Engineer Officer:* As set forth under Course of Instruction on pp. 277–281 of the Annual Register of the U. S. Naval Academy, Annapolis (1917–18), and in addition, Naval Architecture and Marine Engineering. Only such work of the Annapolis curriculum can not be provided for as requires the actual use of Navy equipment not available to the University. This work might be arranged, however, in coöperation with the Commandant of the Twelfth Naval District.

The course in Navigation and Nautical Astronomy, which has been planned to conform as closely as possible to the standards, methods, and general procedure employed by the U. S. Navy, includes the following topics: time and the chronometer; the sextant and its adjustments; the compass (variation, deviation, Napier's diagram, various methods of swinging a ship for deviation, etc.); piloting; the sailings; latitude by meridian altitude, by Polaris, and by circum-meridian altitudes; longitude and time sights; azimuth and amplitude by tables and by computation; the New Navigation (Sumner line, method of St. Hilaire); full day's work at sea. Bowditch's American Practical Navigator, and additionally, the American Nautical Almanac, azimuth tables, tide tables, charts, etc., are used. Sextants, a liquid compass, and other apparatus are available for the practical aspects of the work.

In the Seamanship course, Knight's Seamanship, the Blue Jackets Manual and the Seamanship Department Notes of the U. S. Naval Academy are used. In the course on Naval History, the text-books formerly used by the U. S. Naval Academy were: (1) Short History of the U. S. Navy, by Clark, Stevens, Alden and Krafft; (2) Famous Sea Fights from Salamis to Tsushima by John R. Hale and (3) World Politics at the End of the Nineteenth Century by Paul S. Reinsch. Next term, Naval Powers in the Present War by Gill, Charles and Daran (1917) will be substituted for (3). Besides, the instructors frequently refer to such authors as Mahan, Corbett, etc.

To assist the U. S. Naval Reserve Force in preparing men to pass the examination for the ensign's commission, The University of California has been conducting, since September, 1917, extension courses in (*a*) Navigation and Nautical Astronomy, (*b*) Seamanship and Ordnance and (*c*) Naval Regulations. The equivalent of a high-school education has been a prerequisite; course (*a*) presupposes Trigonometry. A rough approximation of the total enrollment is 800.

NOTES.

At the request of the Commanding Officer of the United States Naval Auxiliary Reserve School at the Chicago Municipal Pier, the Department of Mathematics of the University of Chicago has during the summer conducted

four consecutive courses in Navigation for enlisted men awaiting their call to the Pier School, which coöperates with the Ensign School at Pelham Bay Park, N. Y. For enlistment at the Pier, men were required to be between the ages of 21 and 31 and expected to have had at least a brief course in Trigonometry. The course in Navigation, given three hours daily for three weeks, covered Dead Reckoning and selected topics of Nautical Astronomy (such as the determination of latitude, longitude, altitude, azimuth and time). Bowditch's American Practical Navigator was used as a basis, and numerous problems taken from a specially prepared list[1] were assigned for outside work. After July 9, a course of one hour on signalling was added, instruction being given by a man detailed by the Commanding Officer of the Pier School. The total enrollment for the four courses was approximately 550.

Similar courses of four weeks each were given by members of the Department of Mathematics of Northwestern University to a total of approximately 600 men; but on account of the hurried preparation of insertions in this department, further details could not be accurately obtained for the present issue.

The following extract from a letter sent to the Editor of the MONTHLY by an esteemed correspondent contains suggestions that may prove helpful to those seeking college instructors in mathematics at the present time: "(1) There is practically no visible supply of young instructors of ordinary qualifications for instructorships. (2) Other university teachers may be willing to teach at least trigonometry; but care should be exercised to see that (*a*) the offer of help is really final and not to be presently withdrawn (I have suffered); (*b*) that the same teacher can proceed with higher courses as we reach them; . . . (*c*) that the proposed teacher will not himself be overloaded with classes and hence be unable to help in mathematics. (3) A source of supply of teachers that may be overlooked lies in those high-school teachers of university training who would ordinarily (*a*) not accept on account of low salaries; (*b*) not be acceptable as university teachers. This situation is now quite upset, and teachers may be available because (*a*) ordinary requirements cannot be insisted upon by university authorities (*e.g.*, Ph.D. degree *must* be waived); . . . (*b*) there is a strong patriotic motive for the high-school teacher to accept lower salary, since S. A. T. C. work is essentially *war* service; (*c*) school boards and superintendents will release men more readily for what is essentially *war* work."

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. F. S. NOWLAN, of Columbia University, has been appointed instructor in mathematics at Bowdoin College.

Professor T. O. WALTON, of William and Vashti College, has been elected professor of mathematics at the Colorado School of Mines.

[1] Copies of this list can be obtained by applying to Prof. E. H. Moore, University of Chicago.

Associate Professor D. D. LEIB has been promoted to a professorship of mathematics at Connecticut College.

Professor H. R. PHALEN, of Berea College, has been appointed instructor in mathematics at Armour Institute of Technology.

Mr. G. H. SCOTT has resigned the principalship of Benzonia Academy to accept the professorship of mathematics in Doane College.

Dr. G. W. SMITH, instructor at Beloit College, has been appointed instructor in mathematics at the University of Kentucky.

Assistant Professor LOUIS LINDSEY has been promoted to an associate professorship of applied mathematics at Syracuse University.

Professor F. E. CHAPMAN has resigned his position at Southern University to become head of the department of mathematics in the Gulf Coast Military and Naval Academy at Gulfport, Mississippi.

Professor H. E. HAWKES, who has been acting dean of Columbia College, has been appointed dean of the college, consequent upon the resignation of Dean KEPPEL who has become third assistant secretary of war.

At Colorado College, Professor C. H. SISAM, of the University of Illinois, has been appointed professor of mathematics and head of the department, and Dr. W. V. LOVITT, of Purdue University, has been appointed assistant professor of mathematics.

Professor H. L. RIETZ, of the University of Illinois, has accepted the appointment as professor and head of the department of mathematics at the University of Iowa, succeeding the late Professor A. G. SMITH, whose death occurred in 1916.

Associate E. W. CHITTENDEN of the University of Illinois has been appointed assistant professor of mathematics at the University of Iowa.

Mr. J. W. LASLEY has been promoted from instructor to assistant professor of mathematics at the University of North Carolina, and granted leave of absence to study at the University of Chicago.

At the University of California, Associate Professor D. N. LEHMER has been promoted to a professorship of mathematics and Instructors B. A. BERNSTEIN, THOMAS BUCK and FRANK IRWIN have been promoted to assistant professorships. After forty-five years of service, Professor G. C. EDWARDS has been made professor emeritus.

At the University of Illinois the following promotions have been made in the department of mathematics: Associate Professor J. B. SHAW to become full professor; Assistant Professor R. D. CARMICHAEL to become associate professor; Associates G. E. WAHLIN and A. J. KEMPNER to become assistant professors; Assistant R. F. Borden to become instructor. At the same institution the

following new appointments have been made: Associate Professor A. B. COBLE of Johns Hopkins University to be professor of mathematics; Associate Professor H. BLUMBERG of Nebraska to be associate in mathematics; Instructor JOSEPH B. ROSENBACH of New Mexico to be assistant in mathematics.

Dr. HORACE LAMB, professor of mathematics at the University of Manchester, has been appointed Halley lecturer at Oxford University.

Professor R. D. CARMICHAEL has placed in the hands of the Secretary of the Association his resignation as editor-in-chief of the MONTHLY, the same to take effect immediately after the publication of the December, 1918, number. His resignation grows out of his desire to have a greater portion of his energies free for the pursuit of his researches. A committee consisting of H. E. SLAUGHT, R. D. CARMICHAEL and E. R. HEDRICK has been appointed to select his successor.

Dr. R. S. HEATH, professor of mathematics at Birmingham University, has retired from active service on account of ill health.

Mr. CHARLES E. FLANAGAN, actuary of the Conservative Life Insurance Company, Wheeling, W. Va., a charter member of the ASSOCIATION and a contributor to the MONTHLY, died on April 22, at the age of 63 years.

It is announced in *Science* that Professor MAXIME BÔCHER of Harvard University has died at the age of fifty-one years.

Professor A. L. DANIELS, Williams professor of mathematics at the University of Vermont, died on July 18, at the age of 69 years. After a service of twenty-nine years, Professor Daniels was retired in 1914 as professor emeritus on the Carnegie Foundation.

The following items have been received concerning teachers of mathematics who have entered the service of the government:

Professor W. E. MILNE, of Bowdoin College, has joined the ordnance reserve corps with the rank of first lieutenant.

Dr. PAUL R. RIDER, of the department of mathematics at Washington University, has entered the national service.

Mr. THEODORE DOLL, who has been a fellow in mathematics in Northwestern University, and a member of the Association, is now enlisted in the service and assigned to Camp Grant.

Dr. J. N. RICE, instructor in mathematics in the Catholic University of America, is now on leave of absence, serving in the National Army.

Mr. J. J. NASSAU, instructor in mathematics at Syracuse University, is a member of the 303rd regiment of engineers now in France.

Professor V. H. WELLS, of the University of Pittsburgh, has been commissioned a lieutenant in the science and research division of the signal corps.

Dr. T. M. Simpson, Jr., of the University of Texas, has been granted leave of absence for work with the Y. M. C. A. in France until the end of the war.

Miss Lucy T. Dougherty, a charter member of the Association and teacher of mathematics in the Kansas City (Kansas) High School, has gone to France to engage in Red Cross work.

Dr. E. G. Bill, assistant professor of mathematics at Dartmouth College, has been granted leave of absence in order that he may render service to the Canadian government on the staff of the Military Service Council at Ottawa.

Associate Professor J. E. Rowe, of Pennsylvania State College, is doing mathematical research for the National Advisory Committee for Aeronautics and has been detailed to the Bureau of Standards.

At the University of Wisconsin special intensive courses in trigonometry, navigation and gunnery are to be offered during the coming year. A course in practical gunnery will be given by Assistant Professor H. C. Wolff, who is spending the summer at Fortress Monroe in special preparation for this work. Professor C. S. Slichter will give a course in ballistics.

The concluding number of Volume 24 of the *Bulletin of the American Mathematical Society* contains the twenty-seventh annual list of papers, giving the author, title and place of publication of the 108 papers presented to the Society and published during the year 1917–18.

The name of Professor Felix Klein, of the University of Göttingen, together with those of six other German educators, has been cancelled from the roll of honorary members of the National Education Association in response to a persistent demand from active members of the association, from members of the Council of National Defense, and from others.

The American Association for the Advancement of Science will hold its seventy-first meeting in Baltimore from December 27 to 31, 1918, under the auspices of the Johns Hopkins University. Dr. John Merle Coulter of the University of Chicago, will preside. The address of the retiring president will be given by Dr. Theodore W. Richards, Director of the Wolcott Gibbs Memorial Laboratory, Cambridge. This seventy-first meeting of the American Association, which was established in 1848, will be marked by the importance of its program and by the increased interest manifested in all branches of the natural and applied sciences. It will embrace a program devoted very largely to definite working problems related to the winning of the war. When the Association met last in Baltimore, ten years ago, the membership of the Association was less than 6,000. The membership of the Association at present numbers nearly 15,000 and the coming meeting will be one of the most important gatherings of scientific men hitherto held in this country or elsewhere. The officers in charge of Section A, mathematics and astronomy, are members of the Mathematical Association of America. They are Professor G. D. Birkhoff, of Harvard University, vice-president, and Professor F. R. Moulton, of the University of Chicago, secretary.

The following graduate students have been granted the Doctorate in mathematics at American Universities since June, 1917, presenting disertations as indicated:

UNIVERSITY OF CALIFORNIA: FRANK R. MORRIS, "Classification of involutory cubic space transformations;" MARY HELEN SZNYTER, "The hypersurface of the second degree in four-dimensional space;" JAMES S. TAYLOR, "A set of five postulates for Boolean algebras in terms of the operation 'exception.'"

CATHOLIC UNIVERSITY OF AMERICA: OTTO J. RAMLER, "Three-cusped hypocycloids fulfilling certain assigned conditions."

UNIVERSITY OF CHICAGO: ISREAL A. BARNETT, "Differential equations with a continuous infinitude of variables;" JACOB M. KINNEY, "The general theory of congruences without any preliminary integrations;" ERNEST P. LANE, "Conjugate systems with indeterminate axis of curves;" JAMES E. MCATEE, "Modular invariants of a quadratic form for a prime power modulus;" WILLIAM P. OTT, "The general problem of the type of the brachistochrone with variable end points;" LEVI S. SHIVELY, "A new basis for the metric theory of congruences;" WEBSTER G. SIMON, "On the solution of certain types of linear differential equations in infinitely many variables."

COLUMBIA UNIVERSITY: GLENN JAMES, "Some theorems on the summation of divergent series."

CORNELL UNIVERSITY: H. H. DALAKER, "On the otomorphic functions of the group (0, 3; 2, 4, 6)."

UNIVERSITY OF ILLINOIS: RAYMOND FRANKLIN BORDEN, "On the Laplace-Poisson mixed equation;" HOBART DICKINSON FRARY, "The Green's function for a plane contour;" MERLIN GRANT SMITH, "On the zeros of functions defined by homogeneous linear differential equations containing a parameter."

UNIVERSITY OF PENNSYLVANIA: GEORGE H. HALLETT, "Linear order in three dimensional Euclidean and double elliptic space;" HARRY M. SHOEMAKER, "A generalized equation for vibrating membranes."

SYRACUSE UNIVERSITY: MRS. EDWARD DRAKE ROE, JR., "Interfunctional expressibility problems of symmetric functions."

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FERMAT'S METHOD OF INFINITE DESCENT.

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The method of infinite descent and the method of mathematical induction, which may appropriately be called the method of infinite ascent, both came into use in the seventeenth century in that productive period made famous by Descartes, Cavalieri, Pascal, Wallis and Fermat. Mathematical induction was first brought into prominence by Pascal, although it was invented by Maurolycus[1] and used in his arithmetic which was published in 1575. It is a method of infinite ascent from a special case, for which the theorem to be proved is verified by means of the argument from n to $n + 1$. The method of infinite descent, which was invented by Fermat, is not so widely known. Fermat's own account of it is to be found in a letter entitled "*Relation des nouvelles découvertes en la science des nombres*" which he wrote to Pierre de Carcavi in 1659.[2] In this letter he tells Carcavi that he has discovered a new method of demonstration and applied it successfully to the solution of a considerable number of problems in the theory of numbers. He calls it the method of infinite or unlimited descent (descente infinie ou indéfinie) and says that at first he applied it only to negative propositions such as these two:

"*No number of the form* $3k - 1$ *can be of the form* $x^2 + 3y^2$."

"*There is no right triangle whose sides are integers whose area is equal to the square of an integer.*"

He then gives the following abstract of his proof of the latter: "The proof is made by ἀπαγωγὴν εις ἀδυνατον[3] in this manner: If there is a right triangle

[1] W. H. Bussey, "The Origin of Mathematical Induction," this MONTHLY, vol. 24, pp. 199–207.

[2] *Oeuvres de Fermat*, Vol. 2, pp. 431–436.

[3] Greek for *reductio ad absurdum*.

with integral sides and with an area equal to the square of an integer, then there is a second triangle, smaller than the first, which has the same property; and if there is a second, smaller than the first, which has the same property, then there is, by like reasoning, a third smaller than the second; and then a fourth, a fifth, and so on ad infinitum. But there is not an infinite number of integers less than a given integer. From which one concludes that it is impossible that there should be a right triangle with integral sides and with an area which is the square of an integer." In the next paragraph Fermat says: "It was a long time before I was able to apply my method to affirmative questions, because the way and manner of getting at them is much more difficult than that which I employ with negative theorems. So much so that, when I had to prove that every prime number of the form $4k + 1$ is made up of two squares, I found myself in much torment. But at last a certain reflection many times repeated gave me the necessary light, and affirmative questions yielded to my method with some new principles by which sheer necessity compelled me to supplement it. This development of my argument in the case of affirmative questions takes the following line. If a prime number of the form $4k + 1$ selected at random is not made up of two squares, there will exist another prime number of the same sort but less than the given number, and again a third still smaller and so on descending ad infinitum until one comes to the number 5, which is the smallest of all numbers of the kind in question and which the argument would require not to be made up of two squares, although in fact it is so made up. From which one must infer, by reductio ad absurdum, that all numbers of the kind in question are in consequence made up of two squares."

The foregoing is a good account of Fermat's method although it leaves out all details of the proof. Fermat's proof, like so many others of his, is not extant. The proof was actually made by Euler.[1]

The American college student learns the method of mathematical induction by means of theorems on the summation of such series as

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2$$

and of the theorems about the algebraic divisibility of $x^n \pm y^n$ by $x \pm y$. The proof involves the verification of the theorem in a special case and the ascent argument from n to $n + 1$. But the proof can be made just as easily by means of the verification in a special case and a descent argument from n to $n - 1$. For example the theorem that $x^n - y^n$ is divisible by $x - y$ for all integral values of n may be proved as follows: Suppose that there is an integer n for which the theorem is not true. Then the identity

$$\frac{x^n - y^n}{x - y} \equiv x^{n-1} + \left(\frac{x^{n-1} - y^{n-1}}{x - y}\right)y,$$

obtained by ordinary long division, proves that $x^{n-1} - y^{n-1}$ is not divisible by

[1] For a proof of this theorem, in a form which is a modification of the proof described by Fermat, see H. Weber and J. Wellstein, *Encyclopädie der Elementar-Mathematik*, Vol. 1, p. 285.

$x - y$; that is, the theorem is not true for $n - 1$. This descent argument, repeated a finite number of times, requires that the theorem be not true when $n = 2$; in other words, that $x^2 - y^2$ be not divisible by $x - y$. This furnishes the contradiction needed to complete the proof by reductio ad absurdum. The descent is not an infinite descent but rather a finite descent to the special verified case of the theorem. It is really a proof by mathematical induction turned around. Any proof by mathematical induction can be put in this descent form. The proof is well characterized by the description which Fermat gave of his method in connection with the theorem that every prime of the form $4k + 1$ equals the sum of two squares. But Fermat's proof of this theorem does not involve an argument from n to $n - 1$. Nor is the descent in this case one from a prime number to the next smaller prime number. The essential feature of his method is not any such regular descent as that given by the argument from n to $n - 1$. It is rather a descent from an integer n involved in the theorem to some smaller integer; just how much smaller may not be known. This is very well illustrated by the proof that $x^4 + y^4 = z^4$ has no solution in integers different from zero. The proof in outline is this: If $x^4 + y^4 = z^4$ has a solution in such integers, so has $x^4 + y^4 = z^2$; if there exists a solution (x_1, y_1, z_1) of this equation, there exists another solution (x_2, y_2, z_2) in which z_2 is an integer less than z_1; and by the same argument another solution (x_3, y_3, z_3) in which z_3 is less than z_2; and so forth ad infinitum. The contradiction needed for the completion of the argument by reductio ad absurdum is furnished by the fact that there is not an infinite number of positive integers less than a given one. But the contradiction may just as well be obtained from the special case $x^4 + y^4 = 1^2$, which by inspection has no solution in integers different from zero. The descent argument that $z_2 < z_1$ holds without exception. But in the proof that $x^n - y^n$ is divisible by $x - y$ the descent argument fails when $n = 1$. The fact that it fails makes the proof a finite descent proof rather than an infinite descent proof. But it should be said in this connection that the proof that $x^4 + y^4 = z^2$ has no integral solution may be thought of as a finite descent to the special case $x^4 + y^4 = 1^2$ mentioned above.

The proof that $x^4 + y^4 = z^4$ has no integral solution may also be put in this form.[1] Let it be assumed that $x^4 + y^4 = z^2$ has one or more solutions and let (x_1, y_1, z_1) be a solution in which z_1 is the smallest possible. Then the descent argument proves that there is a solution (x_2, y_2, z_2) in which z_2 is less than z_1. The required contradiction is furnished by the fact that there is thus proved to be a smaller than the smallest. In any true infinite descent proof, that is in a proof in which the descent argument holds without exception, the contradiction needed for the reductio ad absurdum argument may be obtained in any one of the three ways mentioned, namely from the fact that there is not an infinite number of integers less than a given integer, or from the descent to a special verified case of the theorem, or from the fact that there is not a smaller than the smallest. But when the descent argument does not hold without exception, as

[1] See H. WEBER and J. WELLSTEIN, *l. c.*, p. 284.

in the proof that $x^n - y^n$ is divisible by $x - y$, the contradiction must be obtained from the descent to a special verified case of the theorem. It is this finite descent proof which is mathematical induction turned around. In thinking of these two types of descent proof one wonders if there is not another type of ascent proof than the regular ascent from n to $n + 1$ of mathematical induction. The well-known proof, due to Euclid, that there is no greatest prime is one such. The argument is that if p is any prime, the number $N = p! + 1$ is a greater prime or else one of its prime factors is; that is, if p is any prime, there is a greater prime.

It is hard to account for Fermat's trouble in applying his method to positive or affirmative propositions. His method is to assume the opposite of what he wants to prove and to show that the assumption leads to a contradiction. Sometimes the statement of a proof by reductio ad absurdum is awkward, but it may be just as awkward for a negative proposition as for an affirmative one. His difficulty reminds one of the student who can prove by mathematical induction that $x^{2n} - y^{2n}$ is divisible by $x + y$ but is at a loss to know how to use the same method to prove that $x^{2n} + y^{2n}$ is not divisible by $x + y$.

Fermat, in his letter to Carcavi, says or implies that his method of infinite descent is applicable to the following propositions, some of them having already been mentioned in this paper:

1. There is no number of the form $3k - 1$ which is also of the form $x^2 + 3y^2$.
2. There is no right triangle whose sides are integers and whose area equals a square number.
3. Every prime of the form $4k + 1$ is equal to the sum of two squares.
4. Every number is either a square or the sum of two, three or four squares.
5. $x^2 - ay^2 = 1$ has an infinite number of integral solutions if a is not a square.
6. There is no integral solution of $x^3 + y^3 = z^3$.
7. There is one and only one integral solution of $x^2 + 2 = y^3$, namely, $x = 5$, $y = 3$.
8. There are two and only two integral solutions of $x^2 + 4 = y^3$, namely, $x = 2, y = 2$ and $x = 11, y = 5$.
9. Every number of the form $2^{2^n} + 1$ is a prime number.
10. There are only two integral solutions of $(2x^2 - 1)^2 = 2y^2 - 1$, namely, $x = 1, y = 1$ and $x = 2, y = 5$.

Fermat does not say explicitly that he succeeded in proving all of these propositions but his language gives one that impression. He does say that he proved Nos. 1, 2, 3, 4, 5, 10. Of Nos. 6, 7, 8, 9 he says: "I afterwards considered certain propositions which, although negative, present very great difficulty, the method of applying the descent being altogether different from that of the preceding cases." Of No. 9 he says further: "This is a very subtle and very ingenious research and, although it is stated affirmatively, it is negative, since to say that a number is prime is the same as to say that it cannot be divided by any number."

This Proposition No. 9 is not true. For when $n = 5$ the number

$$2^{2^n} + 1 = 4{,}294{,}967{,}297 = 641 \cdot 6{,}700{,}417.$$[1]

It is often stated that Fermat did not claim to have proved this theorem. This statement is based on the fact that in a letter to Pascal[2] in August, 1654, he says that, although he is convinced of its truth, he has not been able to find a valid demonstration; and that in a letter to Digby[3] in June, 1658, he says that he is seeking a demonstration of this beautiful theorem. But the quotation given above is from the 1659 letter to Carcavi and in it Fermat seems to say that he has proved it by the method of infinite descent.

For a brief account of Fermat's method of infinite descent, especially of its relation to the solution of Diophantine equations, the reader is referred to R. D. CARMICHAEL'S "*Diophantine Analysis,*" pp. 14–21. The reader is also referred to the supplement of Sir THOMAS L. HEATH'S "*Diophantus of Alexandria,*" Cambridge University Press, 2d edition, 1910, which contains a number of references to the method of infinite descent.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By JOSEF NYBERG, Hyde Park High School, Chicago, Ill.

In a previous paper[4] I showed how the study of the line in analytic geometry could be presented from the viewpoint of functions and relations between variables. The present article explains how logarithms and exponential relations are introduced on the same basis.

I begin the work by solving some problem leading to an exponential relation between two variables. Most such problems taken from physics or chemistry lead to a negative exponent, which is undesirable at the beginning of the work. However, a suitable problem is that of determining the number N of bacteria in a solution at a time t, knowing the number N_0 present at $t = 0$ and knowing the rate of increase. The conditions are similar to those for finding the length of a rod with increasing temperature, a problem studied under linear functions, but different in that the coefficient of expansion of the rod is independent of the temperature while the bacteria increase at a rate dependent upon t. If the bacteria increase r per cent. in a unit of time, then

$$N = N_0\left(1 + \frac{r}{100}\right)^t.$$

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[1] This was proved by Euler in 1732. For this and other comments on the theorem see W. W. R. Ball, *Mathematical Recreations and Essays*, 5th edition, p. 39. Also see R. C. Archibald's *Remarks on "Klein's famous problems of elementary geometry,"* this MONTHLY, Vol. 21, p. 248.

[2] *Oeuvres de Fermat*, Vol. 2, p. 307.

[3] *Ibid.*, p. 402.

[4] "The Linear Function and the Line," in this MONTHLY, Nov., 1917, page 406. This was the third of a series of related papers of which the other two are:

"The Unification of Freshman Mathematics," April, 1916, page 101;

"The Presentation of the Notion of Function," Sept., 1917, p. 309.

The next work consists in drawing carefully a graph of $y = a^x$ taking $a = 2$ as the most convenient. We find points on the curve by geometrical constructions using the line $y = ax$ and interpolating points by using the lines $y = \sqrt{a}x$, $y = \sqrt[4]{a}x$, etc. (remembering that the logarithms increase in arithmetic progression when the numbers increase in geometric progression). At once curves like $y = y_0a^x$ can be sketched by a proper multiplication of the ordinates, and $y=y_0a^{nx}$ by a multiplication of the abscissas. Since the latter curve is the same as $y=y_0b^x$ where $b = a^n$, any exponential curve can be sketched from the one fundamental one. Moreover a satisfactory value of n can be found from the graph of $y = a^x$ inasmuch as n is the abscissa of the point whose ordinate is b. We call n the logarithm of b to the base a.

Having seen how the exponential function arises, and how it differs from the linear function in its rate of increase, graphed it, and introduced the word logarithm, we are ready to study the properties of the function. It is at once evident that negative numbers have no real logarithms since the curve by our construction can never get below the x axis, that the logarithm of the base is 1, that $\log 1 = 0$, and that numbers less than 1 have negative logarithms. Progressively we prove the fundamental relations for $\log (N_1/N_2)$, $\log N_1N_2$, and $\log_a N = \log_b N \cdot \log_a b$. For exercises in drill I use numbers whose logarithms can be taken from the graph which the student has drawn, or else use integers less than 40 whose logarithms I have written on the blackboard. The student computes log 3, log .04, etc., from the properties of the function. At this point the student also solves such exponential equations as $p^x \cdot q^{x+2} = r$ (p, q, r, < 40). The difficulties in this equation which Mr. Simpson[1] pointed out can never arise under this method of treatment because negative logarithms are no less common than positive ones, and because the notion of the characteristic and mantissa have not yet been introduced. Mantissas and characteristics are in fact unnecessary until we begin using tables.

Before proceeding to the use of tables I like to explain at this point the significance of the constant e as a base. The fact that $y = e^x$ has at any point a slope equal to the ordinate at that point does not appear to be unusually significant, not even to a student of calculus; nor can the student appreciate its significance as the limiting value of a certain series unless we show him *how* and *why* this particular series is *unavoidable* in the problems. This can be done by returning to the original problem of the bacteria which led to the relation

$$N = N_0\left(1 + \frac{r}{100}\right)^t,$$

when the rate of increase is r per cent. per unit of time. If the unit of time is cut in two, then

$$N = N_0\left(1 + \frac{r}{200}\right)^{2t};$$

[1] "Relating to the Teaching of Logarithms," in this MONTHLY, Sept., 1916, page 264.

and if divided into q parts, then

$$N = N_0\left(1 + \frac{r}{100q}\right)^{qt}.$$

We wish to derive the expression for N when q is increased indefinitely (*i. e.*, when the unit is decreased in size). Put $r/100q = 1/u$ so that

$$N = N_0\left(1 + \frac{1}{u}\right)^{urt/100}.$$

We may then show that $\lim_{u=\infty}(1 + 1/u)^u$ is a definite constant, which we denote by e. This work may be done with as much rigor as the individual teacher prefers. Thereafter similar problems are assigned from physics, as for example, the amount of light coming through a pane of glass, or problems from chemistry on the decomposition of elements. r is negative for most of these problems, but this fact is not now a disturbing element to the student. These problems also have the advantage that the constants y_0 and n in the relation $y = y_0e^{nx}$ have a definite physical association.

Not until the previous theory is clear do I consider logarithms in their application to computing. There are two new ideas that need to be impressed on the student: (1) the conventions of the computer of dividing a logarithm, irrespective of whether it is positive or negative, into two parts, an integral characteristic and a decimal mantissa; (2) the conventions of the printer used in abbreviating the work of printing the table. Considering also the definition previously given, we are dealing with a logarithm from three distinct aspects: first, what it actually is, as log .6974 = − .15658; second, what the printer puts into the table, as 84342; third, what the computer puts on his paper, as .84342 − 1.

The textbooks do not clearly differentiate these three steps, and yet every error of the beginner is due to the confusion of these notions. It alone is responsible for the error that Mr. Simpson mentions; it leads to the usual errors in handling numbers less than unity, and to errors in finding the antilogarithm. The first point can be emphasized by having the student check his logarithms (if he is working the customary exercises of finding the logarithms of an assigned list of numbers) from his graph. Under the second item I frequently resort to the homely method of making the student put a decimal point before each mantissa in some column of his table. This cures him permanently of any inclination to write .08434 instead of .8434 − 1 for log .6974, or of writing 212.7 instead of 3.2127 for log 1632. The abbreviation of the printer's work is also the justification of 10 as a base in computing, whereas we use e as a base for exponential relations arising in problems other than computing. This the student will readily see if he computes $\log_e 1.234$, $\log_e 12.34$, etc. In fact there is scarcely a better way for learning the rules for handling logarithms than by comparing the similar rules for e as the base. Under the third idea of what the computer puts on his paper, the teacher can explain the advantage of cologarithms, the necessity of writing 2.4133 − 3 instead of .4133 − 1 for log .259 in finding its cube root, and similar matters.

This tripartite analysis of a logarithm is more powerful than may appear at a first reading. We learn by contrast; and the value of this particular explanation lies in seeing logarithms from the viewpoints of the computer, the printer, and the graph. Most teachers are inclined to believe that the best instruction consists in presenting to the student only one idea at a time. Psychologically, however, it is equally sound to keep together two closely related ideas, and teach by contrast and comparisons. Ordinarily logarithms are taught solely for their use in computing; later in analytic geometry the student draws perhaps a few exponential curves; and then in calculus he learns about the base e. The introduction of both bases simultaneously impresses the significant facts about each; the mind is put in a receptive mood when it sees the necessity of the rules. The writer believes the present paper affords a good illustration of how various parts hitherto separated in the curriculum by years can be taught sequentially when organized on the basis of functions. In addition to unity of subject matter, we also obtain an unusual degree of coherence. Each day's work is intimately linked with the preceding and prepares for the following.

NOTE ON LAGRANGE'S LIKE-PRODUCING QUADRINOMIAL.[1]

By Sir THOMAS MUIR, South Africa.

1. From an identity originally due to Lagrange we know that the product

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)(y_1^2 + bcy_2^2 + cay_3^2 + aby_4^2)$$

can be expressed in the same form as either factor, namely, in the form

$$P_1^2 + bcP_2^2 + caP_3^2 + abP_4^2.$$

For the case, however, where the second factor is the same as the first, the formula is nugatory, all that it then tells us being that $(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^2$ is equal to itself. It seems therefore not a little interesting to know that without introducing any new principle we can show that the product of the *cube* of the first factor by the second factor can be simply expressed in the particular form in question.

2. We recall in the first place that

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^2 = \begin{vmatrix} x_1 & bcx_2 & cax_3 & abx_4 \\ -x_2 & x_1 & -ax_4 & ax_3 \\ -x_3 & bx_4 & x_1 & -bx_2 \\ -x_4 & -cx_3 & cx_2 & x_1 \end{vmatrix}$$

and that each primary minor of this determinant contains the square root of the

[1] Part of the object of this paper is to throw a little light on the subject of Problem 489 of this MONTHLY for Nov. 1917.

determinant as a factor, the adjugate of the determinant being

$$\begin{vmatrix} x_1Q & x_2Q & x_3Q & x_4Q \\ -bcx_2Q & x_1Q & -bx_4Q & cx_3Q \\ -cax_3Q & ax_4Q & x_1Q & -cx_2Q \\ -abx_4Q & -ax_3Q & bx_2Q & x_1Q \end{vmatrix}$$

if we put Q for $x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2$.

3. From this it follows that if we take any three columns of the said determinant along with *any fourth column whatever* to form a new determinant, the latter must have Q for a factor. A special interest attaches to the choosing of y_1, y_2, y_3, y_4 for the fourth column: for the new determinants and their factors then are

$$\begin{vmatrix} y_1 & bcx_2 & cax_3 & abx_4 \\ y_2 & x_1 & -ax_4 & ax_3 \\ y_3 & bx_4 & x_1 & -bx_2 \\ y_4 & -cx_3 & cx_2 & x_1 \end{vmatrix} = Q(x_1y_1 - bcx_2y_2 - cax_3y_3 - abx_4y_4),$$

$$\begin{vmatrix} y_1 & x_1 & cax_3 & abx_4 \\ y_2 & -x_2 & -ax_4 & ax_3 \\ y_3 & -x_3 & x_1 & -bx_2 \\ y_4 & -x_4 & cx_2 & x_1 \end{vmatrix} = Q(-x_2y_1 - x_1y_2 - ax_4y_3 + ax_3y_4),$$

$$\begin{vmatrix} y_1 & x_1 & bcx_2 & abx_4 \\ y_2 & -x_2 & x_1 & ax_3 \\ y_3 & -x_3 & bx_4 & -bx^2 \\ y_4 & -x_4 & -cx_3 & x^1 \end{vmatrix} = Q(x_3y_1 - bx_4y_2 + x_1y_3 + bx_2y_4),$$

$$\begin{vmatrix} y_1 & x_1 & bcx_2 & cax_3 \\ y_2 & -x_2 & x_1 & -ax_4 \\ y_3 & -x_3 & bx_4 & x_1 \\ y_4 & -x_4 & -cx_3 & cx_2 \end{vmatrix} = Q(-x_4y_1 - cx_3y_2 + cx_2y_3 - x_1y_4),$$

where the cofactors of Q turn out to be values of P_1, P_2, P_3, P_4 in § 1 as given by Lagrange's rule. The consequence is that if in the equality of § 1 we substitute for each P the quotient of the appropriate determinant by Q, we obtain the following theorem:

If by $|\,rstu\,|$ *be denoted the determinant whose columns are the* rth, sth, tth, uth *columns of the array*

$$\begin{matrix} y_1 & x_1 & bcx_2 & cax_3 & abx_4 \\ y_2 & -x_2 & x_1 & -ax_4 & ax_3 \\ y_3 & -x_3 & bx_4 & x_1 & -bx_2 \\ y_4 & -x_4 & -cx_3 & cx_2 & x_1 \end{matrix}$$

then

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^3 \cdot (y_1^2 + bcy_2^2 + cay_3^2 + aby_4^2)$$
$$= |\,1345\,|^2 + bc\,|\,1245\,|^2 + ca\,|\,1235\,|^2 + ab\,|\,1234\,|^2.$$

4. From Lagrange's result it of course follows that

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)(y_1^2 + bcy_2^2 + \cdots)(z_1^2 + bcz_2^2 + \cdots)$$

can be expressed in the same form as any one of the factors, say in the form

$$T_1^2 + bcT_2^2 + caT_3^2 + abT_4^2,$$

and there thus arises the problem of finding the T's in terms of the x's, y's, z's and a, b, c. The simplest way of writing the four expressions obtained is by means of the notation for bipartite functions, according to which

$$\begin{array}{ccc|c} x & y & z & \\ \hline a & b & c & \xi \\ d & e & f & \eta \\ g & h & i & \zeta \end{array} \quad \text{stands for } ax\xi + by\xi + cz\xi + dx\eta + ey\eta + fz\eta + gx\zeta + hy\zeta + iz\zeta,$$

each element of the square array being taken along with the outside element in the same column and at the same time along with the outside element in the same row. We then have

$$T_1 = \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline y_1 & bcy_2 & cay_4 & aby_4 & z_1 \\ -y_2 & y_1 & -ay_4 & ay_3 & bcz_2 \\ -y_3 & by_4 & y_1 & -by_2 & caz_3 \\ -y_4 & -cy_3 & cy_2 & y_1 & abz_4 \end{array}$$

and the other T's differing from it merely in having for their outside column a different row of the determinant equivalent for $(z_1^2 + bcz_2^2 + caz_3^2 + abz_4^2)^2$.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

We believe that those of our readers who have followed the discussions of certain solutions of cubic equations as published in the February and June numbers of the MONTHLY will be especially interested in the method given below for approximating nearly equal roots of a cubic.

I. Relating to Approximations to Nearly Equal Roots of a Cubic Equation.

By Paul Capron, U. S. Naval Academy, Annapolis.

If $y = f(x) \equiv x^3 - 3a^2x + 2ma^3$, the equation $y = 0$ has three real roots when $m^2 \ngtr 1$, has a double root $\pm a$ when $m = \pm 1$, and has two roots nearly equal to $\pm a$ when m is nearly equal to ± 1. The discussion that follows is confined to the case in which the nearly equal roots are positive; it is applicable to negative roots if the signs of both m and a are reversed, or if the equation is replaced by $\bar{y} \equiv -f(-x) = 0$.

The method of arriving at the approximation is as follows:

Let the larger of the two roots be x_1, the smaller x_2, and let the graph of $f(x)$ be a curve s of which the points P_1, P_2 correspond to the roots x_1, x_2. As the roots are nearly equal, P_1 and P_2 are joined by a relatively short segment of s, which includes the minimum point, L, and does not include the inflection. (These conditions define the restriction implied by the words "nearly equal.") Suppose a quadratic parabola s', having the equation $y_1 = f_1(x)$, to intersect the cubic parabola s three times at L, three times at infinity in the direction of the y axis, and consequently nowhere else. Then s' crosses s at L, and as it is not as steep as s for large values of x, keeps to the right of s where $x > a$, to the left where $x < a$. (At L, $x = a$.)

Let the roots determined by $y_1 = 0$ be x_1', x_2', of which x_1' is the larger. Then $x_1' > x_1$, $x_2' > x_2$. P_1' and P_2' being the points of s' corresponding to the roots x_1', x_2', let ordinates to the cubic s be drawn through P_1' and P_2', meeting s at Q_1 and Q_2. Now let tangents to s at Q_1 and Q_2 meet the x-axis at P_1'' and P_2'', of which the abscissas are x_1'' and x_2''. As the curvature of s does not change sign from P_1 to P_2, it is evident that $x_1'' > x_1$, $x_2'' < x_2$ and also that $f(a) < 0, f(x_1') > 0, f(x_1'') > 0, f(x_2') < 0, f(x_2'') > 0$.

The values $(x_1' - x_1'')$ and $(x_2' - x_2'')$ to be subtracted from x_1' and x_2', are the quotients of the ordinates of s at Q_1 and Q_2 by the slopes of s at these points; it will be found convenient to use a smaller divisor than the actual slope at Q_1, and a (numerically) larger one at Q_2, with the result that, the corrected roots thus found being x_1''' and x_2''', $f(x_1''') < 0$ and $f(x_2''') < 0$; that is, P_1''' and P_2''' being the points of the x-axis having the abscissas x_1''' and x_2''', P_1''' is enough to the left of P_1'' to be to the left of P_1 and P_2''' is enough to the right of P_2'' to be to the right of P_2.

Thus, of the successive approximations to x_1, viz., a, x_1', x_1'', x_1''', the first and last are too small, the others too large. In each case, x''' is the more convenient of the closer approximations, and the error in using it is less than $|x'' - x'''|$. In case the error is undesirably large, a closer approximation, but without limits of error, is obtained by finding the ratio (practically the same in all cases) in which the segment $P''P'''$ is divided by the chord joining the points of s that have the abscissas x'' and x'''.

The resulting approximation may be formulated:

If in the equation $x^3 - 3a^2x + 2ma^3 = 0$, m is nearly equal to 1, let $\frac{1}{9}(1 - m) = \mu$, $\sqrt{\frac{2}{3}(1 - m)} = \lambda$; then the nearly equal roots are

$$x = a(1 \pm \lambda - \mu)$$

with an error less than

$$a\mu \frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda},$$

the larger approximation being too small, the smaller too large. If this limit of error is unsatisfactory, much closer results may be obtained by applying five-sixths of its value as a correction to increase the larger approximation or decrease the smaller approximation.

As an illustration of the method, consider the equation

$$8x^3 - 36x^2 + 39x + 3 = 0.$$

When the roots are diminished by 3/2, this equation is transformed into $x^3 - \frac{15}{8}x + \frac{15}{16} = 0$. Here $a = \frac{1}{4}\sqrt{10}$, $m = \frac{3}{10}\sqrt{10}$, $\lambda = \sqrt{\frac{2}{3}(1 - m)} = 0.184962$, $\mu = \frac{1}{9}(1 - m) = 0.005702$;

$$x = \frac{\sqrt{10}}{4}(1 \pm 0.184962 - 0.005702) = 0.93229 \text{ or } 0.63984.$$

Adding 1.5, we have 2.43229 (too small) and 2.13984 (too large).

The errors are $< a\mu \cdot \frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} = \frac{\sqrt{10}}{4} \times 0.0057 \times \frac{0.0925}{1 \pm 0.0925} < 0.0005.$

If the results are not as close as desired, we may compute the errors more carefully (they are 0.00038 + for the larger root, 0.00046 − for the smaller), and, increasing the larger root by $\frac{5}{6} \times 0.00038 = 0.00032$, and decreasing the smaller by $\frac{5}{6} \times 0.00046 = 0.00038$, we may say that the roots are probably 2.43261 and 2.13946. (To six decimals the roots are actually 2.432617 and 2.139465.)

In Wentworth's College Algebra, the equation $x^3 - 515x^2 + 1155x - 649 = 0$ is shown to have two very nearly equal roots, approximately 1.1230914 and 1.1270002. If we multiply the roots of this equation by 3 and diminish the roots of the transformed equation by 515 we have

$$x^3 - 785280x - 267845848 = 0.$$

Here

$$a = 8\sqrt{4090}, \quad m = -66961462\sqrt{4090} \div (65440)^2.$$

$$\sqrt{4090} = 64(1 - p),$$

where

$$p = 0.00073269029253238 56,$$

$$(1 + m) = (65440)^{-2}(-3139968 + 4285533568p) = (65440)^{-2}(0.8435952782),$$

$$(1 + m) = 1.969762360 \times 10^{-10}; \quad \mu = \tfrac{1}{9}(1 + m) = 2.18862384 \times 10^{-11},$$

$$\lambda = \sqrt{\tfrac{2}{3}(1+m)} = (65440)^{-1}\sqrt{0.5623968521}$$

$$= 0.7499312316 \div 65440 = 1.145982933 \times 10^{-5},$$

$$x = -a(1 \pm \lambda - \mu) = -512(1-p)(1 \pm \lambda - \mu)$$
$$= -512 + 512p + 512(1-p)(\mp\lambda + \mu),$$

$$x + 515 = 3 + 0.3751374297765814 + 511.6248625702234186(\mp\lambda + \mu),$$

$$\tfrac{1}{3}(x + 515) = 1.123091428324 \text{ or } 1.127000184062.$$

The error is less than

$$\tfrac{1}{3}a\mu\frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} < \tfrac{1}{3} \times 512(1-p) \times 2.19 \times 10^{-11} \times \frac{5.73 \times 10^{-6}}{1 \pm 5.73 \times 10^{-6}} < 2.14 \times 10^{-14}.$$

In this case, the approximation might safely have been carried one or two places further.

The derivation of the formula follows:

Given $y \equiv f(x) \equiv (x^3 - 3a^2x + 2ma^3)$; transforming the origin of coördinates to the minimum point, L, $(a,\ 2(m-1)a^3)$, we obtain $y_1 = x_1^3 \pm 3ax_1^2$. The desired quadratic approximation to y_1 near L is $y_2 = 3ax_1^2$. Transformed to the original system of coördinates, the latter becomes $y_3 = 3ax^2 - 6a^2x + (1+2m)a^3$. Solving $y_3 = 0$ we find $x_1' = a(1 + \sqrt{\tfrac{2}{3}(1-m)})$, $x_2' = a(1 - \sqrt{\tfrac{2}{3}(1-m)})$.

Substituting in $y = f(x) \equiv x^3 - 3a^2x + 2ma^3$, we obtain $f(x_1') = a^3[\tfrac{2}{3}(1-m)]^{3/2}$, $f(x_2') = -a^3[\tfrac{2}{3}(1-m)^{3/2}]$.

The derivative is $f'(x) \equiv 3(a^2 - x^2)$; $f'(x_1') = 2a^2(1 - m + \sqrt{6(1-m)}) > 0$, $f'(x_2') = 2a^2(1 - m - \sqrt{6(1-m)}) < 0$.

$$x_1' - x_1'' = \frac{f(x_1')}{f'(x_1')} = a\frac{[\frac{2}{3}(1-m)]^{3/2}}{2[1 - m + 3\sqrt{\frac{2}{3}(1-m)}]} = \frac{a}{9}\cdot\frac{1-m}{1 + \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}};$$

similarly

$$x_2' - x_2'' = \frac{a}{9}\cdot\frac{1-m}{1 - \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}},$$

$$x_1'' = a\left[1 + \sqrt{\tfrac{2}{3}(1-m)} - \frac{1}{9}\frac{1-m}{1 + \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}\right];$$

$$x_2'' = a\left[1 - \sqrt{\tfrac{2}{3}(1-m)} - \frac{1}{9}\cdot\frac{1-m}{1 - \frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}\right].$$

Neglecting the radical in the denominator, we have:

$$x_1''' = a[1 + \sqrt{\tfrac{2}{3}(1-m)} - \tfrac{1}{9}(1-m)] < x_1'';$$

$$x_2''' = a[1 - \sqrt{\tfrac{2}{3}(1-m)} - \tfrac{1}{9}(1-m)] > x_2''.$$

Also

$$f(x_1''') = \frac{a^3}{27}(1 - m)^2[-5 + \sqrt{\tfrac{2}{3}(1 - m)} - \tfrac{1}{27}(1 - m)] < 0.$$

$$f(x_2''') = \frac{a^3}{27}(1 - m)^2[-5 - \sqrt{\tfrac{2}{3}(1 - m)} - \tfrac{1}{27}(1 - m)] < 0.$$

Therefore the errors are less, respectively, than

$$x_1'' - x_1''' = \frac{a}{9}(1 - m)\left[1 - \frac{1}{1 + \frac{1}{2}\sqrt{\frac{2}{3}(1 - m)}}\right] = \frac{a}{9}(1 - m)\cdot\frac{\frac{1}{2}\sqrt{\frac{2}{3}(1 - m)}}{1 + \frac{1}{2}\sqrt{\frac{2}{3}(1 - m)}},$$

$$x_2''' - x_2'' = \frac{a}{9}(1 - m)\left[\frac{1}{1 - \frac{1}{2}\sqrt{\frac{2}{3}(1 - m)}} - 1\right] = \frac{a}{9}(1 - m)\cdot\frac{\frac{1}{2}\sqrt{\frac{2}{3}(1 - m)}}{1 - \frac{1}{2}\sqrt{\frac{2}{3}(1 - m)}}.$$

Hence the rule given earlier for nearly equal positive roots. For nearly equal negative roots (m nearly -1), if $\frac{1}{9}(1 + m) = \mu$, $\sqrt{\frac{2}{3}(1 + m)} = \lambda$, $x = -a(1 \pm \lambda - \mu)$, with an error less than $a\mu\cdot(\frac{1}{2}\lambda/(1 \pm \frac{1}{2}\lambda))$, the numerically larger root being too small numerically, the other too large numerically.

Whether the approximate roots are positive or negative, the one further from the inflection is too small, the one nearer the inflection too large; in other words, both are on the portion of the x-axis intercepted by the adjacent arch of the graph of $y = 0$.

If we let $\frac{1}{2}\mu = z$, we have $f(x_1''') = -12a^3z^2(5 - 2\sqrt{3z} - 6z)$ as already seen, and further, $f(x_1'') = 12a^3z^2(1 - 3z)^{-2}(1 - \frac{38}{3}z + 21z^2 + 8z(1 - 2z)\sqrt{3z})$; $f(x_2''')$ and $f(x_2'')$ differ only in the sign of $\sqrt{3z}$. Consequently

$$-f(x_1''') : f(x_1'') = 5 : 1$$

very nearly, for either root. Approximating by means of the chord of the graph, we thus obtain a closer result, for which, however, no limits of error are fixed. If the limit of error of the earlier approximation is

$$a\mu\frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} = E_1, E_2$$

(E_1 taking the upper sign, E_2 the lower)

$$x = a(1 \pm \lambda - \mu \pm \tfrac{5}{6}E_{1,\,2})$$

if the roots are positive;

$$x = -a(1 \pm \lambda - \mu \pm \tfrac{5}{6}E_{1,\,2})$$

if the roots are negative.

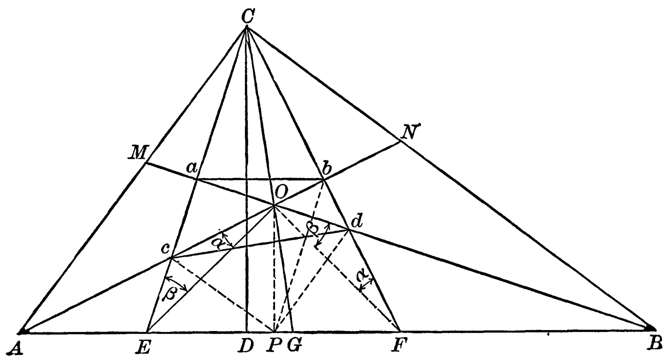
The chief advantages of the approximation are that it gives an easy means of locating two roots, and of separating them if they are very nearly equal, and in the latter case gives an approximate solution more convenient than Horner's method and more accurate than the trigonometric solution. The formulas for

the trigonometric solution are $\cos 3\theta = -m$, $x = 2a\cos\theta$; if $|m|$ is nearly unity, the value of θ is not very accurately determined. The first of the examples cited above is near the border-line; five-place tables will give nearly as accurate results by the trigonometric method as by the approximation, but with more labor. [The roots by this method are -0.0721, 2.4326, 2.1394.]

II. Relating to Some Relations in a Right-angled Triangle.

By Albert Babbitt, University of Nebraska, Lincoln.

In the right-angled triangle ABC (with right angle at C) draw CD perpendicular to AB; also draw CE, CF, BM and AN bisectors of the angles ACD, BCD, ABC and CAB respectively.



We have then the following theorems:

Theorem I. *The bisector of the angle B is perpendicular to CE, and the bisector of the angle A is perpendicular to CF. Moreover, CE is bisected by the bisector of the angle B (in point a) and CF is bisected by the bisector of the angle A (in pt. b), and consequently ab is parallel to AB and is equal to* $\frac{1}{2}EF$.

Proof. Since

$$\underline{|bOd} = \frac{A}{2} + \frac{B}{2} = 45^\circ,$$

$$\underline{|bdO} = \frac{BCD}{2} + \frac{B}{2} = 45^\circ,$$

hence $\underline{|dbO} = 90^\circ$, and AN is perpendicular to CF. Similarly, it may be proved that BM is perpendicular to CE.

From the equality of the right-angled triangles FbA and AbC, we have $Fb = bC$, i. e., CF is bisected by the bisector of the angle A. Similarly, $Ea = aC$, and it follows at once that ab is parallel to AB and is equal to $\frac{1}{2}EF$.

Theorem II. *The bisectors CF and CE cut off on the hypotenuse AB segments AF and BE which are equal to the sides AC and BC respectively.*

Proof. From the equality of the right triangles FbA and AbC, it follows that

$AF = AC$, and from the equality of the right triangles BaE and BaC it follows that $BE = BC$.

THEOREM III. *The radius r of the circle inscribed in the triangle ABC is equal to* $\frac{1}{2}EF = ab$.

Proof. Since $CO = OF$ (ΔCOF is isosceles) and $CO = OE$ (ΔCOE is isosceles), $OE = OF$; and consequently EFO is an isosceles triangle.

Moreover, $\underline{|FOE} = 2\,\underline{|FCE} = 90°$. Whence, $OP \equiv r = \frac{1}{2}EF = ab$.

THEOREM IV. *The radius,* $r = cd/\sqrt{2}$, *i. e., the radius is equal to the side of a square whose diagonal is cd.*

Proof. From the similarity of the triangles aOb and dOc, we have

$$\frac{ab}{cd} = \frac{Oa}{Oc} = \frac{Ob}{Od} = \sin 45° = \frac{1}{\sqrt{2}}.$$

Hence (Theorem III), $r = ab = \dfrac{cd}{\sqrt{2}}$.

THEOREM V. *The line of centers of the circles inscribed in the triangles ADC and BDC is perpendicular to the bisector of the right angle C, i. e.,* $cd \perp CG$.

Proof. Since in the triangle cdC two altitudes cb and da pass through the point O, $CG \perp cd$.

THEOREM VI. *The centers of the circles inscribed in the triangles ADC and DBC, i. e., the points c and d are equidistant from the point P and their distances are equal to* $OP = r$.

Proof. From the equality of triangles OFd and OcE it follows that $OD = Ec$, and hence follows the equality of OPd and PEc and consequently $cP = Pd$. Moreover, $\underline{|OdF} = 180° - (\alpha + \beta) = 180° - 45° = 135°$. Hence, if a circle be drawn with P as the center and $OP = PF = r$ as the radius $\underline{|OdF}$ will be an inscribed angle and $Pd = cP = r$.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

ADDRESS.[1]

By HERBERT E. HAWKES, Columbia University.

In the remarks which I have to make this morning, I do not propose to go into details regarding the history of Mathematics Clubs or to discuss the function of such clubs under ideal conditions. I wish rather to tell briefly of my own experience with such clubs which may possibly serve as a practical help to those who have conducted or are about to organize such organizations in their own institutions.

[1] Delivered at the second summer meeting of the Mathematical Association of America, at Cleveland, Ohio, September 6, 1917.

At the very start let me say that the kind of club I have in mind is not the kind which is of interest merely to the two or three ablest men in college. The exceptional man who even in his undergraduate days shows promise of unusual scientific enthusiasm and ability, the kind of man who appears once in several years in the small college, and of whom there is only one in a class in the large college, must be taken in charge by his instructor in any case, and should have careful personal direction so that if he is the kind of man who should be advised to study mathematics professionally, he may be brought in contact with the real thing in mathematical investigation as early as practicable. Such a man ought to be a leader in the club, but if the club is geared for this man, it will be a flat failure. The pace will certainly be too hot for his less gifted fellows, and the result will be a mutual admiration society of about three somewhat unworldly spirits.

The kind of club I *am* talking about is one in which the able freshman can find some interest, and every student who is interested in interesting things and who has taken up the study of analytics can take an active part.

What is the use and the function of such a club as I have indicated? It seems to me that there are three kinds of purpose that such a club may serve, if it is successful. It surely benefits those students who present papers or solutions, it ought to help those who attend, and it should be a good thing for the college as a whole.

Just a word on this last point, before taking up the first two. One of the most unfortunate traditions among the undergraduates of our colleges is the feeling that they are serving their college only when they are playing baseball, or running the football team. Their loyalty is splendid and genuine, and such a precious enthusiasm should not be allowed to cool, even if it is sometimes observed to run in a direction of secondary importance. It is almost impossible for the undergraduate to realize that in the long run the greatest honor to a college is a competent and well-trained graduate, and the most far-sighted loyalty to his alma mater would lead him to prepare himself to be such a graduate. To be an officer of the mathematics club is an honor, and those who work for the welfare of the club are recognized as working for the college. And since its activities have to do with the intellectual life, it should appeal to the most genuine spirit of loyalty to the institution.

Let me emphasize, however, that unless the activities of the club are kept simple, so that it may have the reputation in college of being well within the domain of rationality of a good many students, this feature of the club will be a failure. For a small body of pale people talking about existence theorems is not a spectacle over which college students as a whole will enthuse. I am certain, however, that this idea of organizing certain extra-curricula activities which have contact with things of the mind helps to divert some of the energy of the students from an overwhelming development of athletics. I do not intend to express any lack of appreciation for athletics. Young men must have them. But few will argue that they are not carried to an extreme in many of our colleges, and that a counter-irritant should be sought.

The benefit of the club to the student who attends, but who has no stated part is fairly obvious. It depends, however, on the way the meetings are conducted, whether the auditor gets much out of it or not. In my experience it has been absolutely necessary to conduct the meetings in the most informal manner possible. The speaker expects to be interrupted, and to explain in greater detail questions of notation or demonstration or implication which anyone requests. Often the more carefully and clearly a paper is presented, the greater curiosity it awakens, and the oftener the speaker is asked to make more detailed explanation.

The fundamental fact is that students cannot be expected to go many times to a mathematics club because they feel that they ought to. They will only go because they want to, and nothing but a good time intellectually and socially will hold them. I do not mean by this that there should be meager food for thought, and much entertainment. Quite the contrary. The intellectual nourishment cannot be too good or too well prepared. But it must be adapted for their powers of digestion, and must be served with good fellowship and simplicity.

The benefit of the club to those who give papers is naturally greater than to any one else, and on them depends the success of the club to a large extent. Our responsibility as teachers of college mathematics is nearly as great in the direction of training our students in the difficult art of self expression as in the technique of mathematics. The regular class room instruction, especially in elementary work, consists to a large extent, for many of us, in patiently requiring our students to express the truth that is in them in the form of an English sentence which has a subject, a predicate and the proper qualifying clauses. Now the club affords a splendid opportunity to give a few good students, men who are worth taking pains with, excellent practice and training in the exposition of a specific topic in mathematics. I would remark parenthetically that this means an appreciable amount of painstaking and tactful work on the part of some one on the teaching staff. The students must feel that they do the whole thing, but they must be far from correct. I have rarely found the student who had really adequate terminal facilities. They always prepare papers from twice to four times as long as they should be, and find the greatest difficulty in condensing to advantage. Even if the student sees his problem in the proper perspective, it requires almost superhuman efforts to get him to express it clearly and concisely. But I am forced to say that assisting a student to get an effective method of exposition without doing the work for him is, in my opinion, scientific education of the most important and useful kind, and any means that enables one to render such service is an important feature of the college.

The second great benefit to the speaker in the mathematics club is the experience which he gets in examining the literature of his topic, and going outside his text books for his material.

The third, and the most important of all is the consciousness that comes with independent search for truth. The truth sought is usually not new to

science, but it is new to the student, and has all of the freshness of spring to him. For college students, what might be called "searchwork," that is, topics dealing with *important* questions which contain enough meat to interest any right thinking person, is much more profitable than the discussion of some topic which he has worked out for himself but which one cannot find in the books, or memoirs. In fact, I think that it is a pity for an able student to get the impression that the discovery of some little point that no one else has happened to mention, is mathematical scholarship and a substitute for a broad and wide knowledge of the subject.

My procedure with a student who has signified his desire to prepare something for the club is somewhat as follows. After finding a topic which is within his range, and in which he is likely to be interested, I give him one or two references which he is to read, and to report upon. Then I often tell him of other books in which he had better look, and urge him to go over all the literature he can find which seems to promise well. By the time this is done he can make an outline of what he proposes to say, specifying what he wishes to prove, and what is to be given without proof. The student always wants to prove almost everything, and cannot understand how a paper can omit proofs, and still be a good one, except when he is listening to it. Finally a few days before the meeting I try to go over with him in more detail the plan of his paper, and often the details of exposition. Of course he never speaks from notes, but puts diagrams on the board before the meeting. I have found that if a paper is planned which the student thinks will take twenty-five minutes, it actually takes forty-five if he is not interrupted badly. If he is interrupted, he never gets through, but has to summarize the last part. I make great efforts, however, to keep the papers down to twenty-five minutes of talk.

Undoubtedly the kind of organization which would be best adapted to students of one institution might not be suited to another. I have had experience with only two types of club at Columbia College. At first there was no organization at all. Meetings were held every two weeks by those who were interested to come, but without undergraduate officers, no dues, and no machinery at all. I always presided, in the most informal manner, and it was a purely spontaneous expression of interest in mathematics on the part of those who attended. After a couple of years of this sort of procedure, the students suggested that they have a regular club with officers, dues of twenty-five cents per year to cover the expense of sending cards to members, and a written constitution.

So far as the essential features of the club are concerned, I cannot see that it makes the slightest difference what kind of organization is in force. The fact from which one cannot get away, is that a mathematics club will not run itself, and that it requires a great deal of careful work to get speakers who can attract a good house. If the organization is such that some member of the staff can really look after this feature of the work and at the same time keep in the background so that the students seem to be in charge, it does not make much difference whether there are many officers or few, or whether there is a constitution or not.

I am inclined to think, however, that the club is of more service to the college community if there are undergraduate officers, and definite organization. If the professors can attend not as professors but as fellow creatures of the students there is no objection. But the moment that the boys feel that another disguised recitation is in progress, and that their professors are watching their accomplishment, the club is no longer a club, but a parade ground.

The most important and at the same time the most difficult feature of the club to manage, is the program. Especially during the first part of the year, papers must be provided which will attract the students and hold their interest. I suppose that the mathematical topics which naturally attract the student who is not far advanced, are those which he thinks have some mystic significance. Something about higher dimensions, or infinity, or paradoxes. Personally I think that the greatest service which a club can perform in regard to these topics is to denude them of their mystic qualities, and to bring the ideas involved squarely and fairly into relations with sound mathematics. But students cannot perform this service, for they have not the perspective, and so far as my experience goes, make a bad matter of any topic relating to these subjects. Consequently, it usually seems wisest for an older person to talk about the fourth dimension and orders of infinity, rather than a student.

One might think papers of an historical nature admirably adapted for presentation at the club. But to my great surprise, I have had very little luck with such papers. It may be my own fault, but I have found that the students do not have a sufficiently broad and scholarly grasp of the subjects whose history they are presenting to make topics of this kind rewarding. Here again, a member of the staff does much better.

A distinguished educator once complained to me that the teachers of mathematics and physics missed a great opportunity in not humanizing the presentation of their subjects. He said that a boy might study physics two years and not know whether Boyle was an Irishman or a very painful swelling on one's person. I have made a serious effort to get students to prepare papers on the "heroes of mathematics." They like the idea immensely but when it comes to preparing a paper showing just where Napier, Newton, Archimedes or Descartes stood in relation to what preceded and what followed, and what specific problem stimulated them to carry forward their epoch making work, I have sometimes been disappointed, owing again to the lack of perspective. But on the whole such papers have been much more successful than those which attempt to treat the development of an idea or set of ideas.

At least one excellent meeting a year can be provided by asking the students to bring paradoxes or puzzles of a mathematical nature. These may be presented, and solutions requested from the audience.

I have found no difficulty in finding enough interesting topics which involve little or no calculus. Such a procedure does not seem to result in the loss of interest on the part of the more mature students, for they feel the need of a more complete perspective over their elementary mathematics, and as a matter of

fact, a paper which involves a good deal of the calculus cannot be followed in detail by as arge body of students as one likes to have at the club. And, unlike the members of more pretentious mathematical societies, students do not enjoy sitting through a long paper, no part of which they understand with the possible exception of the title. And *they* will not do it.

I have mentioned several types of topics which students find some difficulty in presenting effectively. It is much easier to do that than it is to generalize regarding the kind of paper which is successful. For here, as in most questions involving education, it all comes back to the man behind the guns. If the student himself is thoroughly interested in a topic he can do wonders with it. If not, he can spoil the best subject in the world.

On the whole the students do best on topics which are very definite, and in which there is a clear relation between the new idea which they are presenting and some thing with which they are familiar. For example, I recall a very excellent paper on the exponential and logarithmic function of complex numbers, in which the well-known ideas of many-valuedness, and periodicity as found in trigonometric functions were taken for points of departure, and these same features of the functions under discussion were exposed. Also the paradoxes which arise when one takes the wrong branch of a many-valued function affords instruction and amusement.

Imaginary lines in the plane are also interesting. Any student of analytics can prove that the distance between two points on such lines is zero, and a number of other curious features. Here the hope of success of the paper lies in connecting the new ideas with the similar ones regarding real lines.

Another kind of topic which is likely to be successful is the graphical study of processes which are more or less familiar analytically. Such subjects as the graphical solution of equation, the study of constructions which are possible with the use of the compasses and a fixed parabola, construction with a two-sided straight edge are all excellent.

Since these remarks are simply intended to start a discussion in which it is hoped that many will take part, I do not propose to follow the example of the students in talking twice as long as I should. I must, however, express my conviction that a mathematics club in a college is a powerful stimulus on the side of sound scholarship, and to my certain knowledge able men are inducted through it into the serious study of mathematics, and stimulated in their ambition not only to know the subject so far as their abilities permit but to add to it as well. I know of no activity in which a college professor of mathematics can more profitably spend some effort, than in directing a club. For not only is he helping the cause of scholarship, but he is making a subject which has the reputation of being somewhat austere, an object of living and spontaneous interest on the part of a good body of students, and giving it a place in the college community.

CLUB ACTIVITIES.

The Mathematics Club of Albion College, Albion, Mich.

This club was founded in January, 1911, "to create a greater interest in present-day mathematics and allied subjects, and to study best methods of teaching mathematics." Any student showing unusual ability during the first three semesters of the college course may be elected to membership in the club. All nominations are made by the head of the department, Professor Edwin R. Sleight.

Meetings last for about an hour and the usual routine is as follows: (1) roll-call, each member responding to his name by a short discussion (2 or 3 minutes as a maximum)[1] of an assigned topic [there is never any trouble with this part of the program]; (2) short talk of the evening, limited to 10 minutes; (3) topic of the evening, occupying 20 to 30 minutes; (4) general discussion of any of the topics of the evening; (5) critic's report (the critic is usually a student).

Officers are elected at the close of each semester for the one following; those elected for the first semester of 1917–18 were: President, Lucille Ball '18; vice-president, Esther Turnell '18; secretary-treasurer, Myrtle Speese '19; program committee: Professor Sleight, the vice-president, the secretary-treasurer, and Gertrude Landon '19. For the second semester the officers were: President, Alice Money '18; vice-president, Myrtle Speese '19; secretary-treasurer, Floyd Harper '20.

Programs for the past two years are indicated below [(1) roll call, (2) short talk, (3) evening topic]. The short talks for 1916–17 were historical. "The programs for a normal year are so arranged that each member of the club gives one short talk and one long talk each semester. The programs for 1917–18 are not a fair sample of the work done by the club. We are so rushed by having classes six days of the week, and having practically no vacations that we decided that the programs for the year should, for the most part, be merely reviews of articles found in periodicals."

October 3, 1916: (1) None; (2) "Euclid" by Jesse Campbell '18; (3) President's address "The place of mathematics in the world" by Hazel Miller '17.

October 10: (1) "A recent invention to which mathematics was applied;" (2) "Archimedes" by Gertrude Landon '18; (3) "The forward movement in mathematics" by Alice Money '18.

October 17: (1) "Integration" [each person as his name was called was sent to the board and asked to integrate a form]; (2) "Apollonius of Perga" by Myrtle Speese '19; (3) "Some unusual integrals and their solutions" by Ralph Huffer '18.

October 24: (1) "A practical application of a geometrical theorem;" (2) "Diophantus of Alexandria" by Hazel Miller '17; (3) "A comparison of geometry with mechanics" by Gladys Harger '17.

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[1] There were 19 members in 1917–18 and this number necessarily limited the time which each individual might occupy. Of the 19 members only 4 are young men. Before the war membership was about equally divided between men and women.

October 31: (1) None; (2) "Vieta" by Esther Turnell '18; (3) "Fourier's series and its application" by Professor Arthur F. Beal.

November 7: (1) "My mental attitude toward mathematics;" (2) "Descartes" by Alice Money '18; "The development of mathematics in the United States during the nineteenth century" by Myrtle Speese '19.

November 14: "Some interesting celestial objects" (illustrated lecture, open session) by Professor Beal.

November 28: (1) "A peculiar geometrical construction;" (2) "Gauss" by Vera Junkin '19; (3) "A geometrical representation of multiplication and division" by Mildred Chappell '19.

December 7: (1) "Mathematics in poetry" (quotations referring to mathematics); (2) "Newton" by Ralph Huffer '18; (3) "Mathematics and psychology" by Esther Turnell '18.

December 14: (1) "A coördinate system;" (2) "Leibnitz" by Gladys Harger '18; (3) "Homogeneous coördinate systems" by Gertrude Landon '19.

January 16, 1917: This program consisted of four short talks: "Right and wrong definition of a limit" by Lloyd Crippen '19; "Distributive law of multiplication geometrically proved" by Esther Turnell '18; "The teaching of factoring" by Gladys Harger '17; "Judging a teacher of mathematics" by Myrtle Speese '19.

January 23: (1) "Who's who in mathematics in America;" (2) "A simple solution of a Diophantine equation" by Lucille Ball '18; (3) "The circles of Apollonius" by Ralph Huffer '18.

February 6: (2) "Lagrange" by Myrtle Speese '19; (3) "Tests and short cuts in mathematics" by Jesse Campbell '18; (1) Application of the short cuts given in (3).

February 20: (1) "An interesting event in the life of some mathematician;" (2) "Euler" by Gladys Harger '17; (3) "Some simple applications of congruences" by Hazel Miller '17.

February 27: (1) "Mathematical current events;" (2) "Cauchy" by Edna Colwell '19; (3) "War and mathematics" by Vera Junkin '19.

March 6: (1) "Name of some subject in higher mathematics and its application;" (2) "Steiner" by Margaret Courtright '19; (3) "Different kinds of geometries, with special emphasis on descriptive" by Ian Patterson '19.

March 13: (1) "Mathematics applied to some definite problem;" (2) "Abel" by Mildred Chappell '19; (3) "The future of mathematics" by Mary Hutchins '19.

March 20: (1) "Some interesting personal experience in a mathematics classroom;" (2) "Hypatia" by Murray Fox '18; (3) "Relation of the teacher to the community" by Alice Money '18.

April 3: (1) "Some theorem used in algebra—statement and application;" (2) "Galois" by Gertrude Landon '19; (3) "Discussion of Fermat's theorem" by Ralph Huffer '18.

April 10: (1) "A brief review of some new text-book;" (3) Report by Edna

Colwell '19 who had been elected to attend the mathematics section of the State Schoolmasters' Club held at Ann Arbor.

April 17: (1) "Name of some mathematician connected with function theory, and a brief statement of what he did;" (3) "Teaching of mathematics in secondary schools" by Principal Harry R. Atkinson of Battle Creek, Michigan.

May 1: (1) "The name of some educational society, together with a brief statement of its purpose;" (2) "Sylvester" by Jesse Campbell '18; (3) "Variables and limits" by Lloyd Crippen '19.

May 8: (1) "Mathematical fallacies;" (2) "Weierstrass" by Ian Patterson '19; (3) "How to make the teaching of mathematics interesting" by Hazel Miller '17.

May 15: (1) "Some unusual function;" (2) "Lie" by Murray Fox '19; (3) "Eulerian integrals and gamma functions" by Lucille Ball '18.

May 22: (1) None; (2) "Origin of certain typical problems" by Esther Turnell '18; (3) "Poincaré and his contributions to mathematics" by Margaret Courtright '19.

May 29: Social evening and election of officers for the following semester.

October 16, 1917: President's address by Lucille Ball '18.

October 23: (1) "Mathematical current events;" (3) "Geometrical proof of formulas for sine, cosine and tangent of half angles" by Vera Junkin '19; "Algebraical developments in Ancient Greece and Babylonia" by Margaret Courtright.

October 30: (1) "Some theorem in college algebra proved;" (3) "Condition that three lines pass through the same point" by Joyce Hadaway '20; "Freshmen and freshmen algebra" by Esther Pearl '20.

November 6: (1) "Historical theorem known by the name of the discoverer;" (3) "Methods for solving irrational equations" by Esther Turnell '18; "The students viewpoint of calculus" by Alice Money '18.

November 13: (1) "A peculiar logarithmic combination;" (3) "Mathematics of warfare" by Ralph Huffer '18.

November 27: (1) "A peculiar geometrical construction;" (3) "A new proof for the law of sines" by Carla Kennedy '20; "Mathematics as a means of culture and discipline" by Elizabeth Hubert '20.

December 11: (1) "A mathematical fallacy;" (3) "Significance of mathematics" by Lucille Ball '18; "A historical account of mathematical induction"[1] by Myrtle Speese '19.

January 8, 1918: (1) "A recent text book;" (3) "Projective geometry—a historical account and some applications" by Mildred Chappell '19.

January 15: Election of officers.

February 26: (1) "The history and meaning of mathematical symbols;" (3) President's address: "Why mathematics should be studied in the high school" by Alice Money '18.

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[1] Cf. F. Cajori, (1) *Bulletin of the American Mathematical Society*, Vol. 15, pp. 407–408, 1909; (2) "Origin of the Name 'Mathematical Induction,'" in this MONTHLY, Vol. 25, pp. 197–201, 1918.

March 5: (1) "A statement of a problem whose solution is the irrational root of a cubic equation;" (2) "Some suggestions on the teaching of geometry" by Dorothy Tichenor '20; (3) "Solutions of the cubic" by Almira Priest '20.

March 19: (1) "A mathematical puzzle;" (2) "Valid aims and purposes for the study of mathematics in secondary schools" by Don Alexander '20; (3) "The function of mathematics in scientific research" by Carlton Sawyer '20.

April 2: (1) "An original mathematical limerick;" (2) "The Perry method" by Joyce Hadaway '20; (3) "A review of some old arithmetics" by Carla Kennedy '20.

April 16: (1) "Helpful hints to the teacher;" (2) "The heuristic method" by Vera Junkin '19; (3) "The planimeter" by Gertrude Landon '19.

April 30: Social evening, and election of officers for next semester.

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Maryland.

This club of young women was founded in November, 1913, in order "to promote a spirit of comradeship in the department, to stimulate interest in mathematics and to provide an opportunity to discuss many topics which are not included in the regular course." All students who have completed the first courses in analytic geometry and in calculus, given the first semester of the sophomore year, are eligible for membership. During 1917–18 the club had 22 members and the average attendance at the meetings was about 15. There are no officers but Professor Florence P. Lewis usually presides and acts as program committee. "At the close of each meeting we have very informal discussion and very light refreshments furnished usually by the speakers of the evening."

"Previous to 1916–17 we had no course on the history and teaching of mathematics and so devoted the club work to these subjects. We tried to cover the history in a very elementary general way in two years of club work. A secretary was appointed to make of the work a typewritten résumé, copies of which were given to each girl."

November 12, 1917: "The parallel axiom and modern work on foundations of geometry" by Professor Lewis.

November 26: "Trisection of an angle and the duplication of a cube" by A. Marie Whelan '18.

December 10: "A demonstration lesson in the teaching of geometry" by Miss Elizabeth White, teacher of mathematics, Eastern High School, Baltimore, Md.

January 21, 1918: "The triangle and its circles"[1] by Margaret Amig '19.

February 18: "Geometry of four dimensions" by Marguerite Lehr '19.

March 4: "The history of the invention of logarithms" by Teresa Cohen '12, graduate student of Johns Hopkins University.

[1] See this MONTHLY, 1918, p. 231, note 2.

April 15: "Construction of a pentagon" by Anna L. Ellery '20; "Fallacious proof that all triangles are equal" by Ethel H. Fox '19; "Orthogonal circles" by Effie M. Gray '18.

April 29: "Illustration of non-Euclidean geometry on the Poincaré sphere" by Professor Clara L. Bacon.

May 18: Club picnic at Herring Run.

May 20: "Graphic methods of presenting facts" by Mildred Grafflin '20; "A new theorem on equal circles"[1] by Ethel R. Carroll '20; "History of famous problems in algebra" by Ida R. Marshall '20.

TOPICS FOR CLUB PROGRAMS.

13. Constructions with a Double-Edged Ruler.

It is just about one hundred years since the French mathematician and engineer Jean Victor Poncelet wrote his famous *Traite des propriétés projectives des figures*[2] which entitles him to rank as one of the greatest contributors to the development of modern geometry. On pages 187–190 are given indications of the proofs that: *Every geometrical construction with ruler and straight edge is also possible*: (1) *with straight edge alone if a single circle and its center have been drawn in the plane* (a theorem often incorrectly attributed to Steiner as originator[3]); and (2) *with an angle*[4] (of wood say) *of given opening α, alone.*

Particular cases of the double-edged ruler (2) are: (3) a square, when $\alpha = 90°$; and (4) a ruler with parallel edges when $\alpha = 0°$

As a proof of (1) Steiner found it convenient to show that the means at his disposal were sufficient for solving the following problems:[5] (*a*) To draw through a given point a parallel to a given line; (*b*) to produce a given line segment its own length any given number of times or to divide it into any number of equal parts; (*c*) to draw through a given point a line perpendicular to a given line; (*d*) to draw through a given point a line which makes with a given line a given angle; (*e*) to bisect a given angle or to make an angle any number of times greater; (*f*) to draw from a given point in a given direction a segment equal to a given segment; (*g*) to determine the points of intersection of two circles whose radii and centers are given; and (*A*) to determine the points of intersection of a given line with a circle the position of whose center and the length of whose radius alone are given.

Recent proofs of (2), (3) and (4) have been, in effect, reduced to showing that these same constructions could be carried through by the special instruments in

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[1] See this Monthly, 1918, p. 231, note 3.

[2] Paris, 1822; see also 2e éd., tome 1, 1865, pp. 181–184, 413–414. Poncelet tells us that this great work was the result of research made in 1813–14 when in Russian prisons "privé de toute espèce de livres et de secours, sur-tout distrait par les malheurs de ma patrie et les mien propres."

[3] Hence the expression, "Steiner constructions" employed in this connection should be "Poncelet constructions."

[4] "Fausse équerre."

[5] J. Steiner, *Die geometrischen Constructionen ausgeführt mittelst der geraden Linien und eines festen Kreises*, Leipzig, 1833, pp. 2–3, etc.

question. Such is the mode of procedure of Adler in 1890 and 1906,[1] of Enriques in 1903,[2] of Giacomini in 1907[3] and of Killing and Hovestadt in 1910.[4]

For separate discussion of constructions by means of a ruler with parallel edges reference may be given to papers and notes by Lebescond de Coatpont and De Tilly,[5] Marenghi, and Concina.[6]

Since a parallelogram may be drawn at once with a double-edged ruler note that the following famous problem was discussed by Lambert,[7] s'Gravesende,[8] Poncelet,[9] Tractenberg[10] and Child[11]: "Given a parallellogram, construct with straight edge alone a parallel to a given line."

Important practical applications, in surveying and warfare, of constructions involving a double-edged ruler are indicated in the remarkable little book of F. J. Servois, *Solutions peu connues de différens problèmes de géométrie-pratique* . . . A. Metz, An XII [1804], (especially pages 68–79), and in the very interesting work of Gohierre de Longchamps to which reference has been made already. Solutions of surveyors' problems with the square were given also by L. Mascheroni in his *Problemi per gli agrimensori con varie soluzioni,* In Pavia, MDCCXCIII.[12]

With ruler and compasses, only special problems of degree higher than the second can be solved, no difference how many of these instruments may be em-

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[1] A. Adler, (1) "Über die zur Ausführung geometrischer Constructionsaufgaben zweiten Grades notwendigen Hilfsmittel," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der kaiserlichen Akademie der Wissenschaften zu Wien*, Band 99, Abtheilung IIa, Jahrgang 1890, Heft 8; (2) *Theorie der geometrischen Konstruktionen*, Leipzig, 1906, pp. 123–138.

[2] F. Enriques, *Vorlesungen über projektive Geometrie*, Leipzig, 1903, pp. 266–277; 2. Aufl. Leipzig, 1915, p. 254.

[3] A. Giacomini, "Über die Lösung der geometrischen Aufgaben mit dem Lineal und den lineal Instrumenten . . .," pp. 95–103 of *Fragen der Elementargeometrie* gesammelt und zusammengestellt von F. Enriques, II. Teil, Leipzig, 1907; Italian edition, Bologna, 1914, pp. 89–96.

[4] W. Killing und H. Hovestadt, *Handbuch des mathematischen Unterrichts*, Band I, Leipzig, 1910, pp. 194–199.

[5] "Sur la géométrie de la règle," *Nouvelle correspondence mathématique*, tome 3, 1877, pp. 204–208; tome 5, 1879, pp. 439–442; tome 6, 1880, pp. 34–35.

[6] C. Marenghi, "Geometria della riga a due orli paralleli," *Il bolletino di matematiche e di scienze fisiche e naturali*, Bologna, Vol. 2, 1900–1901, pp. 129–145; U. Concina "Resoluzione dei problemi fondamentali relativi al trasporto delle figure piane colla riga a due orli paralleli," *idem*, 1901, pp. 225–237.

[7] J. H. Lambert, *Freie Perspective*, 2e éd., Zürich, Vol. 2, p. 169. This solution is given also in: (1) *Mathematical Questions and Solutions from the 'Educational Times,'* Vol. 57, London, 1892, p. 88; and (2) J. W. Russell, *An Elementary Treatise on Pure Geometry*, New and revised edition, Oxford, 1905, p. 318.

[8] W. J. s'Gravesende [died 1742], *Oeuvres philosophiques*, Amsterdam, 1774, Ire partie, § 312, p. 174.

[9] *Traité des propriétés*, etc., 1822, pp. 106–107; Lambert's solution is also given here. Both of these solutions are reproduced in L. Cremona, *Elements of Projective Geometry*, translated by C. Leudesdorf, Oxford, 1885, pp. 96–97; Poncelet gave his solution of the *Traité* and two others in his *Applications d'analyse et de geométrie,* Paris, 1862, pp. 437–439 (this part of the work was written while he was in prison in 1813–14). One of these solutions is also reproduced in G. De Longchamps, *Essai sur la géométrie de la règle et de l'équerre*, Paris, 1890, pp. 232–235.

[10] M. I. Trachtenberg, *Mathematical Gazette*, 1908, Vol. 4, p. 334.

[11] J. M. Child, *Mathematical Gazette*, 1910, Vol. 5, pp. 283–284.

[12] Second edition with title: *Problemi di geometria colle dimostrazioni del Capitaus Sacchi*, Milano, 1803; third edition, 1832. French translation of the first edition: *Problêmes pour les arpenteurs avec différentes solutions*, Paris, An XI–1803; 2e éd., with title: *Problèmes de géométrie pratique pour les arpenteurs avec différentes solutions*, Paris, 1838.

ployed. In contrast to this it is interesting to note, as Adler (*l. c.*) emphasizes, that the right angle (square) is a much more powerful instrument. Indeed Plato's instrument for the solution of the problem of the duplication of a cube[1] is equivalent to the use of two right angles to solve a binomial cubic equation. So also four right angles could be used to solve a fifth degree binomial. Descartes employed a similar instrument with right angles[2] for the insertion of several geometric means between two line segments.

The fundamental ideas here introduced were adapted by Lill[3] so as to extend to the solution of polynomial equations with numerical coefficients. From this Adler readily formulated, in particular, the result: Problems of the third and fourth degree can be rigorously solved geometrically by means of several right angles.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

SEND WAR SERVICE COMMUNICATIONS TO DR. HENRY BLUMBERG, University of Illinois.

FIRING DATA.

By J. K. WHITTEMORE.

In the first part of this paper, I shall outline a course in "Firing Data," and in the second part, make some suggestions as to the conduct of such a course. The outline is based on the course given in the last college year in the Yale R. O. T. C. The course in firing data includes all the mathematics necessary for an officer of the Field Artillery, and is an extremely important part of the training for a commission in that service. The work, as here described, applies to the U. S. 3-inch gun, the British 18 pounder, and the French 75 mm., but the tables used in the examples apply only to the U. S. 3″ gun; they are given in Danford and Moretti's "Notes on Training Field Artillery Details," Yale University Press, Sixth Printing, Feb., 1918.

The fundamental firing data problem is the computation of data for the battery for the opening salvo in indirect laying from observations made at the battery commander's station. These data are range, site, deflection and deflection difference. Indirect laying is pointing a gun at a target invisible at the gun; both target and battery must be visible at the battery commander's station.

The course begins with certain definitions which must be thoroughly learned

[1] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 3. Auflage, Band I, Leipzig, 1907, pp. 227, 353. *Cf. L'Algèbre d'Omar Alkhayyâmî* publiée . . . par F. Woepeke, Paris, MDCCLI, pp. 94–96; A. Conti, in *Fragen der Elementargeometrie* (Enriques), Band 2, p. 215; and A. Adler, *l. c.*, 1906, p. 237.

[2] *La Géometrie*, livre II, *Oeuvres de Descartes* publiées par C. Adam et P. Tannery, Vol. 6, Paris, 1902, p. 391.

[3] E. Lill, (1) Résolution graphique des équations numériques de tous les degrés à une seule inconnue, et description d'un instrument inventé dans ce but," *Nouvelles annales de mathématiques*, 1867, (2), tome 6, pp. 359–362; (2) "Résolution graphique des équations numériques d'un degré quelconque à une inconnue," *Comptes rendus de l'académie des sciences*, Paris, tome 65, pp. 854–857. See also T. Vahlen, *Konstruktionen und Approximationen in systematischer Darstellung*, Leipzig, 1911, p. 141.

by the student. These are given alphabetically in Drill Regulations for Field Artillery and in Danford and Moretti. This arrangement is convenient for reference, but the definitions should be grouped in proper connection for instruction. For the comprehension of this paper it is necessary to give the following:

Mil, a unit of angular measure, is one sixteen hundredth of a right angle, and is approximately one thousandth of a radian. The numerical measure of an angle in mils exceeds one thousand times its radian measure by about two per cent. The usefulness of the mil depends on the formula for small angles, $m = c/k$, where m is the angle in mils subtended by a line c units long at a distance of 1,000 k units. If c is the side of a right triangle opposite the small angle m mils, and 1,000 k the adjacent side, then m is given by the formula with an error not greater than two per cent. if m is not greater than 330. The formula is exact for a certain value of m nearly equal to 270. The same formula is used for small m when 1,000 k is not the leg but the hypothenuse of the right triangle. In this case the value of m given by the formula is always too small, the error varying from two per cent. at zero to about three per cent. for $m = 300$.

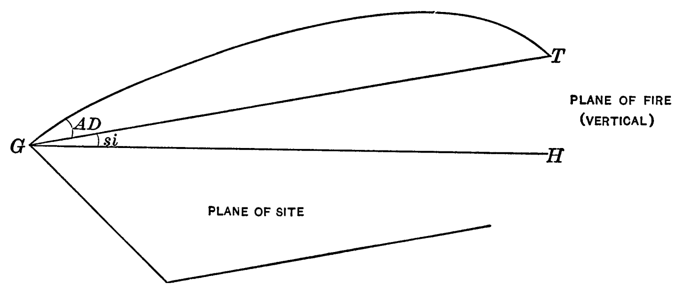

FIG. 1.

Plane of site is the plane containing the line GT and the horizontal line perpendicular to the axis of the bore of the directing gun. G is the muzzle of the directing gun for which the data are computed, and T, the target. (Fig. 1.)

Site (si) is the angle in mils of the plane of site with the horizontal plane. It is positive or negative according as T is above or below G. To avoid the use of negative numbers we use SI, where $SI = 300 + si$. The term site is used for other elevations; thus we speak of the site of the target at the battery commander's (BC) station. (Figs. 1 and 2.)

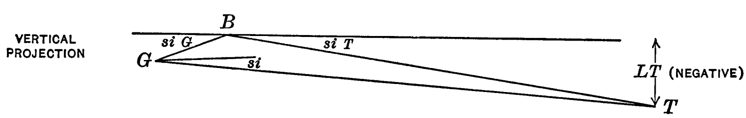

FIG. 2.

Angle of departure (AD) is the angle in mils of the tangent to the trajectory of the projectile at the point of departure from the gun with the plane of site. (Fig. 1.) For most service conditions in the Field Artillery the trajectory may

be regarded as rigid without important error, so that change of site produces no change in the relation of range and AD. Actually the elevation of the gun is given by two mechanisms, on one of which the site is laid off, and on the other, the range.

Aiming point (P) is the object on which the panoramic sight of G is directed. (Figs. 4 and 5.)

Deflection (D) is the angle PGT measured counterclockwise (or the angle TGP measured clockwise) as set off on the panoramic sight. (Figs. 4 and 5.)

Deflection Difference (DD) is the difference in deflection applied to other than the directing gun of the battery necessary to bring it to bear on the proper portion of the target. For parallel fire DD is the parallax of the aiming point. (Fig. 6.)

Distribution is the lateral relation of the shots of a salvo.

After the definitions necessary have been well learned, there should be considerable practice in the use of mils. One important problem depends on nothing else, two others only on the use of mils and the range table. The first of these is the determination of the site from observations at the B. C. station (B). For this the formula (Fig. 2)

$$SI = 300 - \frac{LG - LT}{R}$$

may be used, where LG and LT are the levels of G and T above B, and R is one thousandth of the range. These levels are negative if G and T are below B, and this fact often leads to confusion. I think it better to not use any formula for this problem, rather to use a figure and to describe relative elevations as above or below. Such a problem is usually formulated as follows: SI G at $B = 260$, SI T at $B = 285$, $BG = 270$, $BT = 3000$, $Rn = 3100$. Find SI. We have $LT = -45$, T 45 below B, $LG = -11$, G 11 below B, T 34 below G, $si = \frac{-34}{3.1} = -11$, $SI = 289$.

The next of these problems is that of the clearance of a crest of known height and distance when firing on a target whose site and range are given. The clearance is given in mils by the formula

$$AD + si - \frac{c}{k} - ADC,$$

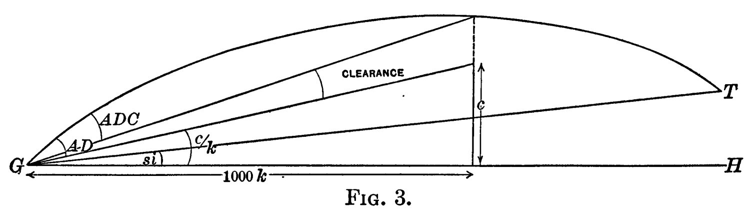

FIG. 3.

where c is the height of the crest in yards, 1,000 k the distance or range of the crest in yards, and ADC the angle of departure for the range of the crest. If

the value given by the formula is negative, the crest is not cleared. If we set the clearance equal to zero and solve for AD we have the angle of departure for the minimum range for the crest. The proof of this formula follows immediately from Fig. 3, provided that all angles involved are small enough to permit the use of the mil formula, as is usually the case in practice. An example of this kind requires evidently the use of a range table connecting AD and Rn (Danford and Moretti, page 254). Such a problem is usually formulated as follows:

Crest 20 yards high, 400 yards from battery; SI 295, Rn 2600. How much is crest cleared in mils and yards? What is minimum range for crest?

The clearance is $76 - 5 - 50 - 7 = 14$ mils, 5.6 yards.

AD for minimum range is 62, 2,250 yards.

The third problem is that of the proper location of the battery to the rear of a crest for a given target. It may be shown, if the tangent to the trajectory for the given target from a gun supposed on the crest is produced backwards to intersect the level ground to the rear of the crest, that a battery placed at the point of intersection will always clear the crest when firing on the given target. Such a problem is the following:

Crest is 15 yards above ground to the rear. Observer on crest finds for T, SI 290, Rn 3500. Where should guns be located? What is SI? How much is crest cleared?

Guns should be to rear of crest $1{,}000 \times \dfrac{15}{115 - 10} = 140$ yards. $SI = 300 - \dfrac{20}{3.64} = 294$. Clearance is $122 - 6 - 105 - 2 = 9$ mils, 1.3 yards.

These three kinds of problems are of considerable importance, furnish good practice in the use of mils, and give the student an opportunity to become familiar with the range table. I do not think that these problems should be once treated and practised and then laid aside, rather the student should be required to work at them occasionally throughout the whole course.

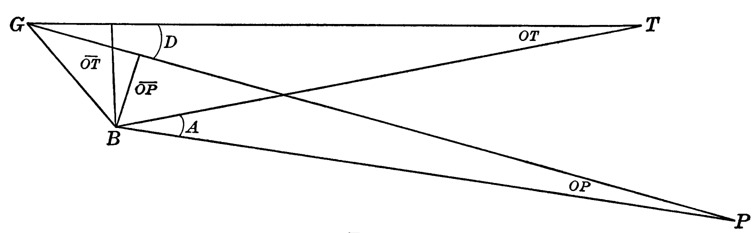

FIG. 4.

The most important of the problems of firing data is the calculation of the deflection for the directing gun, usually no. 2, as we count from the right when facing T. This problem should be introduced as early as possible and should be dwelt upon and hammered in until the students can do one in two minutes and until they consider it the easiest thing in the course. In Figs. 4 and 5, G is the directing gun, T the target, P the aiming point, B the B. C. station; the deflec-

tion D is the angle PGT measured counterclockwise, and is evidently greater or less than 3,200 mils according as P is to the left or right of GT; the angle PBT, which is called A, is measured at B; the angles at T and P, which in practice are small, are the *offsets* (offset angles) of target and aiming point respectively; the perpendiculars from B to GT and GP are also called offsets or offset distances. It is easily seen that

$$D = A \pm OT \pm OP,$$

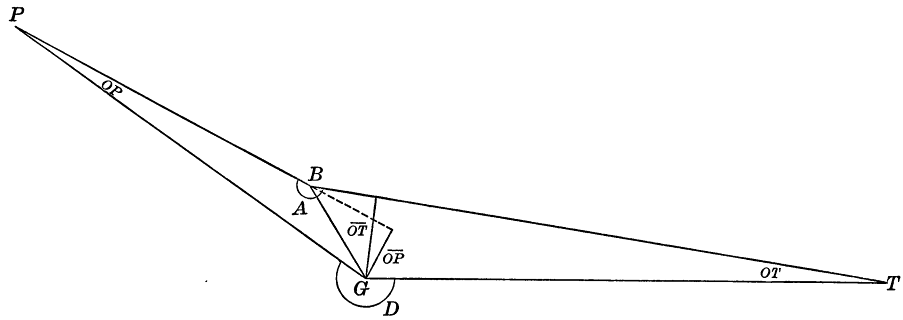

FIG. 5.

OT and OP denoting offset angles of target and aiming point respectively. The offset angles are found in mils from the offset distances,

$$OT = \frac{\overline{OT}}{\frac{1}{1000}BT} \qquad OP = \frac{\overline{OP}}{\frac{1}{1000}BP}.$$

It is clear that these formulas gain in accuracy by choosing B near G and P remote from B. In practice examples the distances BT, BP, $\overline{OT}$, $\overline{OP}$, are at first given. The range GT differs little from BT if B is near G, and may be taken as BT with a small correction, to be added or subtracted according as B is in front of or behind the line of guns, and given to the nearest hundred yards. The chief difficulty of the problem is in giving proper signs to the offset angles in the formula for D. These signs may be determined from a figure but this does not always eliminate the difficulty, as angles greater than a straight angle must frequently be used. I have used the following rules and believe they save time and make for accuracy:

1. B to left of GT, $+ OT$; right, $- OT$.
2. P behind BG, OP same sign as OT; front, opposite sign.

If in Fig. 4, we suppose $BT = 3{,}600$; $BP = 4{,}800$, $BG = 240$, $\overline{OT} = 200$, $\overline{OP} = 160$, $A = 710$, we have $OT = 56$, $OP = 33$, $D = A - OT + OP = 690$ (687), deflection being given to the nearest ten. $Rn = 3{,}700$.

In field practice, of which I shall say more later, the distances BT and BP are measured with a range finder, or may be taken from a map, or in case of necessity, estimated, *not guessed;* BG may be measured or estimated, or computed as follows: the gunner lays off 10 or 20 yards at G in a line perpendicular to BG, the angle subtended is measured at B, and BG is then computed from the mil

formula. But it is not easy to measure or to estimate accurately the offsets OT and OP. For this reason the method is modified (Fig. 5) by drawing the offset lines from G perpendicular to BT and BP; the offsets are then found from $\overline{OT} = BG \sin GBT$, $\overline{OP} = BG \sin PBG$; the range may if necessary be computed more accurately from $Rn = BT - BG \cos GBT$. This method requires the measurement of *two* angles at B. I think it is not desirable in this work to use the trigonometric functions of any but acute angles. The sign of the correction to BT for range can be seen at once, and is negative or positive according as B is to the rear or in front of the guns. The trigonometric work need not be accurate to more than one per cent. A trigonometric table of sines and cosines giving these functions to two decimals for every hundred mils is adequate, and may easily be written on a small card. It is most desirable and important that the student acquire great speed as well as accuracy in working deflection problems.

The deflection difference (DD) for parallel fire given to a battery with guns at equal normal intervals of 20 yards is that for gun no. 3 with no. 2 as directing gun. The DD for no. 4 is double that for no. 3, DD for no. 1 is that for no. 3 with the sign reversed. The DD given is positive or negative according as P is in front or to the rear of the line of guns, and is given to the nearest five mils. It is evident from Fig. 6, which is not drawn to scale, that in mils

$$DD = \frac{20 \cos D}{\frac{1}{1000} GP}$$

In practice average values of cos D, called obliquity factors, are used, each running for 400 mils. These factors are 1, .9, .7, .4, 0; the factor 1 runs from $D = 6{,}200$ through 6,400 to 200. The obliquity factors may conveniently be incorporated in the card trigonometric table whose use is suggested. The

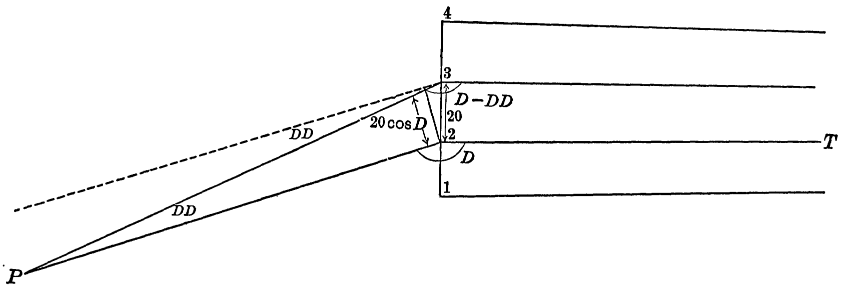

FIG. 6.

simplest method of computing DD is to write first the sign, second the obliquity factor, then $20/\frac{1}{1000}GP$, compute, and give result to the nearest five. In practice, since BP is known, and generally differs little from GP, BP may be used in place of GP. Suppose, for example, DD is required for parallel fire for battery at normal intervals; given GP 2,500 yards, D 2,300.

$$DD = -\ .7 \frac{20}{2.5} = -\ 5,\ (-\ 5.6).$$

An important type of problem, requiring only the use of the mil formula and the idea of *DD*, is the problem of distribution. Such problems are rather in "fire control" than in firing data, having to do with the correction of data from observation of the results of firing, but they have practical value and are of use to the student, and may well be considered in connection with the work in *DD*. They appear to be extremely confusing to the student, and for this reason I give in detail the solution of such a problem explaining what seems to me a good method of arranging the work. The following problem, slightly modified, is given in Danford and Moretti, no. 19, page 265:

Target is a battery at twice the normal interval, *Rn* 4,000. Battery is in position with intervals

.|. 24 .|. 8 .|. 16 .|.

A. P. is 2,500 yards distant, *D* is 2,300, and *DD* for parallel fire, guns at normal intervals, has been given to the guns. At the first salvo, the right gun is on its target. Give commands necessary to put the other guns on their proper targets. Draw diagram.

The diagram I draw as follows: The line of guns is represented on a vertical line at the left of the paper, the line of fire to the right across the paper, the target a vertical line on the right, *T* the point of impact of no. 2, the directing gun, the range written on the line of fire. The next step is to compute *DD* as given. This was found in a preceding paragraph to be -5. Determine next the point of impact of each gun with reference to *T*, points above *T* being represented by $T+$, below *T* by $T-$. We require for this the correct *DD* for parallel fire for each gun. These are respectively for nos. 3, 4, 1,

$$-7\frac{8}{2.5}=-2, \qquad -.7\frac{32}{2.5}=-9, \qquad +.7\frac{16}{2.5}=+4,$$

each computed to the nearest mil. These *DD* if used would give respectively as points of impact $T+8$, $T+32$, $T-16$. The *DD* actually used are respectively -5, -10, $+5$. The differences between the correct *DD* for parallel fire and those used produce changes in yards at 4,000 yards of -12, -4, $+4$, so that actual impact on the line of the target are for 1, 2, 3, 4 respectively

$$T-16+4=T-12, \quad T, \quad T+8-12=T-4, \quad T+32-4=T+28.$$

No. 1 is on its proper target; the proper targets of the other guns are the opposite guns of the enemy battery at 40 yard intervals, and are therefore $T+28$, $T+68$, $T+108$. Guns 2, 3, 4 should then be turned to the left, or their points of impact on the line of *T* raised, by 28, 72, and 80 yards respectively, that is their deflections, for 4,000 yards, increased by 7, 18, and 20 mils respectively. The actual work is conveniently arranged in lines, one for each gun:

No. 3 $DD=-.7\frac{8}{2.5}=-2$ Impact $T+8-12=T-4$ should be $T+68$ left 18

No. 4	$=-.7\,\frac{32}{2.5}=-9$	$T+32-4=T+28$	$T+108$	20
1	$=+.7\,\frac{16}{2.5}=+4$	$T-16+4=T-12$		
2		T	$T+28$	7

The last problem to be considered is also a problem of fire control, the correction of the firing data from the observed effect of a number of shots by an elementary application of the theory of probability. The theory of probability here used has no direct connection with the subject of probability as taught in "college algebra," but is based on the law of errors. Problems dealing with range, deflection, and height of burst are solved in the same way. I shall illustrate the method by solving a problem in the correction of range. The center of impact is a point in the line of fire such that an equal number of shots fall short of it and over it. A zone is an interval on the line of fire of which the center of impact is the center; the sixty per cent. zone, for example, is the zone within which sixty per cent. of the shots fall. The probability factor for any zone is the ratio of the width of that zone to the width of the fifty per cent. zone. A table of probability factors, the same for range, deflection, and height of burst, is given in Danford and Moretti, page 223. The probable error is one half the width of the fifty per cent. zone, and is the distance from the center of impact as often exceeded as not by the points of impact of all shots fired. The probable error varies with the range. A table of probable errors in range, deflection, and height of burst, for ranges from 1,500 to 5,000 yards, is given on page 220 of the same book.

Suppose it is observed in firing 12 rounds (shots) at *Rn* 2,000 that 10 fall short, 2 over the target. What change in range should be made?

We reason as follows: $\frac{1}{6} = 16\frac{2}{3}\%$ of the shots fall over; then T is at the further edge of the $66\frac{2}{3}\%$ zone. For *Rn* 2,000 the probable error in range is 34 yards; the probability factor for the $66\frac{2}{3}\%$ zone is 1.44. Then T is $1.44 \times 34 = 49$ yards beyond the center of impact. The range should be increased by 50 yards, no changes of range other than multiples of 25 yards being possible. Tables of probable errors and probability factors may be and should be dispensed with, for an officer of the Field Artillery should be as far as possible independent of tables. No considerable error will be made if the probable errors for all ranges in range, deflection, and height of burst are taken respectively as 30 yards, 2.4 mils, 1.8 mils. It may be a help to the memory to note that these figures are 5, 4, 3×6. These probable errors apply only to the U. S. 3″ gun. The probability factors for the 50%, 82%, 96% zones are respectively 1, 2, 3 nearly; these zones correspond to 25%, 25% + 16%, 25% + 16% + 7% respectively of shots beyond or short of the center. These figures are easily remembered if we note $2 + 5 = 1 + 6 = 7$. The probability factor required in any example is easily found with sufficient accuracy by interpolation. Thus if we work the example given above without consulting the printed tables

the probability factor is found for the $33\frac{1}{3}\%$ of shots between T and the center of impact as $1 + 8\frac{1}{3}/16 = 1.52$. The probable error being taken as 30 yards, the change to be made in range is again 50 (46) yards. While this work may seem very rough to the mathematician he may feel assured that it is of the greatest practical value.

I wish in the following pages to make certain general remarks concerning the course in firing data, which will, I hope, be of interest to any one planning such a course.

The only mathematical knowledge required of a student taking the course is arithmetic and a very little of each of the following, algebra, plane geometry, plane trigonometry. To do well in the course the student must learn to do accurately and quickly such multiplications and divisions as have appeared in the examples worked in the preceding pages. It is desirable that he should have a sense of accurate approximation, that he should learn to carry his arithmetical work far enough and not waste time by carrying it further. Speed in arithmetical work, as in every part of the work in computing firing data, should be a constant aim in the instruction. In algebra the student must be able to substitute numbers in such simple formulas as have appeared in this paper, to transform simple equations, and to handle negative signs without blundering. No geometry seems necessary beyond an appreciation of the equality of alternate-interior angles. In trigonometry it is necessary only to know how to find the legs of a right triangle, given the hypothenuse and one acute angle.

The time devoted to the course last year at Yale was three hours a week for half the college year. It was felt that this time was not enough for the students, mostly freshmen, to become thoroughly familiar with the problems of the course, and was certainly not enough for the students to learn to work these problems with proper speed. It was planned in the coming year to give to freshmen in the R. O. T. C. a three-hour course throughout the college year on trigonometry and firing data. Probably more than half the time in such a course would be given to firing data.

A very important part of the course is field work. It was not possible last year to give the students nearly enough practice outdoors. My division had five afternoons in the field and should have had at least twice as much. For outdoor work, "fire control" instruments, a range finder, and an aiming circle or scissors are very desirable, but probably cannot be obtained at present. If these instruments are not available the distances from the B. C. station to target and aiming point should be given to the students by the instructor or obtained by the students from a map, or may be estimated, if proper methods of estimation are taught. Such methods are given in Danford and Moretti, pages 146–149. Angles may be measured with some accuracy by a B. C. ruler, which is six inches long divided into sixty equal parts; to the center is attached a string twenty inches long. Each division of the ruler subtends an angle of five mils when the ruler is twenty inches from the eye, the whole ruler thus subtending 300 mils. Angles greater than a right angle should not be measured with a B. C. ruler; if

such an angle is required, a straight line should be run and the supplementary angle measured. To measure an angle of site it is necessary to establish a horizontal line. A simple and fairly satisfactory way to do so is the following: Construct accurately on a sketching board a line perpendicular to one edge and attach a string to a point of this line; in using, attach a weight to the other end of the string and move the board in a vertical plane until the string hangs along the line; the edge is then horizontal. Some distant object in a horizontal line with the observer, as nearly as possible in the same vertical plane with the observer and the point whose site is to be measured, may thus be determined, and the site measured with the B. C. ruler with an error probably not greater than five mils. Angles may be roughly measured by sighting over the knuckles of the clenched fist held at arm's length. Of course each observer must for this purpose know the scale of his own fist.

The course in firing data should if possible be coördinated with other courses, the outdoor work particularly with map making and panoramic sketching and with use of fire control instruments if these are available. For any part of the work, instruction in map reading and matériel (the mechanism of the gun) is useful. The course should be followed by a course in fire control, but the latter can probably be given satisfactorily only by a trained artillery officer.

The tables needed for the course have already been mentioned. They are range table, tables of probable errors and probability factors, and table of sines and cosines for angles in mils. All are given by Danford and Moretti. A brief but sufficiently accurate range table for the U. S. 3″ gun is here given. It seems however undesirable to lay emphasis on memorizing either the range table or the table of probable errors, since these apply only to the U. S. 3″ gun. In the following table the first column is range in yards, the second is the corresponding *AD*, the third, the change in *AD* for 100 yards' increase in range.

1000	20	3
2000	50	4
3000	90	5
4000	140	6
5000	200	7
6000	270	

					SIN	COS					
6400	3200		3200	0	0	1.00	1600	1600		4800	4800
6300	3300	1.	3100	100	0.10	0.99	1500	1700	0	4700	4900
6200	**3400**		**3000**	**200**	**0.20**	**0.98**	**1400**	**1800**		**4600**	**5000**
6100	3500		2900	300	0.29	0.96	1300	1900		4500	5100
6000	3600	.9	2800	400	0.38	0.92	1200	2000	.4	4400	5200
5900	**3700**		**2700**	**500**	**0.47**	**0.88**	**1100**	**2100**		**4300**	**5300**
5800	3800		2600	600	0.56	0.83	1000	2200		4200	5400
5700	3900	.7	2500	700	0.63	0.77	900	2300	.7	4100	5500
5600	4000		2400	800	0.71	0.71	800	2400		4000	5600
					COS	SIN					

In the trigonometric table above the sines and cosines for every hundred mils are given to two decimals. The obliquity factors for corresponding deflections are also given.

Of course it is necessary that the students work a large number of numerical problems. Danford and Moretti give a considerable number (pp. 263–282) taken from examinations held in the non-commissioned officers' course at the school of fire. The instructor will have no difficulty in making up any number of similar problems.

While there exists a large amount of literature on the subject of firing data, I believe that for an introductory course nothing further is necessary than the Drill and Service Regulations for Field Artillery and some text such as that of Danford and Moretti.

Requests for further information concerning the course in training for service in the Field Artillery at Yale may be addressed to the Curriculum Committee, Artillery Hall, New Haven, Conn.

YALE UNIVERSITY.
September, 1918.

COURSES IN NAVIGATION AT THE UNITED STATES NAVAL ACADEMY.[1]

The Department of Navigation at the Naval Academy gives two different courses, the regular course for midshipmen, and a special course for Reserve Officers. The two courses are similar in content, but differ greatly in method of presentation and in the amount of time allotted.

The course for midshipmen is preceded by a thorough course in spherical trigonometry and stereographic projections given by the Department of Mathematics. To this preliminary work about thirty-five recitations are devoted, approximately half of this time being given to the use of stereographic projections and to the logarithmic solution of the same problems treated by projection. The text book is Brown's Trigonometry and Stereographic Projections, published by the Naval Institute, Annapolis, and the tables used are Bowditch's "Useful Tables," from the text on navigation by this author.

The work under the Department of Navigation involves three recitations per week during the last three terms (semesters) of the academic course. In the spring term preceding the first class (senior) year, two months are devoted to a brief course in astronomy, based on White's Theoretical and Descriptive Astronomy (John Wiley and Sons), and the remainder of the term is given to such work in navigation as will best prepare the student for the practical work of the summer cruise. During the last year the whole field of navigation and compass deviations is covered with Muir's Navigation and Compass Deviations (Naval Institute, $4.20) as a principal basis. Use is made of the Practical Manual of the Compass prepared at the Naval Academy (Naval Institute, $1.75), also of Bowditch's American Practical Navigator, the Nautical Almanac, and Azimuth Tables, all publications of the Hydrographic Office, Navy Department, Washington, which may be purchased through the Superintendent of Public Documents.

The special course for Reserve Officers occupies sixteen weeks, five recitations

[1] We are indebted to Prof. R. E. Root for the informaton regarding these courses.

per week. In this course no use is made of stereographic projections, and for text books only the publications of the Hydrographic Office mentioned above are used. Each class is given six periods on board a naval vessel in practical instruction in the use of charts and instruments.

Following is an outline of the 16 weeks' course in Navigation for Reserve Officers as recently given.

1. The assignment of time was as follows:

16 weeks, 5 recitations per week (less 13 recitation periods devoted to written tests, holidays, etc.) .... 67 periods

Practical instruction on U. S. S. Dubuque .... 6 periods

2. During the first nine weeks of the course, the whole class covered the following ground:

(*a*) Geometrical and trigonometrical definitions and use of logarithm tables.
(*b*) Use of chip log, patent log, sounding machine, compass, azimuth, circle pelorus, binnacle, barometers, thermometers, and log book.
(*c*) Application of variation, deviation and compass error.
(*d*) Laying courses, plotting positions and bearings, and measuring distances on chart.
(*e*) Piloting, including cross bearings, two bearings and run between, sextant angles, and use of 3-arm protractor, soundings, lights, tides, etc.
(*f*) The sailings and dead reckoning.
(*g*) Use and adjustments of sextant.
(*h*) Comparison of chronometers, and error and rate of chronometers by noon "tick."
(*i*) Navigational and astronomical definitions.
(*j*) Use of Nautical Almanac.
(*k*) Conversion of arc to time, local to standard time, mean to apparent time, and finding G. M. T.
(*l*) Corrections to observed altitudes.
(*m*) Meridian altitude of sun and constant.
(*n*) Reduction to meridian (sun).
(*o*) Time sight of sun.
(*p*) Azimuth of sun and finding compass error.

3. At the end of eight weeks the class was divided on the basis of progress made; and during the 10th and 11th weeks, two separate courses were pursued:

1st, By the more backward students (about one third of the class), a review of (*c*), (*e*) and (*f*).

2d, By the more advanced students (about two thirds of the class).

(*q*) Conversion of solar to sidereal time.
(*r*) Conversion of sidereal to solar time.
(*s*) Meridian altitudes, reduction to meridian, and time sights of stars.
(*t*) Polaris sights.

4. During the 12th, 13th and 14th weeks, the whole class covered the following subjects:

(*u*) Finding compass error and use of Napier's diagram.

(*v*) Elementary practical compensation of the compass.

(*w*) Lines of position and chart intersections.

(*x*) Naval regulations and instructions on navigational subjects.

5. During the remainder of the time available, separate courses were again pursued by the two divisions of the class:

1st Division (advanced portion of the class).

(*y*) Method of St. Hilaire.

(*z*) Day's work.

2d Division (backward portion of the class).

(*q*), (*s*), (*t*) and (*y*).

6. All of the time assigned to Navigation on board the U. S. S. Dubuque was devoted to piloting, use of charts, use of compass, sextant, etc., and much of the regular section room time was devoted to practical instruction in use of charts, noon "tick," chronometers, barometers, compass, pelorus, three-arm protractors, tide tables, deviascope, etc.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Dr. S. D. ZELDIN of the College of Hawaii has been appointed professor of mathematics in Olivet College.

Professor M. E. GRABER has been elected to the chair of mathematics in Heidelberg University, Tiffin, Ohio.

Professor C. A. BARNHART, formerly of Carthage College, has been appointed professor of mathematics in the University of New Mexico.

Dr. W. O. MENDENHALL, professor of mathematics in Earlham College, has been elected to the presidency of Friends University, Wichita, Kansas.

Assistant professor C. W. WATKEYS has been advanced to a professorship of mathematics in the University of Rochester.

Professor ARNOLD DRESDEN of the University of Wisconsin sailed for France in September for service in the Red Cross.

Dr. F. R. MORRIS has been appointed instructor in mathematics at the University of California.

Mr. H. LYLE SMITH, instructor in mathematics at Princeton University for the past two years, is now in the office of Major F. R. MOULTON of the Ordnance Department at Washington.

Professor S. L. BOOTHROYD, of the department of mathematics and astronomy, University of Washington, took an active part in the Crocker expedition for observation on the solar eclipse the past summer.

Dr. F. D. MURNAGHAN, of Rice Institute, has been appointed associate in applied mathematics at Johns Hopkins University.

Dr. ANNA PELL, associate professor of mathematics at Mt. Holyoke College, has accepted an associate professorship in mathematics at Bryn Mawr College.

Mr. C. A. HUTCHINSON, instructor in mathematics at Wittenberg College, Springfield, Ohio, has accepted an instructorship in mathematics in the College of Engineering of the University of Colorado.

Dr. THOMAS BUCK, assistant professor of mathematics at the University of California, has been commissioned a first lieutenant in the ordnance department of the army, and will be located in Washington doing research work in ballistics.

At the University of Minnesota, Associate Professor R. R. SHUMWAY has been appointed assistant dean of the College of Science, Literature and Arts; and Associate Professor W. H. BUSSEY has been appointed chairman of the department of mathematics for the year 1918–19, and also executive secretary of the School of Chemistry.

The Library Sub-Committee of the Mathematical Association of England has decided to issue a suggestive list of books suitable for mathematical libraries of secondary schools. It is proposed to divide the list into two parts, a Teachers' Library, and a Pupils' Library. A preliminary list of books on algebra and analysis, containing one hundred and sixteen titles, has been issued; it appears as a supplement to the May (1918) number of *The Mathematical Gazette.*

In the recently announced untimely death of Professor MAXIME BÔCHER American mathematicians have lost from their number one of an eminent group of men who have been in the center of American mathematical activity for a quarter century. The most noteworthy recent publication by Professor BÔCHER is his *Leçons sur les Méthodes de Sturm,* a volume which appeared last year as a monograph in the notable series edited by ÉMILE BOREL. It contains an exposition of the author's lectures delivered at the Sorbonne from November, 1913, to January, 1914. It deals with the methods of Sturm in the theory of linear differential equations and their modern developments. In it the author gives an introductory exposition of the range of ideas which have been in the center of his research activity for about twenty-five years; he includes some new results along with his account of previous work of his own and others. This book, together with Professor BÔCHER's address before the fifth international congress of mathematicians at Cambridge, affords the reader a valuable and convenient means of initiating himself into the principal contributions of their author to mathematical science. The writings of few men afford a more suitable introduction to the life-work of their author. Somewhat outside of the main current

of ideas represented in the *Méthodes de Sturm* but nevertheless connected with them is the first work through which Professor BÔCHER attracted attention, namely, his *Reihenentwickelungen der Potentialtheorie*. This appeared first in 1891 as a Göttingen dissertation of seventy pages and was later expanded into a book of 266 pages and published in 1894. Not long after the appearance of this book Professor BÔCHER began to publish his contributions in connection with the work of Sturm. Since that time his main researches have had to do with the interests thus initiated. While he has done work in other directions, it is in this and in the book of 1894 that one must seek his more important contributions.

The American Mathematical Monthly

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The Book Reviews embrace the entire field of collegiate and secondary mathematics.

The Curriculum Content in the collegiate field is carefully considered. Good papers in this line have appeared and are now in type awaiting their turn.

The Notes and News cover a wide range of interest and information both in this country and in foreign countries.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of mathematics for its own sake.

There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual members and over seventy-five institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary at Oberlin, Ohio.

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THIRD SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The Association held its third summer meeting by invitation of Dartmouth College at Hanover, N. H., on Thursday, Friday and Saturday, September 5–7, 1918, in conjunction with and following the summer meeting of the American Mathematical Society. Seventy-one persons attended the sessions, including the following 42 members of the Association:

H. L. AGARD, Williams College.
R. C. ARCHIBALD, Brown University.
R. D. BEETLE, Dartmouth College.
G. D. BIRKHOFF, Harvard University.
DANIEL BUCHANAN, Queen's University.
W. D. CAIRNS, Oberlin College.
W. B. CARVER, Cornell University.
JULIA T. COLPITTS, Iowa State College.
LENNIE P. COPELAND, Wellesley College.
LOUISE D. CUMMINGS, Vassar College.
C. H. CURRIER, Brown University.
E. L. DODD, University of Texas.
C. H. FORSYTH, Dartmouth College.
A. S. GALE, University of Rochester.
O. E. GLENN, University of Pennsylvania.
C. F. GUMMER, Queen's University.
J. G. HARDY, Williams College.
E. V. HUNTINGTON, Harvard University.
W. W. JOHNSON, U. S. Navy.
FLORENCE P. LEWIS, Goucher College.
J. MATHESON, Queen's University.
HELEN A. MERRILL, Wellesley College.
F. M. MORGAN, Dartmouth College.
G. D. OLDS, Amherst College.
H. L. OLSON, New Hampshire College.
F. W. OWENS, Cornell University.
ANNA H. PALMIÉ, College for Women, Western Reserve University.
A. D. PITCHER, Adelbert College.
JESSIE G. QUIGLEY, College of Saint Teresa.
L. H. RICE, Tufts College.
R. G. D. RICHARDSON, Brown University.
E. D. ROE, JR., Syracuse University.
CLARA E. SMITH, Wellesley College.
SARAH E. SMITH, Mount Holyoke College.
H. W. TYLER, Massachusetts Institute of Technology.
OSWALD VEBLEN, Princeton University.
C. A. WALDO, Washington University.
A. G. WEBSTER, Clark University.
J. K. WHITTEMORE, Yale University.
C. B. WILLIAMS, Kalamazoo College.
F. N. WILLSON, Princeton University.
J. W. YOUNG, Dartmouth College.

The increase in railroad rates had its very evident effect in the somewhat reduced attendance as compared with the preceding summer; there were present

nevertheless as members from greater distances—three each from Canada and Ohio, and one each from Iowa, Maryland, Michigan, Minnesota, New Jersey, Pennsylvania and Texas. The presence of Majors Huntington and Veblen, and of Professor Tyler from their governmental activities in Washington, D. C., and Maryland gave a strong suggestion of the unusual times in which we are and brought great aid in the important consultations and results which formed a unique feature of the meeting.[1]

Members and other visitors were comfortably housed in South Fayerweather Hall and College Hall. The latter afforded also a dining room and commodious social rooms which were freely placed at the disposal of all. A leaflet of general information printed by Dartmouth College especially for these meetings was of definite assistance to all. Small parties made brief excursions about the historic village and vicinity. In all respects this was a most pleasant and successful meeting. A unanimous rising vote at the joint session of the two organizations recognized the admirable character of the local arrangements and the warm hospitality shown by the Hanover faculty.

The sessions were preceded by the joint dinner of the Association with the American Mathematical Society on Thursday evening. Fifty-six were present on this occasion, which combined with its pleasant social feature an atmosphere of earnestness in the face of war times and war problems. President Huntington called upon several persons, including Dean Laycock of Dartmouth College, who welcomed the assembled guests to Dartmouth and brought fresh reports from the Plattsburg conference on military education in the colleges; Major Veblen, who urged the call of the ordnance department for able mathematicians to assist in that department, men who are capable of assuming responsibility for independent work in ballistics, as well as those not so fully able to work independently; Secretary Cairns, who read abstracts from the government's circular on the new courses to be instituted in the colleges and universities, and called attention to certain proposed war courses in mathematics which, already sketched out by a few mathematicians, should be perfected at this time; Professor Richardson, who described the instruction in navigation that has already been given in two or three universities and urged the great importance of preparing young men whose choice is the navy; Professor Willson, who at the suggestion of Major Huntington emphasized the place of descriptive geometry as one of the essential courses in the new war mathematics; Secretary Cole, who announced the publication of the first part of the Cambridge Colloquium, that by Professor G. C. Evans; Professor Webster, who gave an engrossing account of the experiences of a civilian enlisted in governmental service; and Dean Olds, who supplemented the report concerning the new war courses.

Sessions of the Association were held as announced on Friday morning and afternoon and on Saturday morning. The joint session of the Association with the Society at half past ten o'clock Friday morning, Professor G. D. Birkhoff pre-

[1] The Secretary-Treasurer regrets that, due to a mistake made by the printer, the members in several states did not receive copies of the program.

siding, was addressed by Professor A. G. Webster on "Mathematics of Warfare." As this well known mathematical physicist is on the U. S. Naval Consulting Board, he was well equipped to tell of current developments in the theory of ballistics. He described the elementary principles of ballistics for long range trajectories, summarizing the contributions to the subject of such men as Danforth, Siacci and Chapel. After developing the "ballistic function" and its differential equation, he closed by touching more briefly on the outstanding problems of interior ballistics. The address when published in its full form will furnish a notable chapter in this theory.

A valuable and suggestive program was arranged by the program committee under the chairmanship of Professor Archibald. A still greater interest and concentration of effort centered about the almost continuous conferences over the formulation of war courses in mathematics. Before this report appears, all institutions concerned will have been informed as to the United States government's recommendations or instructions in this respect. It will however remain as a unique feature of this meeting that thirty or forty professors in collegiate mathematics from representative schools in all parts of the country except the far west met in an informal capacity and after much discussion and earnest consideration came to a united view as to the character and contents of three twelve-week terms for the general military student in college, two twelve-week terms in navigation, and an eight-week course in descriptive geometry, besides suggestions as to further second year courses, so far as there will be men to attend these. The courses as formulated were transmitted through Major Huntington as recommendations made by this unofficial conference to the educational committee of the war department and will doubtless have been incorporated in whole or in part into the suggestions issued by that department and already familiar to the readers of the MONTHLY before the appearance of this report.

ORDER OF TOPICS ON THE SEPARATE PROGRAM.

(1) "The Teaching of Curve Tracing." PROFESSOR F. W. OWENS, Cornell University.

(2) "A Formula in Combinatorial Analysis." PROFESSOR J. W. YOUNG, Dartmouth College.

(3) "Trigonometric Functions—of what?" PROFESSOR W. B. CARVER, Cornell University.

(4) "Firing Data at Yale." J. K. WHITTEMORE, Yale University.

(5) "A Combined Course in Mathematics for College Freshmen." PROFESSOR A. S. GALE, University of Rochester.

(6) Report of the Committee on Mathematical Requirements. PROFESSOR J. W. YOUNG, Chairman.

(7) Presidential Retiring Address: "Plans for a History of Mathematics of the Nineteenth Century." PROFESSOR FLORIAN CAJORI, University of California.

(8) "Some Experiments in the Teaching of Descriptive Geometry." PROFESSOR F. L. KENNEDY, Harvard University.

Abstracts, numbered to correspond with the numbers on the foregoing program, are printed below, together with reports of some further informal discussions.

ABSTRACTS OF PAPERS.

(1) Professor Owens in his paper on the "Teaching of Curve Tracing" urged that the light thrown upon the whole notion of functionality by the introduction of the graph may be much intensified by a much greater use of graphical methods in the *construction* of the curve which exhibits the function. He exhibited some simple methods which he has found to be particularly helpful to students in constructing graphs for functions which are or may be given explicitly, and which make especially clear the genesis of the more complicated function from more simple ones by processes indicated in the expression for the function.

(2) Professor Young considers a class $K = (k_1, k_2, \ldots, k_n)$ on N objects. Any combination C_s of s elements of K, together with the combination $\bar{C}_s$ of the remaining $N - s$ objects is called a *partition* of K, C_s and $\bar{C}_s$ being called the *sides* of the partition. A pair, triple, . . . , i-ad of elements of K is said to occur in a partition if the pair, triple, . . . , i-ad occurs in either side of the partition. A *system* S_m of partitions $(C_{s_i}, \bar{C}_{s_i})$ $(i = 1, 2, \ldots, m)$ consist of any set of m partitions; the s_i are not assumed to be all equal, though they may be. The symbol ξ_i is used to represent any i-ad of K; P_{ξ_i} to represent the number of times the i-ad ξ_i occurs in a given S_m. Thus $P_{k_1k_2}$ means the number of times the pair $k_1\, k_2$ occurs in S_m; $P_{k_j\xi_i}$ the number of times the $(i+1)$-ad $k_j\xi_i$ occurs in S_m. The formula mentioned in the title is as follows:

$$\sum_{{}_j\alpha_{i-1}} [P_{k_j\xi_{i-1}} - P_{k_i\xi_{i-2}} + P_{k_j\xi_{i-3}} - \cdots \pm P_{k_j\xi_1}] = (-1)_i[m + P\alpha_i - P_j\alpha_{i-1}],$$

where α_i is any given set, $k_1, k_2, \ldots, k_i$, of K, when k_j is any element of α_i, and ${}_j\alpha_{i-1}$ represents the set of $i - 1$ elements obtained by removing k_j from α_i. The summation is extended over all $\xi_{i-1}, \xi_{i-2}, \ldots$ contained in ${}_j\alpha_{i-1}$. This formula may readily be summed from $j = 1$ to $j = i$, yielding a second formula.

The above formula for $i = 3$ gives

$$P_{ij} + P_{ik} + P_{jk} = m + 2\, P_{ijk}.$$

In any system S_m, in which all pairs occur the same number of times, all triples will also occur the same number of times. This theorem is readily generalized. Application of the formula is made to certain problems of arrangement —in particular to certain problems connected with a certain type of whist tournaments.

(3) Professor Carver's paper called attention to the difficulty caused for the student by the two different notions of the trigonometric functions, as functions of *angles* and functions of *numbers*. It was suggested that the student might

be helped (1) in trigonometry, by the early introduction, and *frequent use in problems,* of the radian unit; (2) in analytic geometry, by insistence upon the use of the radian unit in the plotting of such curves as $y = \sin x$ and $\rho = a\theta$; (3) in calculus, by a clear presentation of the notion of trigonometric functions of *numbers* just before the development of their derivatives.

(4) Mr. Whittemore's paper, which was an account of the course in Artillery Firing given the past year at Yale University as a three-hour course for a semester, has already appeared in this MONTHLY. It was a very timely address and in its full form offers a great aid to those who have been planning similar courses.

(5) A first step toward a combined course for freshmen was taken at the University of Rochester by Professor Gale in the year 1907–1908, when it was decided to emphasize the graphical significance of the functions occurring in the subjects taught, namely, solid geometry, trigonometry and advanced algebra. Soon thereafter considerable interweaving of these subjects was done. In 1913–1914 the calculus was introduced and a combined course, developed by Professors Gale and Watkeys, has been taught for the past four years.

The central theme of the year's work is the study of the elementary functions, algebraic and transcendental. An attempt has been made to keep in very close contact with applications of mathematics, and to attach greater importance to ideas and the development of the power to think than to purely manipulative processes. An analysis of the properties of functions and graphs consists of relations between a single function and its graph, and of the relations between various pairs of functions and their graphs. These principles may be introduced in connection with simple algebraic functions, and utilized in the presentation of the transcendental functions. Interpretation of the graphs of important functions furnishes the means for organizing and remembering various properties of the functions.

Among the applications, emphasis is laid on the determination of a function whose table of values agrees reasonably well with a given table obtained empirically, the general problem being given as a part of the method of discovery in science.

It is believed that the moulding of the subject matter into a coherent year's work gives the student a better comprehension of what mathematics is and a greater facility in its use, and that the freshmen are more interested in their work than formerly.

Among those who took part in the discussion, Professor Richardson maintained that different courses should be given according as the students do or do not expect to continue the study of mathematics, Professor Williams told of his success in combined courses for freshmen, and Professor Olds recalled the interest which was kindled by the introduction of calculus into the freshman year at Amherst College years ago.

(6) Professor Young made a report of the progress being made along the lines of the committee's activities. (See report in the MONTHLY for December, 1917, page 463, for a statement of these.) Miss Blair's report has been postponed in

its printing in order to assimilate with this report certain material which has come in in connection with two or three recent articles on formal discipline and the transfer of training.

The survey in charge of Professor G. W. Evans of courses in algebra in secondary schools is being correlated with the work of committees appointed by the associations affiliated with the Association in this work. Professor Crathorne has made a preliminary report and is now engaged in studying the results of questionnaires.

It seems altogether probable that within the next few months the committee will be able to present a report embodying definite principles, the report to be submitted to the Association and the affiliated bodies for their discussion and promulgation if the report is adopted.

(7) The address of ex-President Cajori has already been printed in *Science*. The removal of residence of Professor Cajori to California rendered his presence at the meeting impossible, and in his absence the paper was read by Professor Olds.

(8) Professor Kennedy showed by clear and excellent charts how he meets the difficulties of the students as they begin the study of descriptive geometry. He uses a set of large scale diagrams carefully drawn on heavy board in various colors, blue print and problem sheets 8″ by $10\frac{1}{2}$″ issued to the pupils, as well as tests on sheets of the same or double size, such tests of varying length being used once a week to secure concentration of effort and a rating of the student's work.

No text is used in the course. A preliminary sheet shows the method of representing geometrical figures by means of three reference planes (frame of reference) and the three working rules of orthographie projection, *e. g.*, vertical and horizontal projections lie on the same vertical line. Blue prints give successively symbols and conventions, simple and clear perspective and orthographic drawings of the elementary theorems of points and lines, planes and traces, lines and planes (*e. g.*, point of intersection of a given line and a given plane), angle between two planes, the gradual approach by rotations to axonometric and isometric projection, a plate of "facts and pitfalls" (correct and incorrect constructions side by side) mimeographed in blue and red respectively.

The large scale wall charts previously mentioned illustrate the gradual development of a problem (*e. g.*, the problem of passing a plane through three given points was shown in four stages, besides a perspective drawing) and afforded an accurate and neat drawing not feasible in quick blackboard sketching. Colors are freely used to distinguish different situations, as for example given, construction, and required lines. Some of the charts have somewhat the nature of dissolving views, in that lines, etc., not longer of importance are drawn more lightly or are omitted in the later stages.

A complete and concise notation is used both in assignment of problems and in the student's analysis of these; thus, the statement "Given a line *ab* and a point *p*, to pass a plane *M* through *p* perpendicular to *ab*" is given in the following form:

$$\frac{\overline{ab} \text{ and } p}{M \rightarrow p \perp \overline{ab}}$$

The student is expected to note plainly the definite steps in his solution by the use of this notation and other means; for example, an arrow denotes the direction in which he revolves a line. Moreover this notation in connection with given lines, points, etc., mimeographed on problem sheets enables him quickly to attack the assigned problems each day. So far as shades and shadows are taught, a very few charts are used instead of a text, all oral explanation being obviated by the use on these charts of the abbreviated notation already familiar to the class.

Papers are marked with respect to accuracy, clearness and workmanship. Only limited stress is laid on originality since this is apt to develop into eccentricity. Ink is used sparingly and always where it will do the most good.

Professor Kennedy showed a plate embodying a ten minute exercise, a mid-year examination, and a plate of four problems comprising an "inverted test," *i. e.*, four problems were drawn incorrectly, the student being given fifteen minutes to pick out the errors, indicate and correct these.

Dean Randall, who was to have discussed this paper, was unable to be present by reason of the unexpected pressure of duties in inaugurating the new war courses. Professor Willson showed a copy of Monge's *Géométrie Descriptive*, published in 1799 during the French Revolution and dated according to the calendar invented for the new regime by the author and a collaborator; since the use of this calendar ceased soon thereafter this is a very unique book, being the only descriptive geometry and one of the very few books which bear this system of dating.

Meeting of the Council of the Association.

At the meeting of the Council on Friday evening, the following fourteen persons, on applications duly certified, were elected to individual membership:

L. Bianchi. Prof., Univ. of Pisa, Pisa, Italy.
Ugo Broggi, Ph.D. (Göttingen). Prof., Buenos Aires and La Plata Univs., Buenos Aires, Arg.
C. S. Cox, A.M. (Vanderbilt). Prof., Southern Coll., Birmingham, Ala.
P. J. da Cunha. Prof., Univ. of Lisbon, Lisbon, Portugal.
Federigo Enriques. Prof., Univ. of Bologna, Bologna, Italy.
G. A. Gilbert. Prof. of physics and math., St. Ignatius Univ., San Francisco, Cal.
F. D. Murnaghan, Ph.D. (Johns Hopkins). Instr., Johns Hopkins Univ., Baltimore, Md.
H. R. Park, A.B. (Southern Univ.) Teacher, Jun. Coll., Riverside, Cal.
Arthur Pelletier. Prof. of higher alg., École Polytechnique, Montreal, Can.
Salvatore Pincherle. Prof., Univ. of Bologna, Bologna, Italy.
Susan M. Rambo, A.M. (Smith). Asst. prof., Smith Coll., Northampton, Mass.
L. H. Rice, Ph.B. (Syracuse). Instr., Tufts Coll., Tufts College, Mass.

A. V. RICHARDSON, M.A. (Cambridge). Lecturer, Bishop's Coll., Lennoxville, Quebec, Can.

A. G. WEBSTER, Ph.D. (Harvard), Sc.D. (Tufts), LL.D (Hobart). Prof. of physics, Clark Univ., Worcester, Mass.

Several of those elected come as the result of a movement for enlisting foreign members, which has been carried on the past year or more by the Committee on Membership, Professor E. R. Hedrick, chairman.

In this place should be announced a list of twenty-seven persons and three institutions elected to membership by mail vote of the Council in July, 1918:

To individual membership:

R. A. ARMS, Ph.D. (Penna.). Prof., Juniata Coll., Huntingdon, Pa.

R. N. ASHMUN, A.M. (Univ. of Wash.). Computer, Internat. Boundary Commission, Washington, D. C.

H. G. AVERS, A.B. (George Washington Univ.). Coast and Geodetic Survey, Washington, D. C.

H. E. BURTON, M.S. (Iowa). Asst., U. S. Naval Observatory, Washington, D. C.

GENEVIEVE E. COFFREY, A.M. (Univ. of Wash.). Instr. in science, High School, Mackay, Ida.

C. H. GINGRICH, Ph.D. (Chicago). Prof., Carleton Coll., Northfield, Minn.

J. M. HACKLER, Ph.B. (Chicago). Chair of math., Northeastern State Normal, Tahlequah, Okla.

EMMA E. HANTHORN, A.B. (Nebraska). Instr., State Normal, Kearney, Neb.

A. C. HICKMOTT, B.S. (Dartmouth). Statistician, Conn. Gen. Life Ins. Co., Hartford, Conn.

W. G. HUBERT, Sc.D. (New York Univ.). Instr., Coll. of City of New York, New York, N. Y.

C. B. HUGINS, B.S. in M.E. (Carnegie Inst.). Wilkinsburg, Pa.

LAURA M. LUNDIN, B.S. (Mass. Inst. of Tech.). Asst. prof., Wheaton Coll., Norton, Mass.

A. G. MONTGOMERY, A.B. (West Virginia). Instr., Concord St. Normal, Athens, W. Va.

MARY S. MOONEY, A.M. (Bellevue College). Dean of women and prof., Henderson-Brown Coll., Arkadelphia, Ark.

A. L. ONDRAK, A.B. (St. Procopius). Secy., St. Procopius Coll., Lisle, Ill.

R. E. POWELL, E.E. (Ga. Sch. of Tech.). Industr. High Sch., Columbus, Ga.

J. M. RANKIN, A.B. (Maryville Coll.). Instr., Coll. of Idaho, Caldwell, Ida.

PERCIVAL ROBERTSON, Ph.B. (Yale). Instr., The Principia, St. Louis, Mo.

H. E. RUSSELL, A.M. (Wesleyan), Sc.D. (Denver). Prof., Univ. of Denver, Denver, Col.

G. E. F. SHERWOOD, A.M. (Harvard). Asso. prof., Col. Sch. of Mines, Golden, Col.

R. K. STEWARD, C.E. (Maine). Prof. of drawing and design, Mich. Agric. Coll., East Lansing, Mich.

Ella A. M. Thorp, A.B. (Minn.). Asst. instr., Univ. of Minn., Minneapolis, Minn.
G. M. V. Tryon, Fenton, Mich.
G. P. Unseld, A.B. (Colorado). Grad. stud., Univ. of Col., Westminster, Col.
C. B. Watts, A.B. (Indiana). Asst., U. S. Naval Observatory, Washington, D. C.
J. J. Widmayer, Jr., M.S. (St. John's Coll.). Structural designer, Navy Dept., Washington, D. C.
C. C. Wylie, A.M. (Missouri). U. S. Naval Observatory, Washington, D. C.

To institutional membership:

Southwestern College, Winfield, Kan.
College of St. Thomas, St. Paul, Minn.
New Mexico Normal University, East Las Vegas, N. M.

The Council transacted further business in connection with the Monthly and with the annual meeting of the Association. In view of the developments at the Dartmouth meeting it seemed inevitable and of the greatest importance that the subject of mathematics courses for the period of the war should form the predominating feature of the December program. Whether the Association should meet in affiliation with the American Mathematical Society, whether in affiliation with the American Association for the Advancement of Science at Baltimore, whether the Association will best serve the interests of its members by holding one meeting or, instead of this, enable more members to be within reach of the meetings by having duplicate programs in the East and in the West, were the subjects of earnest discussion. By the time this report appears it will doubtless have been decided by the full participation of the Council as to what place is wisest.

W. D. Cairns, *Secretary-Treasurer.*

MATHEMATICAL ENCYCLOPEDIC DICTIONARY.

By G. A. MILLER, University of Illinois.

The following preliminary article is intended to serve as a basis for discussions relating to the nature and the extent of the major articles in the proposed mathematical dictionary. It aims to explain the terms *group* and *group theory* and to define a few of the most important terms which are related thereto. The latter terms should probably appear in their regular alphabetical places with references to the words *group* or *group theory* for their special meanings in this connection. In some cases this special meaning could not be made clear without such preliminary general developments as are here presented.

The object has been to give only such information as is within the range of the first-year graduate student, since the proposed dictionary should clearly not aim

at completeness but should confine itself to rendering very efficient service to certain classes of students. It is evidently a difficult matter to select those elements of a large subject which a student may be supposed to know at a certain stage of his development, in view of the great differences in interest and preparation. On the other hand, brief expositions which aim to present only what is most fundamental have a peculiar charm even for the mature students.

While the chairman of the "Dictionary Committee" of the Association was consulted as regards the desirability of publishing a preliminary article, he did not see the present article itself in manuscript, so that the committee is in no way responsible for its style or content. [See note on page 428. Ed.] The writer believes that the major mathematical terms should appear in five languages, English, French, German, Italian and Spanish, and hence the term group appears in these languages, in order, at the beginning of the article.

Group (grōōp), *groupe, Gruppe, gruppo, grupo.* This term is frequently used in mathematics with its non-technical meaning, denoting a collection composed of a finite or of an infinite number of elements; *e. g.*, a group of terms, a group of points, a group of waves, group insurance, etc. The most general technical mathematical meaning of the word group is obtained by restricting its non-technical meaning by imposing the condition that each of the elements of the collection can be combined with itself and also with every other element of the collection. Moreover, the result obtained by such a combination must also be an element of the collection. This most general technical meaning of the term group includes all its other technical meanings as well as that of domain of rationality. It appears frequently in the geometrical literature, but it occurs also in the literature on analysis. Cf. *Encyclopédie des Sciences Mathématiques*, tome 1, volume 2, page 243.

The most common additional restriction imposed on the elements of a group is that the *inverse* of each element shall also be contained in the collection; i. e., each element of the collection can be combined with at least one element of the collection so as to obtain the *identity*. Among the most common additional restrictions are the following: when the elements are combined, or *multiplied together*, the associative law holds, and if any two of the symbols in the equation

$$xy = z$$

are replaced by elements of the collection the equation has always one and only one root in the collection.

All these restrictions are usually imposed on the elements of a finite collection representing a mathematical group, but frequent exceptions appear when the collection contains an infinite number of elements. Cf. Lie, *Theorie der Transformationsgruppen*, volume 1, page 163, where an infinite group which does not involve the inverse of any one of its elements is discussed. Some writers contend that even an infinite collection does not represent a group unless its elements satisfy all the conditions of combination noted above. Cf. A. Loewy, *Archiv der Mathematik und Physik*, volume 9, (1905), page 105, where it is stated that the definition given on page 218, volume 1, *Encyklopädie der Mathematischen Wissenschaften*, is incorrect as regards infinite groups, because it is satisfied by a collection of elements which does not include the inverses of its elements. A group in the most restrictive sense of the term has been called *ordinary* group by L. Autonne, Paris, *Comptes Rendus*, vol. 143 (1906), p. 671.

E. Galois (1811–1832) seems to have first used the term group with a technical mathematical meaning. In fact, Galois and the other writers before the middle of the nineteenth century practically confined their group theory studies to finite groups whose elements are represented by substitutions, and every finite collection of substitutions satisfies the most restrictive definition of group noted above provided it satisfies the most general technical definition. Difficulties as regards the most desirable definitions of the technical term group began to appear when A. Cayley inaugurated the study of abstract groups (1854), and C. Jordan exhibited the wide usefulness of infinite groups (1868). These difficulties cannot be regarded as solved at the present time.

In a broad way groups have been divided into four categories, as follows: Finite discontinuous, infinite discontinuous, finite continuous and infinite continuous. The first of these categories of groups began to be studied during the latter half of the eighteenth century in connection with the solution of algebraic equations in one unknown, and the terminology thus developed was largely transferred to similar concepts arising in connection with the study of the other categories. Among the important concepts common to all of these

categories are the following: subgroup, invariant subgroup, conjugate groups and quotient group. It may be noted that some groups which are continuous in the terminology of S. Lie (1842–1899) have been called discontinuous by H. Taber, *Bulletin of the American Mathematical Society*, vol. 6 (1900), p. 202.

An illustration of a group in the most general mathematical sense is furnished by the natural numbers when they are combined by addition or multiplication, or by both of these operations separately. When the positive and negative integers together with 0 are combined by addition there results a group in the most restricted sense. Both of these are *infinite discontinuous* groups. An instance of a *finite discontinuous* group is furnished by the 24 movements of space which transform a cube into itself, while the totality of the movements of space which transform a point into itself constitutes an illustration of a *finite continuous* group. The elements of a continuous group are not denumerable. Such a group is composed of one or several families of transformations each depending upon a finite number of parameters.

An infinite continuous group contains an infinite number of parameters, or one or more parametric functions. The theory of infinite continuous groups has been less developed than that of finite continuous groups. In all cases except perhaps that of finite discontinuous groups there is as yet no uniformity of usage as regards the essence of the restrictions to be imposed on the most general technical definition of a group although the restrictions imposed by S. Lie have been very widely adopted in works on continuous groups. It should be emphasized that a collection alone is not a technical group. There must be also some law of combination.

The concept involved in the most general technical definition of the term group is that of a system of elements which is *closed* as regards one or more than one method of combining these elements. This concept is as old as mathematics itself since it appears in the fundamental operations of arithmetic and it also underlies Euclidean geometry. Cf. H. Poincaré, *The Monist*, volume 9, 1899, where it is stated, on page 31, that without the notion of group there would be no geometry. In fact, it appears probable that the successive extensions of the number concept, so as to include fractions, and negative, irrational and complex numbers, was largely due to the fact that the most general technical notion of group is ingrained in us and has influenced our intellectual development. Notwithstanding the great importance of this general technical notion and the need of a special name for it, it is too general to serve as a basis of a special abstract theory at the present stage of mathematical development.

Efforts to formulate an abstract definition of group, sufficiently special to serve as a basis for an autonomous theory, were inauguarated after the most general technical notion of group had been extensively applied to a set of concrete elements (substitutions) which intrinsically obey the associative but not the commutative law of combination. The theory of substitutions, more than the theory of numbers, thus furnished the model for abstract definitions of a group, which were first clearly formulated by H. Weber (1882), and G. Frobenius (1887). Somewhat earlier (1870) L. Kronecker had formulated such a definition for an abstract abelian group. Among later formulations of such definitions we would refer especially to those which appeared in the *Transactions of the American Mathematical Society* and are due to E. H. Moore (1902, 1904, 1905), L. E. Dickson (1905), E. V. Huntington (1903, 1905, 1906). Cf. W. A. Hurwitz, *Annals of Mathematics* (1906–1907), page 94.

An explicit abstract formulation of a definition of group in the most general technical sense, including both continuous and discontinuous groups, was given by S. Lie in 1871, *Forhandlinger Videnskabs-Selskabet*, edited in 1872, page 243; but in the development of his theory of continuous groups Lie imposed additional restrictions so that his groups can be defined by means of differential equations. In the preface to volume 3 of his *Theorie der Transformationsgruppen* (1888–1893), which aims to give a pure abstract theory of finite continuous groups, he directed attention (p. 17) to the fact that F. Klein used the term continuous group in a more general sense than that adopted by Lie.

Among the other terms used for group are the following: *Permutation* (Ruffini), *system of conjugate substitutions* or *conjugate system* (Cauchy), *closed system* (Lie and Klein in their earliest publication only). The term employed by Cauchy has been extensively used by others, especially by French writers. In particular, it is used instead of group in the *Cours d'algèbre supérieure* by J. A. Serret, being retained in the sixth edition (1910). When a part of the elements of a group constitute a group the latter is called a *subgroup* or a *divisor* of the former.

When the elements of a group are symbols of operation having no intrinsic properties the group is said to be *abstract;* when these symbols have intrinsic properties the group is said to be *concrete*. Abstract groups are sometimes called *general* groups. It is therefore necessary to distinguish between general groups and groups satisfying the most general technical definition of this term. When the elements of a group combine according to the commutative law the group is said to be *commutative* or *abelian*. When the elements of a group represent geometric concepts the group is called a *geometric* group. Groups which are not geometric are usually called *algebraic* or *analytic*. When the elements of a group are represented

by substitutions it is called a *substitution* group.

If s_1 and s_2 represent two elements of a group G the element $s_1^{-1}\, s_2\, s_1$ is called the *transform* of s_2 with respect to s_1. When s_2 remains fixed while s_1 is replaced successively by all the elements of G there results a totality of elements known as the *complete set of conjugates* of s_2 under G. The elements of this set when they are combined in every possible manner generate a group which includes the complete set of conjugates of all its elements under G. When this group is not identical with G it is said to be an *invariant* or *self-conjugate subgroup* of G. An invariant subgroup is characterized by the fact that it includes the complete sets of conjugates of each of its elements, and when a group does not contain any such subgroup besides the identity it is called a *simple* group. A group which is not simple is said to be *composite*. Invariant subgroups are also called *normal divisors, proper divisors, monotypic, self-conjugate, etc.*

When all the elements of a group H are transformed by an element s of H, or of a larger group G in which H appears as a subgroup, there results a group H' known as the *conjugate* of H with respect to s. It results directly that H is, in turn, the conjugate of H' with respect to the inverse of s, or s^{-1}. When s represents successively all the elements of G there results a *complete set of conjugates of* H under G. A necessary and sufficient condition that H is invariant under G is that all of these conjugates are identically equal to each other. All the elements of G can be divided with respect to H into sets (known as *co-sets*), and when H is invariant these sets combine as units and constitute a group called the *quotient group* of G with respect to H, which is denoted by G/H. A quotient group is also called a *factor* group, or a *complementary* group.

The elements of an abstract group are often called *operators* or *operations*. When the number of these operators is finite it is called the *order* of the group. On the other hand, the order of a finite continuous group is the number of its arbitrary parameters. A substitution group on n letters is said to be of *degree n*. Some writers use order for degree and vice versa. A substitution group is said to be *regular* when its order is equal to its degree and every substitution besides the identity involves all the letters.

When each letter of a substitution group is replaced by a particular letter in some substitution of the group it is called a *transitive* group. A regular group is necessarily transitive. When a substitution group is not transitive it is said to be *intransitive*. A transitive substitution group whose letters can be divided into sets such that every substitution of the group transforms all the letters of each of these sets either among themselves or into those of another one of these sets is called *imprimitive* or *non-primitive*. All transitive groups which are not imprimitive are said to be *primitive*. While the term primitive is positive it is usually defined negatively and the converse is true as regards the term imprimitive.

It is important to note that in the theory of continuous groups the intransitive groups are classed with the imprimitive groups while this is not done in the case of finite substitution groups. A substitution group which involves all the possible substitutions on its letters is called *symmetric*. The symmetric group of degree n is of order $n!$ The substitution group composed of exactly half the substitutions of a symmetric group is called *alternating*. When a substitution group has the property that it replaces a sub-set of r of the letters by every possible such set of its letters it is said to be *r-times* or *r-fold* transitive. The alternating group of degree n is $(n-2)$-fold transitive while the symmetric group of this degree may be said to be either $(n-1)$-fold or n-fold transitive.

There are groups which are composed of a finite number of families of continuous transformations. For instance, the movements of the plane which transform a point into itself and are represented by the following two sets of equations:

$$x' = x\cos\theta - y\sin\theta,$$

$$y' = x\sin\theta + y\cos\theta,$$

and

$$x' = x\cos\theta + y\sin\theta,$$

$$y' = x\sin\theta - y\cos\theta.$$

These two families of continuous transformations constitute a group, but one cannot pass continuously from a transformation of the former type to one of the latter since the determinants of the former are equal to unity while those of the latter are equal to -1. Such groups are sometimes called *mixed* or *complex* groups.

There is still a considerable lack of uniformity as regards the use of terms in group theory. For instance, the term *principal group* (Hauptgruppe) was used by G. Frobenius (*Crelle*, vol. 86, 1879, p. 219) to represent the group formed by the identical element, while F. Klein employed the same term for the continuous mixed group of movements whose invariants constitute elementary geometry. Similarly, the term *anharmonic* group has been employed for two distinct groups; viz., for a group of order 6 which obeys the same laws of combination as the symmetric group of degree 3, and hence is said to be *simply isomorphic* with this group (Pascal's *Repertorium*, vol. 1, 1910, p. 238), and also for the non-cyclic group of order 4, commonly known as the *four-group* (Capelli, *Istituzioni di analisi algebrica*, 1909, p. 111).

Group theory. The systematic development of theorems relating to properties of

groups is known as group theory. Among the various branches of this science the theory of substitution groups is the oldest, having been founded by A. L. Cauchy, about 1845, and first embodied in the form of a separate treatise by C. Jordan, *Traité des substitutions et équations algébriques*, 1870.

The theory of substitutions contains two large branches. The older of these is sometimes called the theory of permutation groups, and is based on the possible interchanges of letters, while the other branch is involved in the theory of linear transformations, and is commonly known as the theory of linear substitution groups.

The theory of finite abstract groups is intimately connected with the two theories of substitution groups just noted and was first embodied in the form of a separate treatise by W. Burnside, *Theory of groups of finite order*, 1897; second edition with greater emphasis on linear groups, 1911. These three theories are sometimes referred to as *algebraic group theory*. There is, however, no clear line of distinction between algebraic group theory and the group theories of analysis and geometry.

The group theory of analysis may also be divided into three large branches, viz., theory of finite continuous groups, theory of infinite continuous groups, and theory of groups of automorphic functions. The first of these theories was first developed in a systematic manner by S. Lie, *Theorie der Transformationsgruppen*, three large volumes, while the last was treated in volume 1 of *Automorphe Functionen* by R. Fricke and F. Klein, 1897. No systematic treatise on the general theory of infinite continuous groups has yet been published.

Geometric group theory is based on the group theories of algebra and analysis. In geometry the group concept has entered more widely into the various developments than in algebra or in analysis. Among the treatises devoting considerable space to geometric groups we may mention Klein's *Einleitung in die höhere Geometrie*, II, 1893, and Lie's *Geometrie der Berührungstransformationen*, 1896.

C. Alasia prepared a general bibliography on group theory, which was published in volumes 18–22 of *Rivista di fisica matematica e scienze naturali*, Pavia. A bibliography relating to finite groups together with many historical data may be found in the *Constructive development of group theory* by B. S. Easton, 1902. Among the treatises on the theory of groups which were not noted above are the following: E. Netto, *Substitutionentheorie*, 1882; translated into Italian by G. Battaglini, 1885, and into English by F. N. Cole, 1892; S. Lie and G. Scheffers, *Vorlesungen über Differentialgleichungen*, 1891, and *Vorlesungen über kontinuierliche Gruppen*, 1893; G. Vivanti, *Teoria dei gruppi di transformazioni*, 1898; translated into French by A. Boulanger, 1904; L. Bianchi, *Lezioni sulla theoria dei gruppi di sostituzioni*, 1900; L. E. Dickson, *Linear Groups*, 1901; J. E. Campbell, *Theory of Continuous Groups*, 1903; J. A. de Séguier, *Groupes Abstraits*, 1904; G. Fubini, *Teoria dei gruppi discontinui e delle funzioni automorfe*, 1908; H. Hilton, *Finite Groups*, 1908; E. Netto *Gruppen- und Substitutionentheorie*, 1908; J. A. de Séguier, *Groupes de Substitutions*, 1912; Miller, Blichfeldt and Dickson, *Theory and Applications of Finite Groups*, 1916; H. F. Blichfeldt, *Finite Collineation Groups*, 1917.

FUNDAMENTALS IN THE MATHEMATICS OF INVESTMENT.

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§ 1. Introduction.

In most books on the mathematics of investment there is a wealth of formula somewhat forbidding to the casual reader, however necessary it may be to the accountant or actuary. It is the object of this paper to present in rather compact form some of the fundamentals of the subject, with a few general formulas of wide application.

§ 2. Interest and Discount.

The mathematics of investment deals with the *increment* of *value*. If P units of value—say P dollars—at one moment of time are worth or conceived to be worth S units at a later moment, the increment $S - P$ is called the *interest*[1]

[1] For the general theory, it is immaterial whether the change of value is brought about by a loan or by a series of commercial transactions, indeed, whether the increment is positive or negative.

(on P), and this same difference is called the *discount* (on S) for the period of time determined by the given moments. The ratios

$$r = \frac{S - P}{P}, \qquad u = \frac{S - P}{S} \tag{1}$$

are respectively the *rate* of *interest* and the *rate* of *discount* for the period.

Discount is also called *interest in advance*; S is called the *amount* of P, and P the *present value* or *present worth* of S; P is often called the *principal* or *capital*.

If each dollar of P increases by r, the total increase is $S - P$, the given increase. The interest-rate for a period, then, may just as well be defined as the increase of *one* (dollar) during the period. In place of a dollar, any unit of money or value may be used.

Likewise, if from each dollar of S, the value at the end of the interval, the same deduction u is made, the total deduction is $S - P$, the difference between the ultimate and initial value of the money in question. Thus u may be defined as the discount on *one* (dollar) for the period.

The sum of money $1/(1 + r)$ at the beginning of the period becomes *one* at the end of the period. The former is, then, the *present value* or *present worth* of the latter, and may be designated by w. Then

$$\frac{1}{1 + r} = w = 1 - u. \tag{2}$$

Illustration.—A man borrows \$100 for one year at a discount of 4%—or at 4% interest in advance. In this case, he actually receives \$96 and must pay \$100 at the end of the year. Thus, \$1 due in one year is worth \$.96 now; $w = .96$. The discount on \$1 is \$.04; $u = .04$. The interest is \$4 on \$96, the interest rate is about $4\frac{1}{6}\%$; $r = .0416+$.

§ 3. Derived Interest Rates.

Let r be the interest rate for each of n consecutive periods[1] of time. Then an initial P (dollars) becomes $P(1 + r)$ at the end of the first period, $P(1 + r)^2$ at the end of the second period, $\cdots$, $P(1 + r)^n$ at the end of the nth period. For the sake of simplicity, the P may be dropped. The amount of *one* for the entire period or *term*—formed by joining the n consecutive periods—is

$$1 + r_n = (1 + r)^n,$$

where, by the definition of § 2, r_n is the *interest* on *one* for the entire period or term.

In conformity with this, the amount of *one* for t periods,[2] where t is any

[1] The periods need not be of equal length. The interest charged for February may be the same as for March. So far as the theory goes, there is no reason why the periods should be even approximately equal.

[2] Even this does not require the periods to be of equal length. Moments of time are merely to be in one-to-one correspondence with real numbers, the later of two moments to be associated with the number algebraically greater.

positive real number, is defined to be

$$1 + r_t = (1 + r)^t. \tag{3}$$

The interest on *one* for the entire period is then

$$r_t = (1 + r)^t - 1. \tag{4}$$

If w_t and u_t refer to this new period, it follows from (2) and (3) that

$$\frac{1}{1 + r_t} = 1 - u_t = w_t = w^t = (1 - u)^t. \tag{5}$$

Illustration.—Many banks pay a "*nominal* 4%" *convertible semi-annually.* This means that they pay 2% for 6 months. Then \$100 becomes \$102 at the end of 6 months, and this \$102 becomes \$104.04 at the end of the next 6 months. This makes \$4.04 the interest on \$100 for one year, and the interest rate for the year is $4\frac{1}{25}\%$. This is in conformity with (4) where the original period is 6 mo., $r = .02$, $t = 2$, $r_t = .0404$.

§ 4. Compound Interest vs. Simple Interest.

If in Equation (4), the Binomial Theorem is used and only two terms retained, the result is simple interest on *one* (dollar). The error of the approximation, using simple interest for compound, is the sum of the terms after the second in the binomial expansion.

If P at one moment is worth Q at another moment, then P and Q will be said to be *equivalent* to each other. Thus, equivalence involves the notion of time as well as of value.

A fundamental property of *compound interest* is the following:

Two sums of money each equivalent to a third sum of money are equivalent to each other.

Thus P at one time is equivalent to $P(1 + r)^t$ after the lapse of the time t. This in turn is equivalent to $P(1 + r)^{t+t'}$ after the further lapse of time t'. But the latter is also equivalent to the original principal P after the lapse of the time $t + t'$.

Thus the initial and the ultimate value are each equivalent to the middle value, and they are equivalent to each other.

But, if *simple interest* is used, two sums of money each equivalent to a third are not equivalent to each other.

For major financial computations, simple interest would be absurd—although it often gives a permissible approximation for a fraction of a year.

Thus, the amount must be an exponential function of the time; it can not be a linear function of the time.

The amount is, indeed, a linear function of the principal. Thus often we may ignore the principal at first, and merely use it as a multiplier as the concluding step in a problem.

Illustration.—If at 5% *simple* interest, \$100 is loaned for 4 years, it becomes \$120; and if this is reloaned for 6 years, it becomes \$156. Whereas, if the \$100 were loaned for 10 years straight the amount would be only \$150.

But if \$100 is kept continuously at strict 5% *compound* interest for 10 years, the amount will be exactly \$100 $(1.05)^{10}$, even if the money changes hands a dozen times.

§ 5. Perpetuities.

A *perpetuity*[1] is an infinite series of values, associated with moments of time which extend indefinitely into the future. These moments are usually thought of as the end moments of the periods into which they divide time. When it is desirable to associate the values—usually called *payments*—with the initial moments of the periods, the perpetuity is called a *perpetuity-due*, or, often, an *immediate perpetuity*. Unless otherwise sta ed, the payments of the perpetuity are to be taken as all equal; indeed, frequently it is understood that each payment is a payment of *one* (dollar).

If r is the interest rate for each period, the *present value of a perpetuity* of *one* per period, payable at the end of each period forever is

$$b_{\infty} = \frac{1}{r}; \tag{6}$$

and the *present value of a perpetuity-due* of *one* per period, payable at the beginning of each period forever is

$$B_{\infty} = \frac{1}{u}. \tag{7}$$

For $1/r$ will yield as interest *one* at the end of each period forever; and $1/u$ will yield as interest in advance *one* at the beginning of each period forever. The interest is to be withdrawn as soon as it falls due.

Illustration.—A building must be reconstructed at the end of every 25 years, at an expense of \$10,000. What sum of money put out at 4% compound interest will pay for the renewals forever? Let r_{25} be the interest rate for the period of 25 years. Then from (6) the required endowment is

$$10{,}000\left(\frac{1}{r_{25}}\right) = \frac{10{,}000}{r_1}\left(\frac{r_1}{r_{25}}\right) = 250{,}000\ (.0240120), = \$6003,$$

as found by using a monetary table.

A perpetuity deferred t periods is a series of perpetual payments, the first payment to be made after the lapse of $1 + t$ periods,—thus the first payment is made t periods later than it would normally be made. This t may be any positive real number,—indeed t may be negative, if the *forborne perpetuity*, to be considered presently, is counted as a special case of a deferred annuity. It follows from (5) and (6) that

[1] If \$4 is to be collected as interest on \$100 at the end of each year forever, this is called a perpetuity.

$$ {}_{t|}b_\infty = w^t\left(\frac{1}{r}\right) \tag{8} $$

is the present value of *one* per period forever, the first payment to be made after the lapse of $1 + t$ periods.

Likewise, for the *perpetuity-due deferred t periods*,

$$ {}_{t|}B_\infty = w^t\left(\frac{1}{u}\right) \tag{9} $$

is the present value of *one* per period forever, the first payment to be made after the lapse of t periods.

Now $(1 + r)^t b_\infty$ at any given moment is equivalent to b_∞ for that moment ea lier by t periods, and hence is the "present value" of a perpetuity whose first payment was made $t - 1$ periods earlier. Thus, for this *forborne perpetuity*,

$$ {}_{-t|}b_\infty = (1 + r)^t\left(\frac{1}{r}\right) \tag{10} $$

is the sum of the "amounts" of the earlier payments of *one* per period, begun $t - 1$ periods earlier, and the "present values" of the later payments of *one* per period ad infinitum.

Likewise, for the *perpetuity-due forborne t periods*,

$$ {}_{-t|}B_\infty = (1 + r)^t\left(\frac{1}{u}\right) \tag{11} $$

is the sum of the amounts of the earlier payments of *one* per period, begun t periods earlier, and the present values of the later payments of *one* per period in regular continuation forever.

§ 6. Annuities.

An *annuity* is a series of periodic payments. The payments are usually equal and limited in number.

A perpetuity may be considered as an annuity with an infinite number of payments. *And an annuity may be considered as the difference between two perpetuities starting at different times.*

The value of an annuity may be required at any time. But usually its value is required (1) at the time of the first payment, or (2) one period before this payment is made, or (3) at the time of the last payment, or (4) one period later than the last payment. Taking n as the number of payments, these four values, in order, are

$$ B_n = \frac{1}{u} - w^n\left(\frac{1}{u}\right) = \frac{u_n}{u}, \tag{12} $$

$$ b_n = \frac{1}{r} - w^n\left(\frac{1}{r}\right) = \frac{u_n}{r}, \tag{13} $$

$$c_n = (1 + r)^n \left(\frac{1}{r}\right) - \frac{1}{r} = \frac{r_n}{r}, \tag{14}$$

$$C_n = (1 + r)^n \left(\frac{1}{u}\right) - \frac{1}{u} = \frac{r_n}{u}, \tag{15}$$

where each payment is *one*, as is seen by referring to Equations (3)–(11).

These four formulas can be proven without reference to perpetuities. To prove (14), note that an initial *one* (dollar) is worth r per period, payable at the end of each period for n periods, together with *one* at the end of the nth period—this terminal *one* is principal returned. But an initial *one* is also worth $(1 + r)^n$ at the end of the nth period. Hence r per period for n periods is worth $(1 + r)^n - 1$ at the end of the nth period. Thus *one* per period for n periods is worth $\{(1 + r)^n - 1\}/r = r_n/r$, at the end of the nth period.

The most common proof of these formulas involves the summing of a geometric progression.

Illustration.—If a man deposits, in a bank that pays 4%, \$100 at the end of each year for 25 years, the accumulation to his credit at the end of the 25 years will be \$4,164.59, as found from a table based upon (14).

The terms *deferred* and *forborne* are applied to annuities in the same way as to perpetuities. Thus, the present value of an annuity *deferred* t periods is $w^t b_n$.

B_n and C_n are respectively the "present value" and "accumulation" of an *annuity-due,*—here a payment is made at the *beginning* of each interval.

In (12)–(15), the r and u are respectively the interest-rate and discount-rate for the *period between payments.* If an interest rate is given for some other period—say for a period of length t—then by (3) and (2) these formulas (12)–(15) may be transformed to involve r_t explicitly.

The following interesting and useful relations may be easily proved algebraically or arithmetically:

$$b_n = w^n c_n, \qquad B_n = w^n C_n,$$

$$\frac{1}{b_n} = \frac{1}{c_n} + r, \qquad \frac{1}{B_n} = \frac{1}{C_n} + u.$$

§ 7. Varying Annuities and Perpetuities.

It has been found that $1/u$ will furnish *one* at the beginning of each period forever. Required the capital that will furnish *one* at the beginning of the first period, *two* at the beginning of the second period, and so on, increasing forever. These payments will be furnished if a perpetuity-due is started at the beginning of each period. The capital required for this perpetuity-due of perpetuities-due is

$$(IB)_\infty = \frac{1}{u^2}. \tag{17}$$

The formula for a perpetuity-due of perpetuities is

$$(Ib)_\infty = \frac{1}{u} \cdot \frac{1}{r} = \frac{1}{r^2} + \frac{1}{r}. \tag{18}$$

This yields *one* at the *end* of the first period, *two* at the *end* of the second period, and so on, increasing forever.

An *increasing* annuity or perpetuity is one in which the successive payments are in order, *one, two, three,* etc., as in counting. Thus, an increasing annuity of n payments is an increasing perpetuity minus an increasing perpetuity started n periods later, minus also n times a perpetuity of *one* started likewise at the latter moment.

A *varying* annuity or perpetuity may have payments forming an arithmetic progression of second or higher order. A reader interested in such annuities is referred to "The Institute of Actuaries' Text Book, Part I," pages 40–48.

§ 8. Complete Annuities.

Suppose that the interest r on *one* (dollar) is collected at the end of each of m consecutive periods. If the loan is continued for the fraction t of a period, and the "*face*" of this loan—viz. *one*—is then collected, interest for that fraction t of a period is also due, to the extent of r_t, as per (4). Now the interest payments of r each form an annuity, and the single payment of r_t is said to *complete* this annuity. Likewise, if $1/r$ is loaned, an annuity of *one* per period results, which is *completed* by the payment of r_t/r.

Let the whole term of the loan be n periods. Then $n = m + t$. Now an initial $1/r$ has at the end of n periods the value $(1 + r)^n/r$. Thus, reasoning as in § 6, *one* per period for m periods, followed by the completing payment of r_t/r, is worth at the end of n periods,

$$\frac{1}{r}(1 + r)^n - \frac{1}{r} = \frac{r_n}{r},$$

as in (14). Formula (14) has thus been made valid for all positive real values of n provided that the completing payment is r_t/r when n is not integral. This completing payment approaches one when t approaches one, as obviously it should.

Indeed, formulas (12)–(15) are valid for all positive values of n if the completing payment of r_t/r is made at the end of the n periods.

Likewise they hold if the set of m regular payments of *one* each is preceded by an initial completing payment of u_t/u, made earlier than the first regular payment by the fraction t of a period.

Indeed, these formulas will remain valid if any set of k consecutive regular payments is replaced by a single payment of u_k/u at the beginning of the term of the special set, or by a single payment of r_k/r at the end of the term of the special set.

If the t above is a fraction of a small period of time, such as a year, a fair approximation for r_t/r, or indeed for u_t/u, will be t itself, in most cases—as easily seen by using the Binomial Theorem.

Illustration.—On a loan of \$2,500 at 4%, interest payments of \$100 have been collected regularly for a certain number of years. The lender wishes the loan of \$2,500 paid 6 months after the last interest payment has been made. In practice \$50, the simple interest on the \$2,500 for 6 months would be collected. But this is not the equitable interest. \$100 payable at the end of a year is equivalent to \$49.50 payable at the end of 6 months and \$49.50 at the end of the year. The interest on \$49.50 for 6 months makes up the extra dollar. An annuity of \$100 per year is completed (exactly) by \$49.50 at a moment 6 months later than a regular payment. This illustrates the fact that *for a fraction of one period, simple interest is greater than compound interest,* so that a money lender can well afford to substitute simple interest for compound interest for a fraction of the specified interest period.

§ 9. Common Notation.

To gain generality, certain commonly accepted symbols have been avoided in this paper thus far.

When the year is taken as the unit of time, the usual symbols are as follows:

$$r = i, \qquad u = d, \qquad w = v,$$

$$b_n = a_{\overline{n}|}, \qquad c_n = s_{\overline{n}|}, \qquad B_n = \mathbf{a}_{\overline{n}|}, \qquad C_n = \mathbf{s}_{\overline{n}|}.$$

Thus

$$a_{\overline{n}|} = \frac{1 - v^n}{i}, \qquad s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

The corresponding rate of interest for one one-mth of a year is j_m/m; and the corresponding rate of discount is f_m/m. Here j_m is called the "*nominal rate of interest,*" and f_m the "*nominal rate of discount.*"

The equations

$$1 + i = \left(1 + \frac{j_m}{m}\right)^m = v^{-1} = (1 - d)^{-1} = \left(1 - \frac{f_m}{m}\right)^{-m} = e^{\delta}$$

connect the most fundamental of these quantities where the force of interest

$$\delta = \log_e (1 + i) = \lim_{m=\infty} j_m = \lim_{m=\infty} f_m,$$

with i constant.

§ 10. Conclusion.

No attempt has been made to display all the formulas most frequently used, or to introduce the reader to the many fascinating applications in connection with premiums on bonds, amortization schedules, wearing values of machinery, sinking funds, etc.

Few students of mathematics have any conception of the beauty or difficulty of certain problems arising in actual business transactions.

But an intense arithmetic appreciation of the important relations underlying annuities and perpetuities will go far toward equipping a student to solve problems of this kind.

BOOK REVIEW.

SEND ALL COMMUNICATIONS ABOUT BOOKS TO W. H. BUSSEY, University of Minnesota.

Introduction to Mathematical Statistics. By CARL J. WEST, Ph.D., Assistant Professor of Mathematics, Ohio State University. R. G. Adams and Co., Columbus, 1918.

In spite of the wide divergence between the original sources and purposes of mathematical statistics, the present development of the subject seems to be along two main lines—that is, either it presents statistical information consisting usually of numerical measurements in a form easily and quickly interpreted by the eye, or it derives and applies formulas for the purpose of measuring various phenomena presented by the measurements.

The first line of development calls for very elementary mathematical knowledge and has been well treated by several authors, especially Brinton. The second line of development has been treated in its various phases in scientific journals and a few English and German books; it involves mathematical principles ranging from the most elementary to the most abstruse but was never presented by an American statistician in approximately complete form until Dr. West's *Introduction to Mathematical Statistics* appeared. The usefulness of the German books in this country is impaired by the language while all the other foreign books on the subject are too voluminous to be used as textbooks in American colleges.

As may be expected, Dr. West's book gives a treatment of both lines of development and, considering the great amount of literature of the second kind in various scientific journals, shows good judgment in the selection of important principles. No doubt, Dr. West's book will be widely used—especially as a textbook in colleges—with few rivals for many years to come, although any author of a statistical textbook would merit considerable praise even though he did little more than encourage wider study and help to standardize methods.

The reviewer agrees with Dr. West in most of his introductory statements but insists that a knowledge of at least the calculus is indispensable to a full understanding of the principles involved in the second line of development mentioned above, especially the work of Pearson and his followers, some of which is treated by Dr. West.

The printers show a lack of experience with scientific books or else the proof was not carefully read; the alignment is poor in places—for example, letters are out of line in eighteen different places on page 18—and typographical errors are fairly frequent.

A few paragraphs are poorly written. For example, the last paragraph on page 131 is surely not in the form intended. Also, the expression *Since* a_1 *and* a_2 *are not integers,* in the last paragraph on page 135, seems to be out of place. The repetition of the formula for K on page 138 seems unnecessary—it is given on the preceding page. The formulas for S_3, S_4 and S_5 on page 98 are almost too badly mixed and the expression for $1/y \cdot dy/dx$ in terms of the moments is almost too poorly aligned to be unravelled by the student. But the trouble in both cases is typographical.

The latter part of the book is a little too condensed, especially the part of the appendix devoted to the Pearson types of frequency curves, which cannot be used except for purposes of reference unless supplemented by explanations and illustrations by an experienced instructor. Reference should have been made to the types of curves recently developed by Pearson as well as to the interesting abacus given in Pearson's Tables to be used in distinguishing the various types.

The treatment of the various kinds of averages is good although the reviewer believes that the weighted arithmetic average should be regarded as a special form of the ordinary arithmetic average—where the various conceptions of *weight* take the place of *frequencies.*

Correlation theory is treated at great length, as it deserves, although the important correlation surface and its equation, with its important relation to the correlation coefficient, is unfortunately omitted in order to avoid more advanced mathematical principles. The distinction between the correlation coefficient and the correlation ratio is well explained.

Most of the errors found in the book are trivial and of the kind which it is almost impossible to avoid in a book dealing with a virgin field. On the whole, Dr. West's book promises to fill well the long felt want for a textbook on mathematical statistics and no longer need instructors depend solely upon their own notes and lectures. The fact that the book is well provided with good examples should make it especially successful as a text-book.

C. H. Forsyth.

Dartmouth College.

PROBLEMS FOR SOLUTION.

Send all communications about problems to B. F. Finkel, Springfield, Missouri.

It is to be understood that problems proposed for solution or solutions of problems which have been proposed in the Monthly are welcomed from all readers, whether subscribers or not. Single copies containing these problems or solutions will be sent to those contributing, provided their addresses are known to the Secretary of the Association.

2727. Proposed by H. J. WOODALL, Stockport, England.

Having given $2^k \equiv +k$ (mod. p) and ξ the haupt-exponent of 2 for mod p (ξ is the least power of 2 whose residue, for modulus p, is plus unity) solve $x \cdot 2^x + 1 \equiv 0$ (mod p).

Also, as regards the inverse problem, what do we know about the factors of $N \equiv b \cdot 2^b + 1$, when b is a known positive integer, e. g., $b = 141$.

2728. Proposed by NORMAN ANNING, Somewhere in France.

A material triangle of uniform density and thickness is of such a shape that when suspended from the vertices in succession, the lower sides have slopes of $1:1$, $1\frac{1}{2}:1$, and $3:1$. Construct the triangle given that the shortest side is 10 inches.

By definition, an $a:1$ slope makes an angle with the vertical whose tangent is a.

2729. Proposed by N. P. PANDYA, Sojitra, India.

Solve in integers $x^3 + 3y^4 = z^2$.

2730. Proposed by W. E. HEAL, Washington, D. C.

If $z^n = x^n + y^n$, where x, y, z, n are integers, and n prime, prove that

$$\{z^{(n-1)/2}[(z-x)x^{(n-1)/2} + (z+y)y^{(n-1)/2}]\}^2 + \{x^{(n-1)/2}[2x^{(n+1)/2} + y^{(n-1)/2}(x-y)]\}^2$$
$$= \{z^{(n-1)/2}[(z+x)x^{(n-1)/2} + (z-y)y^{(n-1)/2}]\}^2 + \{y^{(n-1)/2}[2y^{(n+1)/2} - x^{(n-1)/2}(x-y)]\}^2$$
$$= [z^{n+1} + (x^{(n+1)/2} - y^{(n+1)/2})^2] \times [z^{n-1} + (x^{(n-1)/2} + y^{(n-1)/2})^2].$$

Also prove that $z^{n+1} + (x^{(n+1)/2} - y^{(n+1)/2})^2$ and $z^{n-1} + (x^{(n-1)/2} + y^{(n-1)/2})^2$ have no common factors except the divisors of $z^{n-1}(z+x)(z+y)$.

2731. Proposed by JAMES K. WHITTEMORE, Yale University.

A bowl is in the form of a paraboloid of revolution. If for a given volume the surface is a minimum, prove that the ratio of the diameter of the top to the depth is approximately 1.86.

SOLUTIONS OF PROBLEMS.

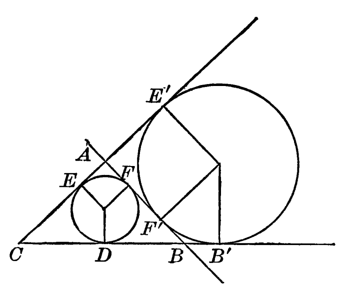
2660. Proposed by JOSEPH E. ROWE, State College, Pa.

Prove that the distance measured along the side of a triangle, from the point of contact with the inscribed and escribed circle, is equal to the side of the triangle between the two circles.

SOLUTION BY GEORGE F. WILDER, Brooklyn, N. Y.

Let ABC be the triangle and s its semi-perimeter. From the equalities $AE = AF$, $CE = CD$, and $BD = BF$, we have $s = AE + CD + BD$, or $s = AE + a$, since $CD + BD = a$. Hence, $AE = s - a$, (1). Also, since $AE' = AF'$ and $BF' = BD'$, $CE' + CD' = 2s$. Hence, $CD' = s$, since $CE' = CD'$. Hence, $BD' = CD' - CB = s - a$, (2). From (1) and (2) we have $AE = BD'$.

Now $EE' = EA + E'A = BD' + AF'$, since $AE = BD'$ and $AE' = AF'$, and hence $EE' = BF' + AF' = AB$.



Similar and various other solutions were received from the following contributors: J. V. BALCH, HORACE OLSON, PAUL CAPRON, S. W. REEVES, C. P. SOUSLEY, J. F. CONNELLY, ELIJAH SWIFT, and the PROPOSER.

This problem was proposed in an examination test in Cooper Union, New York, and solutions similar to the one above were received from G. J. HARRIS, H. F. MATHUSSEN, I. J. FAJANS, C. J. SCHMITT, H. GLADSTONE, MORRIS DEVORKEN, ANGELO J. SAJIN, DAVID TENNENBAUM, JACOB GREISMANN, I. MILLENKY,

MAX SILVERMAN, G. VANDERBOON, J. LEBEDEFF, P. WEINBERG, MOSE KAHN, MORRIS GROSS, PETER PUMO, HYMAN MILBERG, S. L. RADER, O. C. SIMONSEN, ABRAHAM PLETMAN, FRANK PATLI, M. J. MONTFERRANTE, JOSEPH MENT, MORRIS BUNSIS, SAMUEL COHEN, and CLIFFORD STRAIN.

2668. Proposed by B. F. FINKEL, Drury College.

Show that

$$v = \frac{2}{9}\frac{ga^2\sigma}{\eta},$$

where a is the radius of a droplet, σ its density, η the viscosity of the air and v the velocity under gravity g. *Stokes's law.*

SOLUTION BY ALBERT R. NAUER, St. Louis, Missouri.

Stokes finds analytically a formula determining the resistance of a fluid medium to the movement of a sphere through it,

$$F = \tfrac{4}{3}\pi\rho an\sqrt{-1}[af_1'(a) + 2f_2(a)]e^{\sqrt{-1}nt}, \tag{I}$$

Stokes's *Mathematical and Physical Papers,* Vol. III, p. 33. From a previous development (p. 28) is given

$$f_1''(r) - \frac{2}{r^2}f_1(r) = 0, \qquad \text{and} \qquad f_2''(r) - \frac{2}{r^2}f_2(r) - m^2f_2(r) = 0. \tag{II}$$

Then the equations of condition become on putting $f(r)$ for $f_1(r) + f_2(r)$, and $r = a$,

$$f'(a) = ac, \qquad f(a) = \tfrac{1}{2}a^2c, \tag{1}$$

$$f'(b) = 0, \qquad f(b) = 0. \tag{2}$$

The integration of the differential equations (II) has no difficulties, the first one is of a well-known form and the second one is a Riccati equation solvable in finite terms.

The integrals are

$$f_1(r) = \frac{A}{r} + Br^2 \qquad \text{and} \qquad f_2(r) = Ce^{-mr}\left(1 + \frac{1}{mr}\right) + De^{mr}\left(1 + \frac{1}{mr}\right).$$

Here $D = 0$, since otherwise the velocity would be infinite at an infinite distance; $B = 0$, otherwise the velocity would be finite when $r = \infty$.

From Eq. (1), we get

$$A = \tfrac{1}{2}a^3c - aCe^{-ma}\left(1 + \frac{1}{ma}\right).$$

Putting $b = \infty$, we get from Eq. (2),

$$\frac{d}{da}\{\text{Equation (1)}\} = 0 = \frac{3a^2c}{2} - aC(1 - m - 1)e^{-ma}, \qquad C = \frac{3ac}{2m}e^{ma},$$

and then

$$A = \tfrac{1}{2}a^3c + \frac{3a^2c}{2m}\left(1 + \frac{1}{ma}\right).$$

Substituting in (I),

$$F = -\tfrac{2}{3}\pi a^3cn\sqrt{-1}\left(1 + \frac{9}{ma} + \frac{9}{m^2a^2}\right)e^{\sqrt{-1}nt}. \tag{III}$$

Substituting for m its value, which is the particular square root of $(n\sqrt{-1})/\mu'$ which has its real part positive, and write V for $ce^{\sqrt{-1}nt}$. Then

$$F = -6\pi\mu'\rho aV, \tag{IV}$$

where $\mu' = \mu/\rho$, called by Stokes, the *index of friction* and ρ is the density of the fluid. Whenever the motion is so slow that the part of the resistance which depends on the square of the velocity may be neglected, we have, supposing V to be the terminal velocity;

$$-F = \tfrac{4}{3}\pi ga^3(\sigma - \rho),$$

where g is the force of gravity, and σ, which is supposed to be greater than ρ, the density and a the radius of the sphere.

Hence,

$$6\pi\mu'\rho aV = \tfrac{4}{3}\pi g a^3(\sigma - \rho),$$

or

$$V = \frac{2}{9}\frac{g}{\mu'}\left(\frac{\sigma}{\rho} - 1\right)a^2 = \frac{2}{9}\frac{ga^2\sigma}{\eta},$$

by placing

$$\frac{\sigma - \rho}{\mu\rho} = \frac{\sigma}{\eta}.$$

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas.

DISCUSSIONS.

Concerning Two Fifth-Power Problems in Diophantine Analysis.

By Cyrus B. Haldeman, Ross, Butler County, Ohio.

I. *To resolve*

$$p^5 + q^5 + p^5 + s^5 = t^5 + u^5 + v^5 \qquad (1)$$

where p, q, r, s, t, u, and v are rational.

Solution. Let $p = 2x$, $q = y - x$, $r = 2a - b^2 - x$, $s = 2a + b^2 + x$, $t = x + y$, $u = 2a - b^2 + x$ and $v = 2a + b^2 - x$.

Substituting in (1) we get

$$\begin{aligned}(2x)^5 + (y - x)^5 + (2a - b^2 - x)^5 + (2a + b^2 + x)^5 \\ = (x + y)^5 + (2a - b^2 + x)^5 + (2a + b^2 - x).^5\end{aligned} \qquad (2)$$

We may write (2):

$$\begin{aligned}(x + y)^5 + (x - y)^5 + [(2a + b^2) - x]^5 - [(2a + b^2) + x]^5 \\ + [(2a - b^2) + x]^5 - [(2a - b^2) - x]^5 - (2x)^5 = 0.\end{aligned} \qquad (3)$$

Expanding, regarding each binomial in parenthesis as a single quantity, and adding, we have from (3)

$$\begin{aligned}2x^5 + 20x^3y^2 + 10xy^4 - 2x^5 - 20x^3(2a + b^2)^2 - 10x(2a + b^2)^4 \\ + 2x^5 + 20x^3(2a - b^2)^2 + 10x(2a - b^2)^4 - 32x^5 = 0.\end{aligned} \qquad (4)$$

Equation (4) reduces to

$$2x^2y^2 + y^4 - 16ab^2x^2 - 64a^3b^2 - 16ab^6 - 3x^4 = 0.$$

Solving for y^2, we get

$$y^2 + x^2 = \sqrt{4x^4 + 16ab^2x^2 + 64a^3b^2 + 16ab^6} = 2x^2 + 4a\,(b^2 + d^2), \text{ say.}$$

From this we find

$$x^2 = \frac{4a^2b^2 - (b^2 + d^2)^2a + b^6}{d^2},$$

and

$$y^2 = \frac{4a^2b^2 - (b^2 + d^2)^2a + b^6 + 4d^2(b^2 + d^2)a}{d^2}.$$

To rationalize the expressions for x and y, let

$$d^2x^2 = 4a^2b^2 - (b^2 + d^2)^2\, a + b^6 = (n - 2ab)^2, \tag{5}$$

and

$$d^2y^2 = 4a^2b^2 - (b^2 + d^2)^2\, a + b^6 + 4d^2\,(b^2 + d^2)\, a = (n - 2ab - Z)^2. \tag{6}$$

From (5) we get

$$a = \frac{n^2 - b^6}{4bn - (b^2 + d^2)^2},$$

and from (6), since the square of $n - 2ab$ is assumed equal to the first four terms of the left-hand member, we have

$$a = \frac{2nZ - Z^2}{4bZ - 4d^2\,(b^2 + d^2)}.$$

Equating the values of a and clearing of denominators, we get

$$4b^6d^2\,(b^2 + d^2) - 4b^7Z - 4d^2\,(b^2 + d^2)n^2$$
$$= Z^2\,(b^2 + d^2)^2 - 4bZ^2\,n + 4bZn^2 - 2Z\,(b^2 + d^2)^2\,n.$$

Place $4bZn^2 = -4d^2\,(b^2 + d^2)n^2$ and solve for

$$Z = -\frac{d^2\,(b^2 + d^2)}{b};$$

then

$$n = \frac{4b^7Z - 4b^6d^2\,(b^2 + d^2) + Z^2\,(b^2 + d^2)^2}{4bZ^2 + 2Z\,(b^2 + d^2)^2}.$$

Substituting the value of Z in n gives

$$n = \frac{d^2\,(b^2 + d^2)^3 - 8b^8}{2b\,(d^4 - b^4)}.$$

The value of Z and the value of n in terms of b and d, substituted in either of the above values of a gives

$$a = \frac{d^4(b^2 + d^2)^2 - 4b^8}{4b^2(d^4 - b^4)}.$$

From (5) and (6) we have

$$x = \frac{n - 2ab}{d},$$

and

$$y = \frac{n - 2ab - Z}{d}.$$

The values of a, n and Z substituted in these expressions for x, and y, give

$$x = \frac{bd^2(b^2+d^2)^2 - 4b^7}{2d\,(d^4-b^4)} \quad \text{and} \quad y = \frac{bd^2(b^2+d^2)^2 - 4b^7}{2d(d^4-b^4)} + \frac{d(b^2+d^2)}{b}.$$

From the values of a, x and y we obtain

$$p = \frac{bd^2(b^2+d^2)^2 - 4b^7}{d(d^4-b^4)}, \qquad q = \frac{d(b^2+d^2)}{b},$$

$$r = \frac{2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)}{2b^2d(d^4-b^4)},$$

$$s = \frac{2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)}{2b^2d\,(d^4-b^4)},$$

$$t = \frac{d^4(b^2+d^2) - 4b^8}{bd(d^4-b^4)},$$

$$u = \frac{2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)}{2b^2d(d^4-b^4)},$$

$$v = \frac{2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)}{2b^2d(d^4-b^4)}.$$

By substitution of the expressions just found for p, q, r, s, t, u and v in (1); and after clearing of denominators, we have the identity

$$\begin{aligned}
&[2b^3d^2(b^2+d^2)^2 - 8b^9]^5 + [2bd^2(b^2+d^2)(d^4-b^4]^5 \\
&\qquad + [2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)]^5 \\
&\qquad + [2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)]^5 \\
&= [2bd^4\,(b^2+d^2)^2 - 8b^9]^5 \\
&\qquad + [2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)]^5 \\
&\qquad + [2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)]^5
\end{aligned}$$

Take $b = 1$, $d = 2$, and we have, after dividing by 12^5,

$$16^5 + 50^5 + 53^5 + 79^5 = 63^5 + 66^5 + 69^5.$$

II. *To resolve the equality*

$$\begin{aligned} p^5 + q^5 + r^5 + s^5 + t^5 + u^5 + v^5 + w^5 \\ = (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \ldots + (2g_n)^5, \end{aligned} \tag{1}$$

where $p, q, r, s, t, u, v, w, g_1, g_2, g_3, \ldots, g_n$ *are rational.*

Solution. Let $p = a + b + c$, $q = a - b - c$, $r = b - a - c$, $s = c - a - b$, $t = d + e + f$, $u = d - e - f$, $v = e - d - f$, $w = f - d - e$.

Then, by substitution in (1), we have

$$(a + b + c)^5 + (a - b - c)^5 + (b - a - c)^5 + (c - a - b)^5 \\ + (d + e + f)^5 + (d - e - f)^5 + (e - d - f)^5 + (f - d - e)^5 \quad (2) \\ = (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \dots + (2g_n)^5.$$

For convenience, we may write (2):

$$[a + (b + c)]^5 + [a - (b + c)]^5 + [(b - c) - a]^5 - [(b - c) + a]^5 \\ + [d + (e + f)]^5 + [d - (e + f)]^5 + [(e - f) - d]^5 - [(e - f) + d]^5 \quad (3) \\ = (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \dots + (2g_n)^5.$$

Expanding, regarding each binomial in parenthesis as a single quantity and adding, we get from (3)

$$2a^5 + 20a^3(b + c)^2 + 10a(b + c)^4 - 2a^5 - 20a^3(b - c)^2 - 10a(b - c)^4 \\ + 2d^5 + 20d^3(e + f)^2 + 10d(e + f)^4 - 2d^5 - 20d^3(e - f)^2 - 10d(e - f)^4 \quad (4) \\ = 32\,(g_1^5 + g_2^5 + g_3^5 + \dots + g_n^5).$$

Equation (4) reduces to

$$10a^3[(b + c)^2 - (b - c)^2] + 5a[(b + c)^4 - (b - c)^4] \\ + 10d^3[(e + f)^2 - (e - f)^2] + 5d\,[(e + f)^4 - (e - f)^4] \\ = 16(g_1^5 + g_2^5 + g_3^5 + \dots + g_n^5);$$

and by further expansion and reduction, we have

$$5(a^3bc + ab^3c + abc^3 + d^3ef + de^3f + def^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5). \quad (5)$$

Let $d = a$ and (5) becomes

$$5a^3\,(bc + k^3\,ef) + 5a\,(b^3c + bc^3 + ke^3f + kef^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5).$$

Place $bc + k^3ef = 0$ and have

$$5a(b^3c + bc^3 + ke^3f + kef^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5).$$

From these equations we obtain

$$b = -\frac{k^3ef}{c}, \qquad a = \frac{2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5)}{5(b^3c + bc^3 + ke^3f + kef^3)},$$

which give the requirements.

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by R. C. Archibald, Brown University, Providence, R. I.

CLUB ACTIVITIES.

Denison Mathematics Club, Denison University, Granville, Ohio.

This Club came into existence through well attended meetings of an informal nature in 1915 shortly after Professor Forbes B. Wiley went to Denison University as head of the department of mathematics. In January, 1916, a constitution was approved and a club formally organized "to bring before its members matters of interest in mathematics that are not regularly discussed in courses offered in the curriculum." Any college student or member of the faculty who desires to join the club is eligible to do so. The average attendance at meetings last year was about 25 as against 30 the year before.

Officers 1917–18: President, Grace Jefferson '18; vice-president, Ruth Phillips '19; secretary-treasurer, Sterling Abell '20; assistant secretary-treasurer, Lawrence Curl '20. The program committee consists of these officers together with the head of the department.

October 3, 1916: Election of officers for the year.

November 7: The purposes of the club explained by the president, and various short topics presented to illustrate to new-comers some of the possibilities of such an organization.

November 21: "Problems in the mathematics of astronomy" by Professor Wiley.

December 19: "History of the Denison Mathematics Club"[1] by Marie Tilbe '17; "Integral curves" by Professor Anna B. Peckham.

January 23, 1917: "Mathematical societies and journals" by Professor Wiley; attention was drawn to problems in this Monthly.

February 20: Solution of problems proposed at the previous meeting by Lawrence Curl '20 and Virgil Traxler '19; "Spherical projection" by Harlan C. Reynolds '17.

March 6: "The possible use of parallel axes in the plane of intersecting axes of coördinates" by Professor Wiley.

March 20: "The concept infinity" by Professor Paul Biefeld of the astronomy department.

April 17: "Hyperbolic functions" by Professor Wiley.

May 1: Officers for 1917–18 elected.

September 25: "The probability function" by Professor Wiley.

October 23: "The fourth dimension" discussed by Sterling Abell '20, George Read '18, Clifford Marshall '18 and Charles T. Bumer '19.

November 6: "Theorems of elementary number theory" by Professor Wiley.

November 20: "Inversion" by Professor Peckham.

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[1] Referring, presumably, to meetings before the formal organization.

December 18: "Constructions with compasses alone and with ruler alone" by George T. Street Jr., instructor in mathematics.
January 22, 1918: "The algebra of number pairs" by Grace Jefferson '18.
February 19: "The possible use of parallel axes in the plane of intersecting axes of coördinates" (continued)[1] by Professor Wiley.
March 5: "The different methods of defining the trigonometric functions and the identification of the functions so defined" (primarily for freshmen) by Professor Wiley.
March 19: "A problem in invariants" by Mildred Hunt '08.
April 16: "Introduction to the study of groups" by Professor Wiley.
May 14: Election of officers for 1918–19.

"During the past year our Club has felt increasingly the loss of strong members who have entered the national service."

UNDERGRADUATE MATHEMATICS CLUB, University of Illinois, Urbana, Ill.

The first mathematics club at the University of Illinois was organized during the year 1899–1900. Members of the faculty, graduate students and undergraduates constituted the membership. At the bi-weekly meetings for several years the topics discussed were suitable for undergraduates but gradually the club became an experiment station for trying out papers to be presented later before the American Mathematical Society. As a result in 1909 the Club was divided into two sections: (1) the graduate section with its tri-weekly meetings devoted mostly to the presentation of doctor's dissertations and original papers by faculty members; and (2) the undergraduate section which met monthly, for about an hour, "for the consideration of questions of general mathematical interest and the solution of problems." During several years the average attendance at meetings of this latter section was 30–35. While the management of the section was largely in the hands of graduate students a majority of the speakers were undergraduates. This arrangement did not seem to give entire satisfaction and still further bifurcation occurred in the autumn of 1917.

There are, then, now three organizations: I. Mathematics Club—Graduate Section, all members of the department staff and all graduate students being *eo ipso* members. The meetings are open to all interested and are devoted exclusively to reports on research work done in the department and to occasional reports on scientific meetings. II. The Mathematical Round Table, consisting of all graduate students in mathematics (and a few selected seniors) but not open to men with the Ph.D. degree. At the weekly meetings the average attendance is 12–15. Each member presents two papers a year on some semi-advanced topic not taught in the regular courses. Visitors are not encouraged. III. The Undergraduate Mathematics Club from which the department staff, as well as graduate students, are excluded. The members consist of juniors and seniors majoring in mathematics, and freshmen and sophomores of excellent stand-

[1] *Cf.* this MONTHLY, 1918, page 255.

ing in mathematics courses. All undergraduates are welcome as visitors. The average attendance is 8–12 from a membership of 36. Officers, 1917–18: President, Ruth Andrews '18; vice-president, Mary D. Craigmile '18; secretary, Harry W. Penhallow '19; program committee: the president, George Williams '18, and Dr. Aubrey J. Kempner (faculty adviser).

Apart from the solving of problems less difficult than those in the MONTHLY the programs of the Undergraduate Mathematics Club for last year were as follows.

January 23, 1918: "On paper folding" by Ruth Anderson '18.
February 6: "The value of high school mathematics" by Winifred White '18.
February 20: "Mathematical puzzles" by Fannie McMurray '19.
March 13: "On Rolle's Theorem" by George Williams '18.
April 3: "Constructions with the double ruler" by Mary Craigmile '18.
April 17: "The Theorem of Pythagoras" by Irene Doyle '19.
May 1: "The fundamental theorem of algebra" by C. T. G. Ching '18.
May 15: "On algebraic numbers" by Jesse E. Wilkins '18.

The programs of the Mathematics Club—Undergraduate Section—in 1915–16 were as follows.

November 7, 1915: "Who's who in mathematics in America" by Nathan C. Grimes, assistant in mathematics.
December 5: "Construction possibilities" by Katherine Lackey '16.
January 9, 1916: "Fourier Series and the harmonic analyzer" by Paul L. Bayley Gr.
February 13: "Nomography" by Joe Langueville Gr.
March 13: "Origin of calculus" by Erma Elliott Gr.
April 10: "Graphic solution of equations" by Albert E. Babbitt Gr.
May 9: "Mathematical models" by Mauritz Hedlund Gr.

THE MATHEMATICAL CLUB OF THE KANSAS STATE AGRICULTURAL COLLEGE, Manhattan, Kansas.

This club was organized in September, 1913 "to stimulate larger interest in mathematics on the part of students." Any student of collegiate rank was eligible for membership, which totalled 75 in 1916–17; the average attendance at meetings was about 20. Professor Benjamin L. Remick the head of the mathematics department acted as "officer in charge;" there were no student officers. "For various local reasons such as decrease in enrollment due to the war and the desire of several members of the department to engage in special mathematical work for themselves, it was decided not to carry on the usual club activities during 1917–18."

The programs for the first four years of the Club's existence were printed annually. They are reproduced below.

October 25, 1913: "Organizations and journals for the study of mathematics in America" by Professor Remick; "History of π" by Professor Alfred E. White.
November 8: "Euclidean constructions" by Professor William H. Andrews; "The history of logarithms" by Arthur Fehn, instructor.

November 22: "The teaching of secondary algebra" by Ina E. Holroyd, assistant; "The algebraic treatment of the evolute of a conic" by Professor Harrison E. Porter.

December 6: "History of a few proofs of the Pythagorem Theorem" by Charles H. Clevenger, instructor; "The place of limits in geometry" by Daisy D. Zeininger, instructor.

January 17, 1914: (*a*) "Note on problem 411, AMERICAN MATHEMATICAL MONTHLY,"[1] (*b*) "The perfect magic square for 1914" by Professor Remick; "The tendencies in modern mathematics" by Professor William T. Stratton.

January 31: "Concerning regular polyhedra" by Professor William H. Andrews; "The development of irrational numbers" by Professor White.

February 14: "The development of the decimal system" by Mr. Fehn; "Calculating machines" by Professor Porter.

February 28: "A mathematical notation applied to a few problems" by Mr. Clevenger; "The place of mathematics in the education of women and girls"[2] by Miss Holyroyd.

March 14: "Mathematics as a universal study" by Professor Stratton; "What is the laboratory method in the study of mathematics?" by Miss Zeininger.

April 11: "Non-Euclidean geometry" by Professor Remick; "Some fundamental ideas in mathematics" by Professor Andrews.

April 25: "The theory of duality" by Professor Porter; "The mathematical properties of maps" by Professor White.

May 9: "The history of trigonometry" by Mr. Fehn and Miss Zeininger; "Mathematics in English public schools" by Professor Stratton.

May 23: "The problem of failures" by Miss Holyroyd; "Coördinate systems" by Mr. Clevenger.

November 7: "Recent movements in mathematics" by Professor Remick; "Mathematical paradoxes" by Elliott Ranney '16; "History of the duplication of the cube" by Roy W. Haege '18.

November 21: "A problem in partial payments" by Professor Porter; "Graphical railroad time tables" by Gabe A. Sellers '17; "Plato as a mathematician" by Edith L. Alsop '16.

December 5: "Number bases among the primitive races" by Professor Stratton; "Mathematics at Kansas State Agricultural College from a student's standpoint" by William A. Lathrop '15; "History of algebraic symbolism" by Dilts S. McHugh '18.

January 16, 1915: "History of angular measurement" by Professor White; "History of the trisection of an angle" by Jefferson H. Flora '17; "Magic square for 1915" by Rufus S. Kirk '17.

January 30: "Short methods in the four fundamental operations" by Lee R.

[1] Vol. 20, p. 136: "*ABCD* is a rectangle of known sides. *BC* being produced indefinitely, it is required to draw a straight line from *A* cutting *CD* and *BC* in *X* and *Y*, respectively, so that the intercept *XY* may be equal to a given straight line. (Unsolved in *Educational Times*.)" Cf. *Proceedings of the Edinburgh Mathematical Society*, Vol. 28, pp. 152–178, 1909–10.

[2] Published in *School Science and Mathematics*, June, 1914.

Light '15; "Curves that trisect an angle" by Earl E. Swenson '17; "Origin of number symbols" by Caroline R. Packard '17.

February 13: "Number concept and generalization in algebra" by Mr. Fehn; "Mathematical games" by Zeno C. Rechel '18; "Elementary mathematics in evening schools" by Helen Mitchell '18.

February 27: "The mathematics of mineralogy" by Lyle M. Dean, assistant; "Life and works of Newton" by Louis R. Parkinson '16; "History of arithmetic in the United States" by Mae V. Hildebrand Sp.

March 13: "Correlation of mathematics and physics" by Eustace V. Floyd, assistant professor of physics; "History of the quadrature of the circle" by Robert F. Mirick '16; "Systems of quadratic equations" by James A. Hull '17.

April 10: "The logical and the psychological in geometry" by Miss Zeininger; "The controversy between Leibnitz and Newton" by Charles A. Willis '16; "Mystic properties of numbers" by Charles H. Zimmerman '16.

April 24: "Mathematical symmetry in nature"[1] by Miss Holyroyd; "Construction of logarithmic and trigonometric tables" by Donald D. Hughes '18; "Life and works of Descartes" by Russel H. Oliver.

May 1: "Thought versus rule in mathematics" by Joseph I. Kirkpatrick, assistant in veterinary medicine; "Euclid and his geometry" by Carl D. Hultgren '17; "Hypatia, last of the Greeks" by Sarah K. Kimport '18; "Geometry for engineers" by Leroy N. Miller '18.

May 15: "Historical solution of the quadratic equation" by Professor Andrews; "Relation of mathematics to statistics" by Lester Tubbs '17; "Geometry and construction work" by Basil A. Greene Sp.

November 6: "The place of mathematics in the study of heredity" by Edward N. Wentworth, professor of animal breeding.

November 20: "The function idea in and outside of mathematics" by Professor Remick; "The relation of mathematics to wireless telegraphy" by Harold M. McClelland '16; "Mathematics in verse" by Helen Mitchell '18.

December 4: "Generalizing some theorems in elementary geometry" by Professor Andrews; "The United States coast and geodetic survey" by Hubert A. Dawson '19; "The life and work of Poincaré" by Harry L. Robinson '18.

January 15, 1916: "The application of the prismatoid formula to the solids of elementary geometry" by Professor White; "History of arithmetic in the United States" by Gordon W. Hamilton '19; "The Pythagorean brotherhood" by Elizabeth A. Cotton '19.

January 29: Address by John O. Hamilton, professor of physics; "Zero and infinity" by Elliott Ranney '16.

February 12: "Means of measuring mathematical ability in students" by Professor Stratton; "Graphical solution of quadratic equations having complex

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[1] In this connection, reference may be given to the somewhat unscientific work by S. Colman and C. A. Coan entitled *Nature's Harmonic Unity. A Treatise on its Relation to Proportional Form,* New York, Putnam, 1912.

roots" by Frank M. Sisson '18; "Proof and history of the fundamental theorem of algebra" by Otto B. Githens '17.

February 26: "The mathematics of investment" by Professor Porter; "The mathematics of chemistry" by Herbert H. King, assistant professor of chemistry.

March 4: "Mathematical symbolism and the economy of thought" by Miss Zeininger; "The solution of the biquadratic equation" by Charles A. Willis '16; "The problem of one cent" by Leroy N. Miller '18.

March 18: "The influence of French mathematics in America" by Miss Holyroyd; Problem discussion by Ivor O. Mall '18; "The solution of the cubic equation" by Andrew M. Harvey '18.

April 8: "History and development of Fermat's last theorem" by Mr. Fehn; "Some noteworthy series for the value of π and their derivation" by Lowell E. Baldwin '18; "Development of imaginary numbers" by Wilbur Lane '19.

April 22: Address by Siebelt L. Simmering, assistant professor of steam and gas engineering; "The cattle problem of Archimedes" by Charles A. Frankenhoff '19; "Two American mathematicians" by Earl V. Kesinger '17.

May 13: "The ancient and the modern treatment of proportion" by Lyle McF. Dean Gr. and assistant; "Mathematics and the science of war" by Jefferson H. Flora '17; Problem discussion by Joseph P. Ball '19.

November 14: "Recent developments in high school mathematics" by Miss Holyroyd.

December 5: "The mathematics of life insurance" by Professor Porter.

January 16, 1917: "The problem of three bodies—history and progress" by Professor Remick; "The construction of logarithmic tables" by Harry Dunham '18.

February 16: "Formulæ for the area of a triangle" by Miss Zeininger; "Some problems in gearing" by Myron R. Bowerman, assistant professor of mechanical drawing and machine design.

March 6: "Elementary geometry of the triangle" by Professor Stratton.

April 10: "Some fundamental ideas in mathematics" by Professor Andrews; "The application of mathematics to civil engineering" by Lowell E. Conrad, professor of civil engineering.

May 8: "Circles of the triangle" by Mr. Fehn; "The relation of mathematics to chemistry" by Herbert H. King, professor of chemistry.

June 5: "Construction of the polygon of seventeen sides" by Professor White; "Graphical solution of the quadratic with complex roots" by Donnelly J. Tarpey '19.

Mathematics Club of the University of Montana, Missoula, Montana.

So far as the editor is aware this club was the first one to be founded in 1918. It resulted from a suggestion of its first president. Last semester anyone interested in mathematics could belong to the club but during the present year definite rules for members, as well as a statement of the purposes of the club, are

to be formulated. There were 22 members (3 of the faculty) last year and the average attendance was 18.

Officers 1917–18: President, Gretchen Van Cleve '19; secretary, Adele Maerdian '20; treasurer, Harry Rooney '21. The program committee consists of the president and secretary.

March 11, 1918: Organization of the club.

March 27: Address by Professor Nels J. Lennes, head of the department of mathematics.

April 10: "What other clubs are doing" by Doris Thetge '21; "Russian peasant method of multiplication and other methods" by Adele Maerdian '20.

April 24: "Method of constructing curves on roads and railroads" by Tom Swearingen '19; "Money value of an education" by Bessie Rutledge '20. Election of officers for 1918–19.

May 15: "Mathematics of war" by James Friauf '18.

May 29: "The slide rule" by Radcliffe Beckwith '21.

"At each one of our meetings we have had a 'feed' (a strictly war 'feed,' I assure you,—nothing unpatriotic about us out here in Montana). The young people seem to take interest and pride in getting up a good 'feed,' and they tax us an amount something like fifteen cents to a quarter, each time, to cover the expense. One Sunday we got four automobiles and drove about thirty miles into the country for an afternoon picnic."

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Ill. [*1918*, 132–134].[1]

March 7, 1918: "The history of the teaching of mathematics in the United States" by Mae Campbell '18; "Some mathematical fallacies" by Helen Maloney '18.

March 28: "Bomb throwing" by Frank D. Danielson '18.

April 18: "Gauss's method of quadratures" [subject of Master of Arts thesis] by Theodore Doll '17.

May 2: "Descriptive Geometry" by Milby R. Hammer.

May 18: Election of officers for 1918–19.

MATHEMATICAL CLUB OF ROCKFORD COLLEGE, Rockford, Ill. [*1918*, 187–188].

February 20, 1918: Debate, "Resolved that one year of mathematics should be required in college" by Dorothy Jamison '20, Virginia Schneider '20 (affirmative), and Lila Dole '20, Doris Volland '20 (negative).

March 6: Social meeting.

April 17: "Women in mathematics" by Estle Russell '18.

April 24: Special business meeting.

May 8: "Social meeting open to those freshmen who intend to go on with mathematics and who, therefore, will be able to join the club next year."

[1] This form of abbreviation will be used in the future to indicate earlier pages of this MONTHLY where other items concerning the club may be found.

THE NEWTONIAN SOCIETY OF THE STATE COLLEGE OF WASHINGTON, Pullman, Wash.

This society was founded as The Mathematical Society in November, 1911, but just five years later "after hearing claims made for more than a score of mathematicians, the society voted to name itself in honor of Newton."

"Its purpose is to afford an opportunity for students to form the habit of reading up on assigned topics which, for the most part, do not come up under regular courses. Broadly speaking, it is the aim of this society to lay the foundation for individual investigation and research. The students get some information and much inspiration from attendance upon these meetings. As a rule, one member of the faculty and one student take part in each meeting."

Any member of the faculty and any student interested in mathematics is eligible for membership. The total number of members last year was 14 and the average attendance at meetings was 10. "There are no fees, dues, or other items of expense, in connection with the society."

Officers 1917–18: President, Dorothy Neff '19; vice-president, Dorothea Sorenson '20; reporter (who looks after publicity in the college paper), Flossie Folsom '19; program committee, Professor Charles A. Isaacs, Florence Evans '20 and Rachel Shuman '18.

October 27, 1916: "The continuum" by Professor Isaacs.

November 10: "Life of Euclid" by Rachel Shuman '18; "Life of Archimedes" by Clarence L. Hix, instructor; "Life of Lagrange" by Professor Elmer C. Colpitts; "Life of Newton" by Marie Weldin '17; "Life of Euler" by Frank M. Bryant, instructor; "Life of Descartes" by Professor Isaacs.

November 24: "Mathematical Literature" by Elsie Worthen '20; "Mathematical books in the college library" by Rachel Shuman '18; "Mathematical periodical literature in the college library" by Professor Colpitts.

December 8: "Arithmetic and geometric progressions" by Corrine Barclay '19; "Convergent and divergent series" by Ina Craig '19; "Hypergeometric series" by Mr. Hix.

January 5, 1917: "Methods of Diophantus" by Edith McBride '19; "Arithmetica" by Mr. Bryant.

January 26: "The determinants in algebra" by Dorothy Neff '19; "The determinants in geometry" by Frank Hamelius '18; "Hessians and Jacobians" by Professor Isaacs.

March 2: "Permutations and combinations" by Blanche Lowary '20; "The theory of probability" by Flossie Folsom '19; "Insurance" by Professor Colpitts.

March 16: "Concurrent lines of a plane triangle" by Florence Evans '20; "Properties of a plane triangle" by Marie Weldin '17; "Circles connected with a plane triangle by Mr. Hix.

April 20: "Projectiles" by Rachel Shuman '18; "Gunnery" by Mr. Bryant.

October 18: "Points on a line" by Professor Isaacs; "Properties of a tetrastigm" by Ina Craig '19.

November 15: "Pythagorean astronomy" by Professor Colpitts; "Singular points on a curve" by Edith McBride '19.

December 13: "The ancient and modern abacus" by Mr. Hix.

January 24, 1918: "Symbolic logic" by Professor Isaacs; "Bertrand Russell" by Florence Evans '20.

February 25: "Some problems in modern geometry" by Mr. Bryant and Dorothy Neff '19.

March 21: "What, if any, mathematics should be required for graduates from secondary schools?" by Professor Isaacs; round table discussion by the whole society.

April 1: "Funny figures" by Amy Kelso '21; "Card tricks" by Mr. Hix.

April 11: "The sine law in plane geometry" by Dorothea Sorenson '20; "Addition and subtraction of logarithms" by Professor Colpitts.

May 13: "Arithmetical prodigies" by Flossie Folsom '19; "The nebular and other hypotheses" by Ina Craig '19.

May 16: "The history of the calculus" by Rachel Shuman '18; "Improper integrals" by Elsie Dallas '18.

"The number of programs for the past year was reduced so as not to interfere with the Red Cross work in which many students were engaged."

TOPICS FOR CLUB PROGRAMS.

14. The Cattle Problem of Archimedes.

During the last ten years of his life Gotthold Ephraim Lessing, the German critic and dramatist, occupied himself almost exclusively with the treasures of the library at Wolfenbüttel, Northern Germany, where he was librarian. The results of these researches were embodied in a series of volumes, *Beiträge zur Geschichte und Literatur*, the first being published at Braunschweig in 1773. In this volume first appeared the Greek epigram[1] (in verse form, 44 lines, from the

[1] Page 421 f. The text is followed by Lessing's commentary, a purported solution in Greek, by a scholiast, and a mathematical discussion by Christian Leiste (numerous misprints and errors in connection with the numbers). The same is to be found in the standard edition of Lessing's works: *Sämmtliche Schriften*, herausgegeben von K. Lachmann, besorgt durch F. Munker, Leipzig, Band 12, 1897, pp. 100–107, 110–115. (In this edition the misprints and errors of the 1773 edition have been corrected)—The Greek text of problem and scholium are also given (with Latin translation of the problem) by J. L. Heiberg, *Archimedes opera omnia*, iterum edidit, Vol. 2, Lipsiæ, MDCCCCXIII, pp. 528–534. (See also commentary in J. L. Heiberg, *Quæstiones Archimedeæ*, Hauniae, MDCCLXXIX, pp. 26f., 66f.)—For German translations see: (1) G. F. Nesselmann, *Die Algebra der Griechen*, Berlin, 1842, pp. 481–491; (2) B. Krumbiegel, "Das problema bovinum des Archimedes," *Zeitschrift für Mathematik und Physik*, hist. literar. Abt., 1880, Vol. 25, pp. 121–136. (This comprehensive discussion of earlier work contains German translations of both epigram and scholium. Krumbiegel's paper is followed, pages 153–171, by A. Amthor's mathematical discussion. The two present a masterly presentation of the facts and are fundamental for every student of the problem.)—French translations by Terquem and A. S. C. Vincent in *Bulletin de bibliographie, d'histoire et de biographie mathématiques*, tome 1, 1855, pp. 113–124, 130, 165–173; tome 2, 1856, pp. 39–42.—Italian translation by G. Loria in *Le scienze esatte nell' antica Grecia*, 2a ed., Milano, 1914, pp. 932–939 (Il "problema dei buoi" di Archimede).

The cattle problem was also carefully studied by:

(*a*) J. Struve and K. L. Struve, Vater und Sohn, *Altes griechisches Epigramm mathematischen*

original[1] discovered by Lessing in the Wolfenbüttel library) headed[2]: "A problem which Archimedes found among (some) epigrams and sent, to be solved by those in Alexandria who occupy themselves with such matters, in his letter to Eratosthenes of Cyrene."

In abbreviated and partly symbolic form the problem is as follows: Compute, O friend, the host of the oxen of the Sun, giving thy mind thereto, if thou hast a share of wisdom, compute the number which once grazed upon the plains of the Sicilian isle Thrinacia,[3] and which were divided according to color into four herds, one milk white, one black, one yellow and one dappled. The number of bulls formed the majority of the animals in each herd and the relations between them were as follows:

1 . . . White bulls (W) $= (1/2 + 1/3)$ black bulls (X) + yellow bulls (Y),
2 . . . Black bulls (X) $= (1/4 + 1/5)$ dappled bulls (Z) + yellow bulls (Y),
3 . . . Dappled bulls (Z) $= (1/6 + 1/7)$ white bulls (W) + yellow bulls (Y),

As to the cows:

4 . . . White cows (w) $= (1/3 + 1/4)$ black herd $(X + x)$,
5 . . . Black cows (x) $= (1/4 + 1/5)$ dappled herd $(Z + z)$,
6 . . . Dappled cows (z) $= (1/5 + 1/6)$ yellow herd $(Y + y)$,
7 . . . Yellow cows (y) $= (1/6 + 1/7)$ white herd $(W + w)$.

Inhalts von Lessing erst einmal zum Drucke befördert, jetgt neu abgedruckt und mathematisch und kritisch behandelt, Altona, 1821 (47 pp.).

(*b*) J. G. Hermann (the German classical scholar and philologist), *De archimedis problemate bovino*, Leipzig, 1828 (12 pp.). (Also in *Godofredi Hermanni Opuscula*, Vol. 4, Lipsiæ, 1831, pp. iii–v, 228–238; contains some notes not in original pamphlet.)

(*c*) P. Tannery: (1) "Sur le problème des boeufs d'Archimède," *Bulletin des sciences mathématiques et astronomiques*, 1881, tome 5, part 1, pp. 25–30 (also in P. Tannery, *Mémoires scientifiques*, tome 1, Toulouse, 1912, pp. 118–123); (2) "L'arithmétique des grecs dans Pappus" (1880), *Mémoires scientifiques*, tome 1, 1912, pp. 103–105.

(*d*) F. O. Hultsch im Pauly-Wissowa's *Real Encyclopädie der Classischen Altertumswissenschaft*, Band 2, Stuttgart, 1896, cols. 531–535, 1110.

Students may find T. L. Heath's somewhat brief discussions more accessible: (1) *The Works of Archimedes*, Cambridge, 1897, pp. xxxiv–xxxv, 319–326. (Also in *Archimedes' Werke mit modernen Bezeichnungen herausgegeben und mit einer Einleitung versehen* von Sir Thomas L. Heath. Deutsch von F. Kleim. Berlin, 1914, pp. 471–477). (2) *Diophantus of Alexandria, A study in the History of Greek Algebra*, 2d edition, Cambridge, 1910, pp. 11, 12, 121–124, 279.

[1] There is another copy of the epigram in the Bibliothèque nationale, cod. Paris Gr. 2448 saec. XIV, fol 57. Heiberg indicates all the small differences of the codices.

[2] In this connection I have chosen the translation which Krumbiegel and Heath regard as literal, though somewhat unsatisfactory in meaning.

[3] The problem seems to hark back to the twelfth book of Homer's *Odyssey* where the following lines occur: "Next, you will reach the island of Thrinacia where in great numbers feed kine and the sturdy flocks of the Sun" (G. H. Palmer's translation, Boston, 1894, pp. 188–189); or in Pope's translation

"Thence to Trincaria's shore you bend your way,
Where graze thy herd, illustrious source of day!"

As to the name τρινακρία for Sicily, and the θρινακίη of Homer and of the epigram, see *Encyclopaedia Britannica*, article "Sicily," and Homer's *Odyssey* edited by W. W. Merry and J. Riddell, 2d ed., Vol. 1, Oxford, 1886, p. 516.

If thou canst give, O friend, the number of bulls and cows in each herd thou art not unknowing nor unskilled in numbers, but still not yet to be counted among the wise.

Consider, however, the following additional relations between the bulls of the Sun:

8 . . . White bulls (W) + black bulls (X) = a square,
9 . . . Dappled bulls (Z) + yellow bulls (Y) = a triangular number.

When thou hast then computed the totals of the herds, O friend, go forth as conqueror, and rest assured that thou art proved most skilled in the science of numbers.

The first seven equations, in eight unknowns, lead to the solution:

$$\begin{aligned} W &= 10366482\,n, & X &= 7460514\,n, & Y &= 4149387\,n, & Z &= 7358060\,n, \\ w &= 7206360\,n, & x &= 4893246\,n, & y &= 5439213\,n, & z &= 3515820\,n, \end{aligned}$$

where n is any integer. The solution given without explanation by the scholiast corresponds to $n = 80$, and the total number of cattle of the Sun 4,031,125,560. These numbers do not, as he affirms, satisfy conditions 8 and 9.

It is generally accepted that condition 8 is equivalent to $W + X =$ a square *number*; but an ambiquity in the language of the epigram in this connection makes possible the strained interpretation that $W + X =$ a square *figure* or rectangle (i. e. the product of two factors), since a bull is longer than it is broad.

With this latter interpretation of condition 8 the problem was completely solved by J. F. Wurm in 1830,[1] and the total number of cattle of the Sun was then found to be 5,916,837,175,686. If it were desirable to distribute these cattle uniformly over all the dry land of the earth, one animal would come about every 25 square feet.

Amthor attacked the problem on the basis of the more generally accepted reading and this called for the solution of the equation

$$t^2 - 4729494\,u^2 = 1 \text{ [2]}$$

After elaborate and arduous work, especially with continued fractions, he arrived at the conclusion that $W = 1598\,\langle\boxed{206541}\rangle$, where $\langle\boxed{206541}\rangle$ represents the fact that there are 206541 more digits to follow, and that, with the same notation the whole number of cattle $= 7766\,\langle\boxed{206541}\rangle$. "It is easy to show that a sphere having the diameter of the milky way, across which light takes ten thousand years to travel, could contain only a part of this great number of animals even if the size of each is that of the smallest bacterium."

There has been much debate as to whether Archimedes really propounded the

[1] Jahn's *Jahrbücher für Philologie und Pädagogik*, Vol. 14, 1830, p. 194 f. Review of Hermann's pamphlet. Amthor gives the solution of Wurm's problem.

[2] This equation has been referred to as a Pellian equation, but there is not the slightest ground for so designating any equation of this form. See *Encyclopédie des sciences mathématiques*, tone 1, Vol. 3, fasc. 1, 1906, p. 27.

cattle problem. In 1880 Krumbiegel gave a pretty complete account of arguments for and against this theory. As a result he concludes: The exact form of the problem is probably later than Archimedes, but as to the problem itself, not only is it very possible, but very probable that it really originated with the celebrated geometer of Syracuse. Such too is the opinion of Tannery, and of Heiberg[1] who is the greatest authority on the texts of Archimedes. Hultsch discusses "a most attractive suggestion" (Heath) that the unmistakable vein of satire in the opening words of the epigram, in the transition from the first to the second parts, and in the last lines, was a shaft directed towards Apollonius.

This problem, originating before the beginning of the Christian era is remarkable in the history of Greek algebra since it is a problem in indeterminate analysis of the second degree, a problem the solution of which is more complicated than that of any in extant works of Diophantus.

K. B. Mollweide (who was born in Wolfenbüttel) is quoted as authority for the statement that his friend Gauss had solved the Cattle Problem completely.[2] If Gauss really obtained the solution he is the only person known to have done so. For, although it has been held very unsafe to claim that Archimedes had at his disposal no method sufficiently powerful to cope with such a question, no one contends (nor was it necessary for giving to Apollonius his quietus in accordance with Hultsch's suggestion) that he actually carried through all the necessary computations.

As final reference to the literature of our topic I note three papers published in America, two by A. H. Bell,[3] and one by Mansfield Merriman[4] to which the student who is unable to read the admirable and comprehensive German memoirs of Krumbiegel and Amthor will naturally turn. Bell's papers set forth the results of nearly four years of computation by himself and two others who constituted the "Hillsboro Mathematical Club" of Hillsboro, Ill. They computed thirty or thirty-one of the left-hand figures and twelve of the right-hand figures for each of the eight unknowns, as well as for the total number of the Cattle of the Sun. Amthor seems to have contented himself with computing the first four of the left-hand figures of (*a*) one of the unknowns, (*b*) the total (although similar values for other unknowns follow readily from his work). The corresponding printed results of Bell and Amthor do not agree.

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[1] *Queæstiones Archimedeæ*, l. c., and *Philologus*, Göttingen, Band 43, 1884, p. 486.

With these authorities Heath too seems to range himself. It is interesting to compare his discussion of the Cattle Problem in the first edition of his *Diophantos of Alexandria*, Cambridge, 1885, pp. 7, 142–147, with that given in the second.

[2] Hermann, *l. c.*, p. 230 of *Opuscula*.

[3] (1) "On the celebrated 'Cattle Problem' of Archimedes," *The Mathematical Magazine*, Washington, January, 1895, Vol. 2, pp. 163–164; (2) "The 'Cattle Problem' by Archimedes 251 B. C.," AMERICAN MATHEMATICAL MONTHLY, May, 1895, Vol. 2, pp. 140–141.

[4] M. Merriman, "The Cattle Problem of Archimedes," *Popular Science Monthly*, 1905, Vol. 67, pp. 660–665.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

Communications for this department should be sent to H. Blumberg, University of Illinois.

SOME DRAWINGS AND GRAPHICAL SOLUTIONS IN NAVIGATION.

By Wm. H. Roever.

1. *Introductory Note.*—During the past summer the U. S. Naval Auxiliary Reserve School, Municipal Pier, Chicago, made arrangements whereby its students were to take a course in navigation at the University of Chicago. As a member of the teaching staff at the University during the summer, I offered my services as instructor in navigation to these students. Professor Wilczynski and I were asked to give the work in nautical astronomy. Most of the students, while very intelligent, had only a limited knowledge of mathematics,—not extending beyond plane trigonometry,—and no knowledge whatever of astronomy. For this reason I at first made frequent appeal to their geometric intuition and solved by graphical processes the spherical triangles which arise in the determination of time, latitude, longitude, azimuth, and in the problem of great-circle sailing. I also drew a number of pictures, illustrating space relations. These pictures and graphical processes proved to be so helpful to my students that it seemed worth while to those who had charge of the course in Navigation at the University of Chicago to have them put in a form whereby they will be available to others. Hence this article. It should not be assumed from what has been said, that I did not also give the customary methods of computation.

2. *Time Determination.*—The determination of time reduces itself to the mathematical problem of finding an angle (the hour angle) of a spherical triangle of which the three sides (complements of the latitude, altitude and declination) are known. To solve graphically such a triangle let us consider the trihedral formed by the planes of the sides of the spherical triangle. Fig. 1 represents such

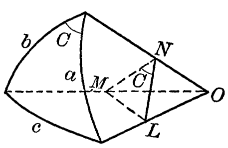

Fig. 1.

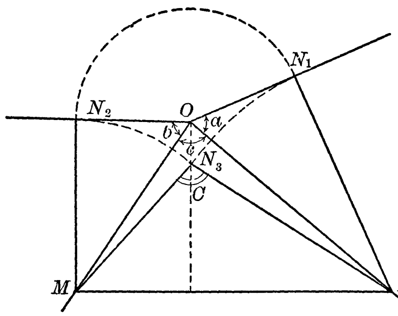

Fig. 2.

a trihedral in which a, b, c denote the sides of the spherical triangle or the face angles of the trihedral, and C denotes the required angle of the spherical triangle

or the corresponding dihedral angle of the trihedral. The point O represents the center of the sphere or the vertex of the trihedral, and N represents any point on the edge of the required dihedral angle. At N perpendiculars are erected to the edge ON in the faces which meet in ON. These perpendiculars meet the other edges of the trihedral in the points L and M. Then, in the triangle represented by LMN, the angle at N is equal to the required angle C of the spherical triangle. To find C graphically, let us cut the trihedral along the edge ON and open it up, or develop it, as shown in Fig. 2. Then N will assume the two positions N_1 and N_2, which lie at equal distances from O on the two positions assumed by the edge ON. The perpendiculars erected to these positions at the points N_1 and N_2, respectively, meet the new positions of the other edges in L and M, respectively. Hence ML, LN_1 and MN_2 in Fig. 2 are the true lengths of the sides of the triangle

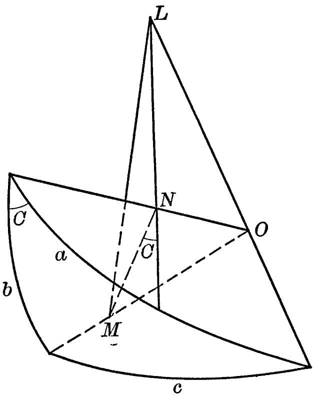

FIG. 3.

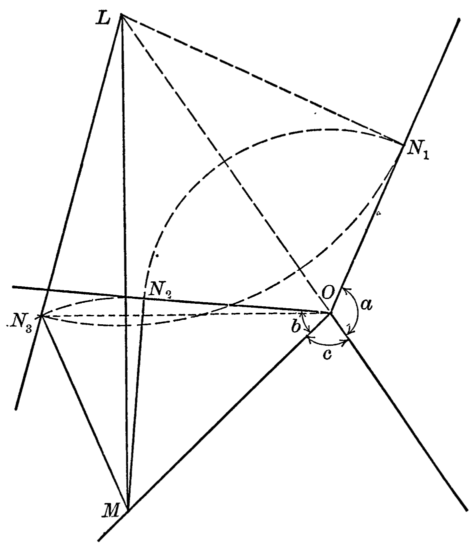

FIG. 4.

represented by LMN in Fig. 1. Thus we are led to the *Construction:* With O (Fig. 2) as vertex lay off the angle c equal to the side of the spherical triangle which lies opposite to the required angle C. Adjacent to the angle c and on either side of it lay off the angles a and b equal to the other two given sides of the spherical triangle. Then on the terminal lines of these last two angles take points N_1 and N_2 at equal distances from O. At these points erect perpendiculars N_1L and N_2M to these sides, meeting the sides of the angle c in the points L and M, respectively. With L as center and LN_1 as radius draw a circle, and with M as center and MN_2 as radius draw another circle. These two circles intersect in a point N_3, which, together with the points L and M forms the triangle represented by LMN in Fig. 1, and hence the angle LN_3M of Fig. 2 is the required angle C of the given spherical triangle. A check on the construction is furnished by the fact that the points N_3 and O must lie on the same perpendicular to the line LM.

In navigation, the sides b and c (complements of the latitude and altitude, respectively) are always acute, but the side a (complement of the declination) may be acute or obtuse. For the case in which the side a is obtuse the point L will lie on the produced edge, as shown in Fig. 3, and then the required angle C is the supplement of the angle MNL of the triangle MLN. The corresponding construction is shown in Fig. 4.

If the sides a and b of the given spherical triangle are nearly 90°, the points L and M of the constructions shown in Figs. 2 and 4, will fall beyond the limits of the drawings. In this case the construction explained in § 4 can be used to advantage.

3. *Solution of the triangle in the Saint Hilaire Method.*—To find the latitude and longitude of a ship by the Sumner Method, two *lines of position* must be determined. The Saint Hilaire Method for finding such lines, involves a spherical triangle of which two sides and the included angle are known and of which the side opposite the given angle is required.[1] To solve graphically this spherical triangle, let us consider Fig. 5, in which the edge of the given dihedral angle C

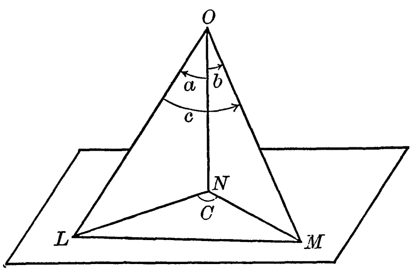

FIG. 5.

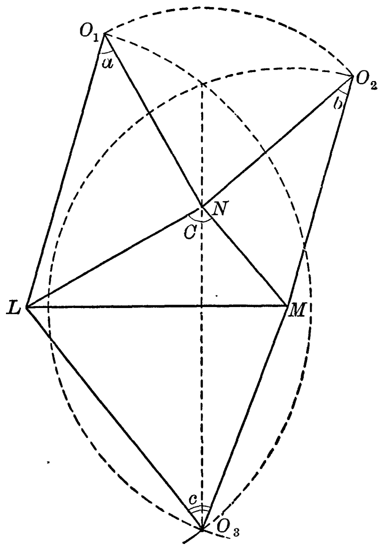

FIG. 6.

of the corresponding trihedral is represented as being in a vertical position, *i. e.*, perpendicular to the plane (let us say) of a table top. The points in which the edges of the trihedral meet this plane are represented by the points N, L, and M, of which the first lies on the edge of the given dihedral angle C. The center of the sphere, or the vertex of the trihedral, is represented by the point O. Let us now think of this trihedral as cut along its three edges ON, OM, OL and

[1] This side is the computed zenith distance of the observed object. The azimuth is usually obtained from an azimuth table, but it can easily be found graphically as shown in the *Remark* at the nd of § 4.

of its faces as turned down around the lines NM, ML, LN into the plane of the table top. They then assume the position shown in Fig. 6. Hence the *Construction:* With N (Fig. 6) as vertex lay off the angle C equal to the given angle of the spherical triangle. At N erect perpendiculars to the sides of this angle and on these take points O_1 and O_2 at equal distances from N. Through O_1 and O_2 draw lines making with O_1N and O_2N the angles a and b, respectively, equal to the given sides of the spherical triangle, and meeting the sides of the angle C in the points L and M, respectively. Then with L as center and LO_1 as radius draw a circle, and with M as center and MO_2 as radius draw another circle. These circles meet in a point O_3, which with L and M forms the triangle represented in

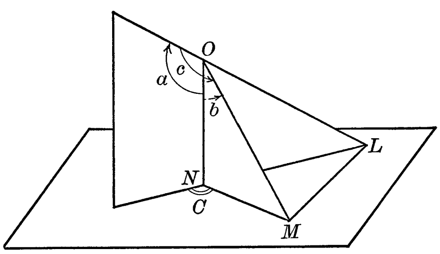

FIG. 7.

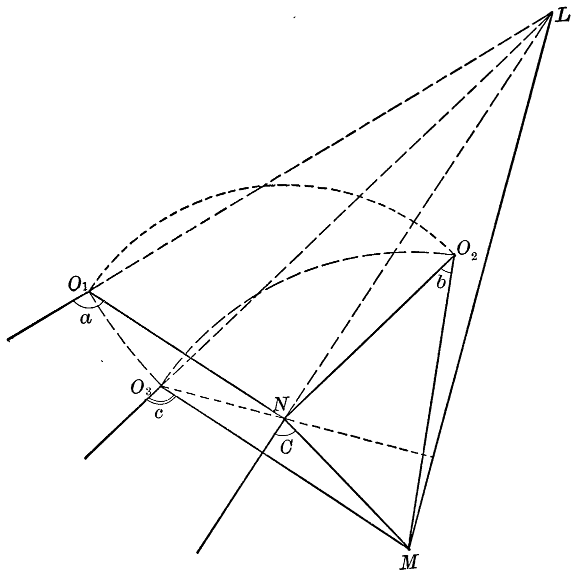

FIG. 8.

Fig. 5 by OLM and of which the angle at O is equal to the required side c of the spherical triangle. Hence the angle LO_3M of Fig. 6 is the required angle c of the spherical triangle. A check is furnished by the fact that O_3 and N should lie on the same perpendicular to LM.

In the navigation problem the side b (complement of the latitude) is always acute, but the parts C and a may be acute or obtuse. In the case where the part a is obtuse the point L lies on the produced edge, as shown in Fig. 7, and then the required part c is the supplement of the angle LOM in the triangle LOM. The corresponding construction is shown in Fig. 8.

If the parts a and b are nearly 90° the points L and M of Figs. 6 and 8 will fall beyond the limits of the drawings, and in this case the construction of § 4 can be used.

4. *Constructions available in failing cases of the above.*—Constructions which have the advantage over the constructions just given, in that all the points re-

main within the limits of the drawings, will now be given. These constructions are perfectly general and may be used whenever the given parts of the spherical triangle are less then 180°. These constructions will now be given, but for their proof the reader is referred to *Loria, Vorlesungen ueber Darstellende Geometrie,* Vol. II, p. 6.

Construction (Fig. 9 for a acute, Fig. 10 for a obtuse).—The three parts a, b, c (*i. e.*, the sides of the spherical triangle or the face angles of the corresponding trihedral) are laid off from a point O as shown in Figs. 9 and 10. A circle Γ of center O and any convenient radius is then drawn cutting the sides of the angles a, b, c in the points $\bar{C}$, $\bar{A}$, $\bar{B}_1$ and $\bar{B}_2$ as shown in the figures. From the points $\bar{B}_1$ and $\bar{B}_2$ perpendiculars are dropped to the rays $O\bar{C}$ and $O\bar{A}$, respectively,

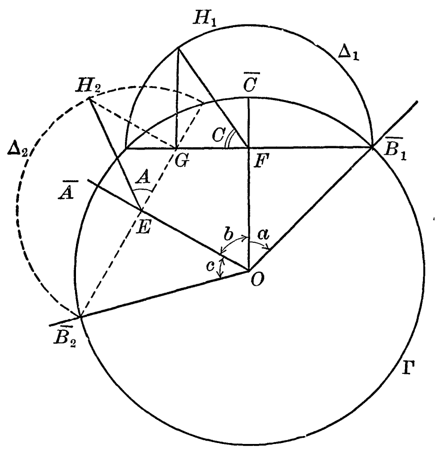

FIG. 9

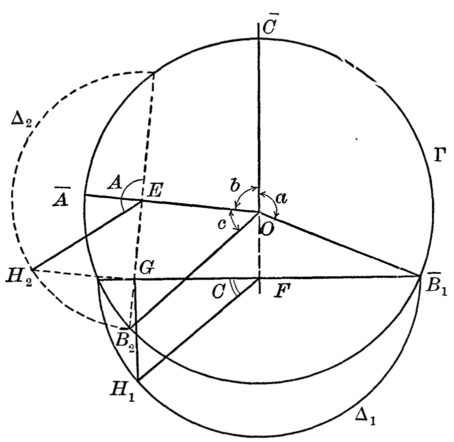

FIG. 10.

cutting these in the points F and E, respectively, and each other in the point G. With F as center and $F\bar{B}_1$ as radius a circle Δ_1 is drawn. At G a perpendicular is erected to the line $\bar{B}_1F$ cutting the circle Δ_1 in H_1. Then the supplement of the angle $\bar{B}_1FH_1$ is the required angle C of the spherical triangle of which the sides are the angles a, b, c.

Construction (Fig. 9 for a acute, Fig. 10 for a obtuse).—If a, b, C are the given parts of the spherical triangle, the circle Γ, the perpendicular $\bar{B}_1F$ and the circle Δ_1 are drawn as in the above preceding construction. The point H_1 is now taken on the circle Δ_1, so that the angle $\bar{B}_1FH_1$ is the supplement of the given angle C. From H_1 a perpendicular is dropped to $F\bar{B}_1$, meeting it in G. From G a perpendicular is dropped to $O\bar{A}$ meeting it in E, and meeting the circle Γ in $\bar{B}_2$. The angle $\bar{A}O\bar{B}_2$ is then equal to the required side c of this spherical triangle.

Remark.—In order to find the angle A (which is the azimuth in the nautical problem), we have merely to draw the circle Δ_2 of center E and radius $E\bar{B}_2$. At G we then erect a perpendicular to $\bar{B}_2E$ cutting Δ_2 in H_2. The supplement of the angle $\bar{B}_2EH_2$ is then equal to the required angle A.

5. *The great-circle-sailing problem.*—The problem of great-circle sailing consists in determining the latitude and longitude of various points Q of the great circle which connects two given points A and B on the surface of the earth. The points Q are then plotted on a Mercator chart[1] from which the course of the ship between consecutive points is determined. To devise a graphical solution of this problem let us first think of the points A and B as both situated in the same hemisphere (Fig. 11).

Now think of a plane tangent to the earth at its pole O' and upon this plane let us project the points A and B from the center O of the earth, denoting the projections by A' and B', respectively. Then the straight line $A'B'$ of this plane is the projection of the great circle path AB, and the lines $O'A'$ and $O'B'$ are the

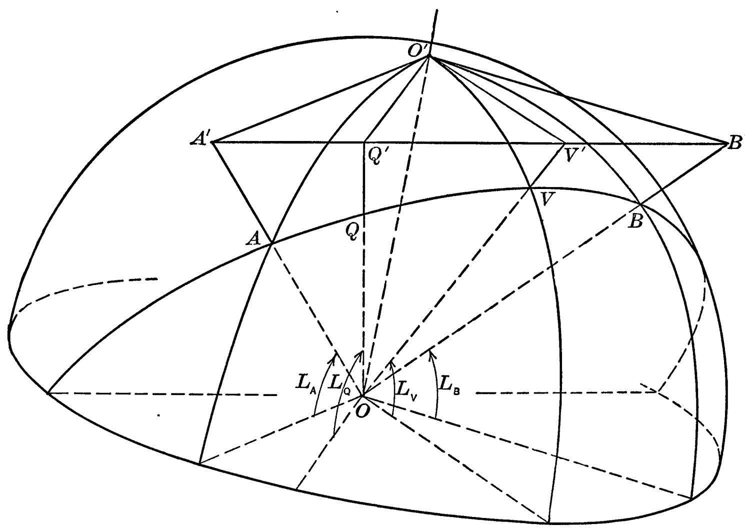

FIG. 11.

projections of the meridians of A and B. Let V denote the point of the great circle AB which is nearest the pole O'. Evidently the projection of the point V is the foot V' of the perpendicular from O' to $A'B'$ in the triangle $O'A'B'$. A general point Q of the great circle AB projects into a point Q' of the line $A'B'$. The tetrahedron $OA'B'O'$ represented in Fig. 11 is the inverted position of the tetrahedron $OLMN$ represented in Fig. 5, so that the parts a, b, C of Fig. 5 correspond respectively to the co-latitude ($90°$-L_A) of A, the co-latitude ($90°$-L_B) of B and the difference in longitude (equal to the angle $B'O'A'$) of the points A and B. Hence the part c of Fig. 5 corresponds to the angular distance AOB of Fig. 11. Thus the construction of Fig. 6 determines graphically (in that it determines the part c) the angular distance between the two points A and B when the latitudes and the difference in longitudes of these points are known. In order to find the latitude, longitude and distance from A (or from B) of the

[1] For the definition of Mercator projection see § 7.

point V and of the general point Q, we will begin by repeating the construction of Fig. 6. In Fig. 12 this much is shown by the heavily drawn lines; the points O', A', B', O_A, O_B, O''' of Fig. 12 correspond to the points N, L, M, O_1, O_2, O_3, respectively, of Fig. 6. In Fig. 12 the triangle $O'A'B'$ is the true shape of the triangle which is shown in Fig. 11 by the same letters. Hence in Fig. 12 the foot of the perpendicular from O' to $A'B'$ is the point V', and any point of $A'B'$ is a point Q'. The angles $A'O'''V'$ and $A'O'''Q'$ are then the angular distances of the points V and Q, respectively, from the point A. Evidently the angle $V'O'A'$

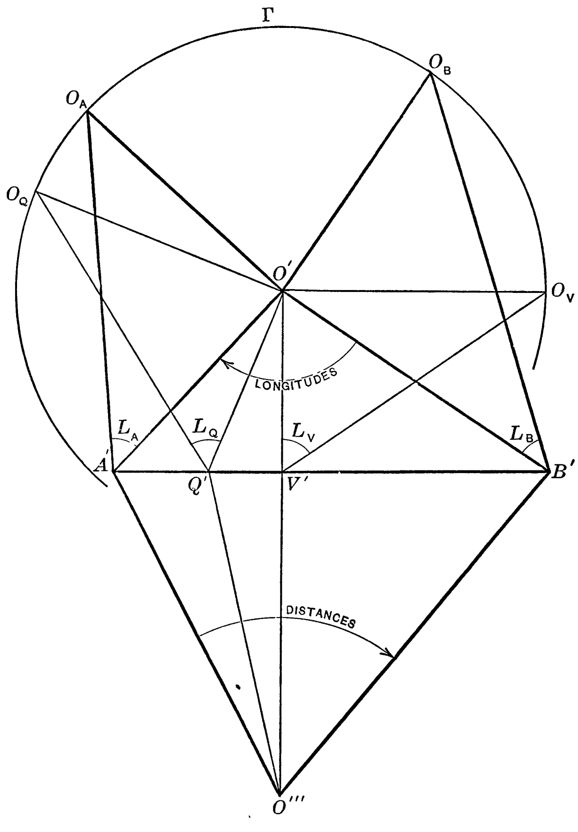

FIG. 12.

(Fig. 12) is the difference in longitude between the points V and A, and the angle $V'O'Q'$ is the difference in longitude between V and Q. To find the latitudes of V and Q let us first draw a circle Γ of center O' and radius $O'O_A = O'O_B$ (equal to the distance represented by OO' in Fig. 11). To get the latitude of V draw in Fig. 12 a line through O' perpendicular to $O'V'$ and cutting Γ in O_v. Then the angle $O'V'O_v$ is the latitude of V. Similarly, to get the latitude of Q draw a line through O' perpendicular to $O'Q'$ and cutting Γ in O_Q. The angle $O'Q'O_Q$ is then the latitude of Q. This is evident from the fact that the triangles $O'O_vV'$ and

$O'O_QQ'$ of Fig. 12 are the revolved positions (turned into the plane of the triangle $O'A'B'$) of the triangles represented by $O'V'O$ and $O'Q'O$ in Fig. 11.

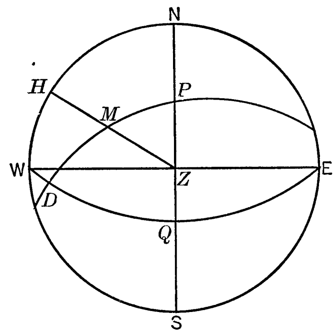

Fig. 13.

Just as we started with Fig. 6 the construction of Fig. 12 corresponding to the case where A and B are in the same hemisphere, so for the case where A and $B$are in different hemispheres we start the construction with Fig. 8.

6. *Definition of a good picture.*—In the absence of models, good pictures serve very well to convey an adequate notion of space relations. By a good picture of a space object is meant a plane representative which, when properly placed, produces upon the retinal surface of the eye an image which does not differ much from that produced by the object itself in a position in which one is accustomed to see or think of the object. It can easily be shown that a central

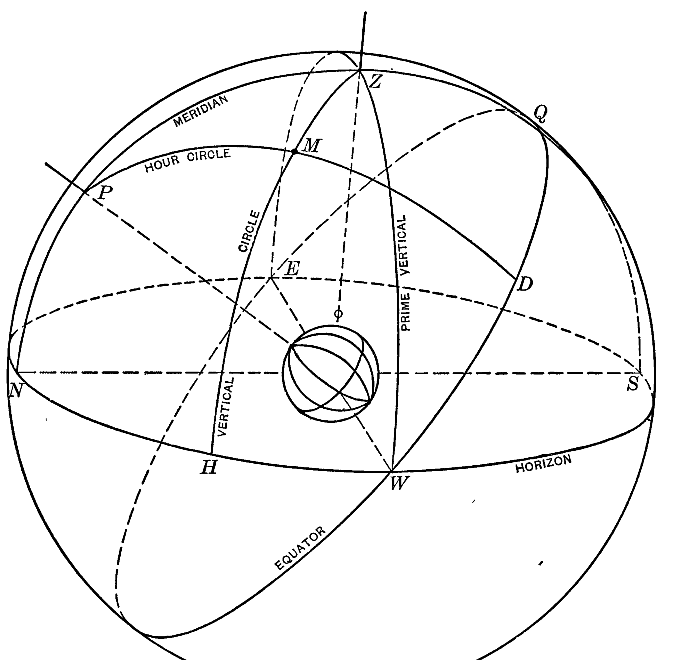

Fig. 14.

or parallel (orthographic or oblique) projection in general satisfies this criterion[1] and therefore such a projection of an object in an accustomed position furnishes a

[1] See for instance, pages 148 and 152 of the article on Descriptive Geometry in Vol. 25 of this Monthly.

good picture. What is here meant by an accustomed position will be made clear from the statements which follow. A stereographic projection of a sphere is a central projection for a particular position of the center of projection.[1] However, the stereographic projection of the celestial sphere with its circles of reference does

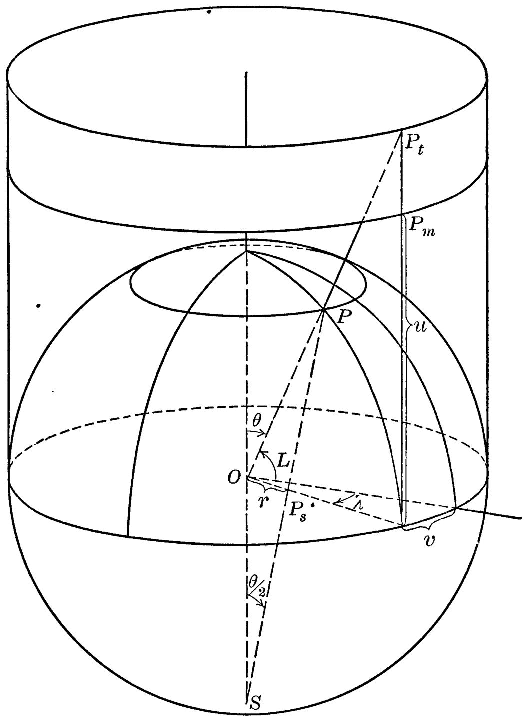

FIG. 15.

not produce nearly so good a picture as does an orthographic projection on a general plane. This will be evident to the reader when a comparison is made of Figs. 13 and 14 which are stereographic and orthographic projections, respectively. Hence a stereographic projection, although it is a projection which satisfies the criterion of producing the same retinal image as the object itself, does not furnish a good picture, because it is projected from a point from which one is not accustomed to seeing the sphere.

7. *Stereographic and Mercator projections defined.*—We have already used the terms Mercator projection and stereographic projection. For the purpose of defining these, as well as showing a simple relation which exists between them, we will make use of the foregoing picture (Fig. 15) which is an orthographic projection.

The stereographic projection of a sphere is the central projection of the points of the surface of the sphere upon a diametral plane from a pole of the great circle

[1] For the definition of stereographic projection see § 7.

in which this plane cuts the sphere. In Fig. 15 the plane of projection is the equatorial plane, and the center of projection is the south pole. The projection of the point P of the sphere is the point P_s of the equatorial plane.

The Mercator projection of a sphere is the development of a right circular cylinder, tangent to the sphere along the equator, the points of which are obtained from those of the sphere in the following manner. The point P of the sphere is projected from the center O into a point P_t of the cylinder (Fig. 15). Upon the element of the cylinder which passes through the point P_t the Mercator projection P_m of the point P lies, and its distance u from the equator is equal to the Naperian logarithm of the reciprocal ($1/r$) of the distance (r) of the stereographic projection P_s from the center O of the sphere, the radius of which is taken to be unity.

Thus if of a point P of the sphere, L = latitude and λ = longitude, and θ is the co-latitude, so that

$$\theta = \frac{\pi}{2} - L \text{ and hence } \theta/2 = \pi/4 - L/2,$$

and if for the stereographic projection P_s the polar coördinates are r and λ, and for the Mercator projection the rectangular coördinates are u and v, then

$$\begin{cases} r = \tan\dfrac{\theta}{2} = \tan\left(\dfrac{\pi}{4} - \dfrac{L}{2}\right), \\ \lambda = \lambda, \end{cases}$$

and

$$\begin{cases} u = \log_e \dfrac{1}{r} = -\log_e \tan\dfrac{\theta}{2} = -\log_e \tan\left(\dfrac{\pi}{4} - \dfrac{L}{2}\right), \\ v = \lambda. \end{cases}$$

Thus one easily sees the simple relation which exists between these two types of map.

8. *Mean sun and equation of time defined.*—In the definitions of mean time and the equation of time, a good picture showing the relation which exists between the mean and apparent suns is very helpful. In the orthographic projection shown in Fig. 16 the celestial sphere with the equinoctial and ecliptic are represented. In this figure, the vernal equinox is represented by the point r, the position which the true sun occupies at perihelion (which occurs about January first) is represented by S_0, and the position which the true sun occupies at any particular instant (the position shown in the figure corresponds to about February 12) is represented by S. Before defining the mean sun, let us first define a fictitious sun which moves in the ecliptic at a uniform rate in the same direction as the true sun does and coincides with the true sun at perihelion (and therefore also at aphelion). The point F represents the position which this sun occupies when the true sun is at S. The *mean* sun may now be defined as the point which moves along the equinoctial at a uniform rate and coincides with the fictitious sun at the vernal equinox (and therefore also at the autumnal equinox).

The position which the mean sun occupies when the true sun is at S is represented by M. In other words, the right ascension of the mean sun M is equal to the mean celestial longitude of the fictitious sun F, these two artificial suns being in coincidence at the vernal equinox. Fig. 16 also represents the center O and the north pole P of the celestial sphere and the hour circles of S, F and M. The hour circles of S and F intersect the equinoctial in the points

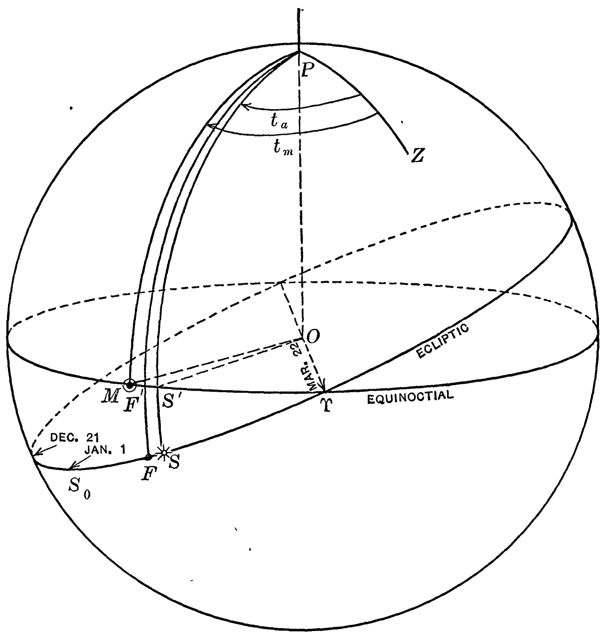

FIG. 16.

S' and F', respectively. The difference in right ascension between S and M, *i. e.*, the angle MOS' is called the equation of time. It is the difference between mean and apparent solar times. Thus if Z represents the zenith of a particular place of the earth at a particular instant,

$$t_m, = \text{angle } ZPM, \text{ is the local mean time of this place,}$$
$$t_a, = \text{angle } ZPS, \text{ is the local apparent solar time of this place,}$$

and

$$\epsilon, = t_m - t_a = \text{angle } MOS', \text{ is the equation of time.}^1$$

9. *The Sumner Method.*—In describing the Sumner Method of determining the position of a ship at sea, a picture representing both the celestial and the terrestrial spheres proves to be very helpful. In Fig. 17 such a picture is represented in orthographic projection. The positions on the celestial sphere, of the pole, zenith and two positions of the sun are represented by P, Z, S_1, S_2, respectively. The positions on the terrestrial sphere, of the points immediately below

[1] It should not be forgotten that hour angle (time) and right ascension are measured in opposite directions.

these, are denoted by p, z, s_1, s_2, respectively. Then z is the position of the ship. and s_1, s_2 are called the subsolar (or in the case of a star, the substellar) points, The meridian of Greenwich is shown on both of the spheres as is also the equator. It is thus evident that the terrestrial longitude of a subsolar point is equal to the Greenwich apparent time and that the latitude is equal to the declination of the true sun. A small circle of the terrestrial sphere which passes through the position of the ship (z) and has as pole the subsolar point is called a Sumner circle. It is

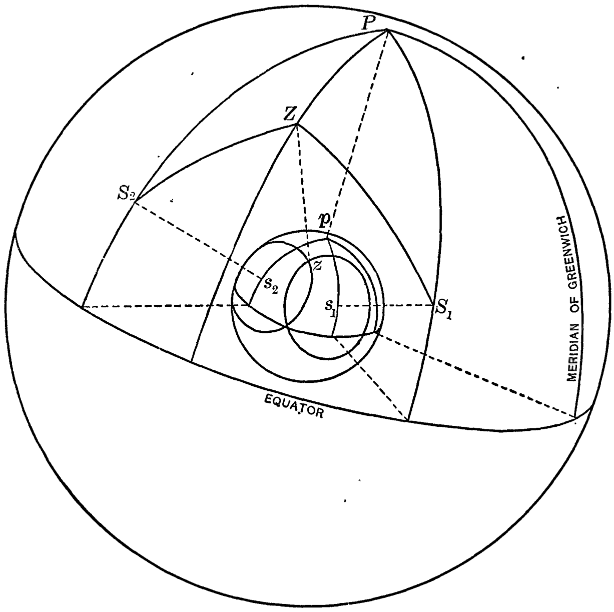

FIG. 17.

easy to see from the figure that the angular radius of this circle is equal to the zenith distance of the sun. The figure shows the two Sumner circles which correspond to S_1 and S_2 and which intersect in the position z of the ship.

10. *The Gnomonic Chart.*—By a gnomonic chart is meant the projection of the points of a spherical surface upon a tangent plane to this surface from the center of the sphere. A straight line of such a chart corresponds to a great circle of the sphere. Such charts are used in solving graphically the problem of great-circle sailing, of which one graphical solution has already been given in § 5. In fact the projection $O'A'B'$ (Fig. 11) on the plane tangent to the sphere at the point O' is a gnomonic chart of the triangle $O'AB$ of the sphere for the case where the plane of projection is tangent to the sphere at the pole O'. In this case the meridians project into straight lines radiating from the point of tangency and the parallels of latitude project into concentric circles whose common center is the point of tangency. If, however, the plane of projection is tangent to the sphere at a point not coincident with a pole, the meridians project into straight lines which

radiate from the point where the axis of the earth pierces the plane of projection, and the parallels of latitude project into conics. This is the case because the planes of the meridians all pass through the axis, which contains the center of projection, and because the projecting rays of a parallel of latitude form a right circular cone. Fig. 18 enables one to see this more clearly. If on such a chart a sufficient number of meridians and parallels of latitude are represented, it is an easy matter to read off from the chart the latitude and longitude of various points of the straight line representative of a great-circle path of the sphere. The method of doing this as well as that of finding the distance and course (direction of sailing) is explained on the great-circle charts which are published by the U. S. Hydrographic Office.

The construction of a gnomonic chart is identical with that of a horizontal sun-dial for a place whose latitude is equal to that of the point of tangency of the plane of projection of this chart. This can be easily seen by the aid of Fig. 18, in which we may now consider the horizontal table top as the dial plate and the axis of the sphere as the gnomon, or style, of the dial. The lines representing the meridians are then positions of the shadow of the style upon the dial plate, and the conics representing the parallels of latitude are the diurnal paths of the

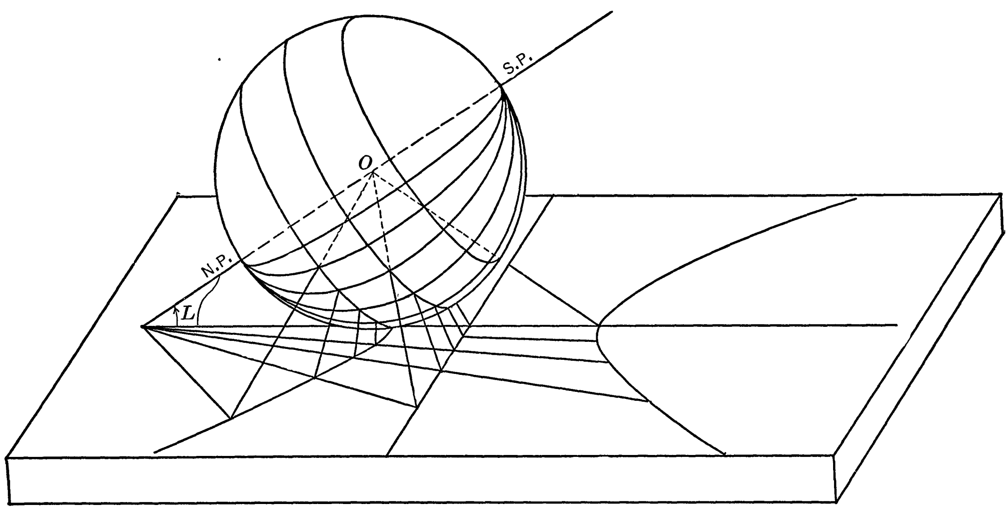

FIG. 18.

shadow of a knob (situated at the point O) of the style on the dial plate. In the dialing problem it is required to find the position of the shadow of the style for the hours of the day. The diurnal paths of the knob are the conics lying between the conics representing the Tropics of Capricorn and Cancer. It is on these conics that the equation of time is laid off from the position of the shadow of the style at apparent solar noon in order to make possible the use of a sundial for the determination of mean time.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

To the list of graduate students who have been granted the Doctorate in mathematics at American Universities since June, 1917, as printed in the September number of this MONTHLY, should be added the following: H. E. BRAY, Rice Institute, "A Green's theorem in terms of Lesbesgue's integral"; TERESA COHEN, Johns Hopkins, "An investigation of plane quartic curves"; ANNA M. HOWE, Cornell, "The classification of plane involutions of order three"; L. J. ROUSE, Michigan, "A contribution to the question of linear dependence in linear integral equations."

After the present issue of this journal was in type the EDITOR received from Professor E. R. HEDRICK, Chairman of the Association's Dictionary Committee, a few remarks concerning the article of Professor MILLER on pages 383 to 387 of this issue (seen by Professor HEDRICK in galley proof). These remarks are as follows: "While the committee on a Mathematical Dictionary has taken no steps concerning Professor Miller's article and while the material was not submitted to it for approval, the chairman of the committee feels sure that the committee would unquestionably sanction a statement approving the style and the nature of the contents of Professor Miller's article. The only possible question that suggests itself is concerning the length to be assigned to each department in order to keep the dictionary as a whole within reasonable limits. No final decision has been arrived at concerning this matter. Sample statements of this character should be of greatest assistance to the committee and it is suggested that others follow Professor Miller's example in submitting definitions."

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THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual members and over seventy-five institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary at Oberlin, Ohio.

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THE CONTENT OF A SECOND COURSE IN CALCULUS.[1]

By E. J. MOULTON, Northwestern University.

The second course in calculus, as presented in different colleges and by different instructors, varies considerably in content. In view of its position on the one hand as often the culmination of the study of mathematical analysis both for students specializing in science and engineering and for students who will be teachers of mathematics in our secondary schools, and on the other hand as a necessary prerequisite for much of the later work in mathematics, it merits careful consideration. A general discussion of the course at a meeting like this should prove stimulating to all of us who have been faced with its difficulties; and it is my desire to learn the views of other teachers as much as to argue my own views that has led me to speak on this subject to-day.

Before considering the content of the course we should think of what its general aim should be. This general aim in turn depends upon the students who are expected to take the course, their preparation and their requirements. As to preparation, the minimum may be assumed to consist of courses in trigonometry, college algebra, analytical geometry and a three-hour year course in calculus. This is doubtless quite generally exceeded, but closely approximates the most common actual prerequisites for the second calculus course; and it is not desirable to assume a much better preparation. In my own classes, for instance, many of the students have had, or are taking, a course in mechanics, but I find it impossible to assume that course as a part of the student's preparation. At best the student's outlook on the field of mathematics is very restricted, and even his working knowledge of calculus is limited to the very simplest portions.

[1] This paper is a revision of one presented to the Mathematical Association of America at the meeting in Chicago on December 27, 1917.

As to the student's requirements, these are best discussed by dividing the students into three groups: first, those who are specializing in science or engineering; second, those for whom this is the last course in analysis and who will presumably have no direct use for the calculus in their subsequent work; and third, those for whom this is a preparation for more advanced mathematics.

For the first group the ability to use mathematics easily and vigorously is clearly the great need. Mathematics for them is primarily a tool, and secondarily a mental discipline; and the general utility of any topic might well be the basis for deciding whether it should be given. For them it is less important to give painstaking proofs of important theorems than to emphasize the importance of those theorems by using them in the solutions of problems which they recognize as practical problems or applications of mathematics. And we may furthermore suggest that the mental discipline is as great in a careful use of a theorem as in the proof of it.

In the second group are prospective teachers and students who may be described as taking mathematics for its cultural value. They are, in general, students who are making mathematics their major subject in college, and will accordingly take other courses of about the same grade, probably in projective geometry or the theory of equations. In these latter subjects mathematics as a systematic, logical development of certain lines of thought is well illustrated, but mathematics as a tool for the discussion of the physical world is largely, if not completely, neglected. It is important that these courses be given in this way; but it is also important that the student shall appreciate the great utility of mathematics. The topics which one associates with the calculus are, at least in part, well adapted to be given so as to show this utility, and accordingly it seems that this aspect of the subject should receive a considerable emphasis. Moreover, the students of the second group require less formal work than those of the first group; they need instead to cover a wide range of topics with only sufficient formal drill to be able to follow the course readily.

Concerning the third group, composed of future mathematicians, there will doubtless be a diversity of views as to their needs. My own feeling is that it is too early in the student's career to introduce the critical attitude of higher mathematics, that it is far better at this stage of the student's development to develop his technique in handling the great algorisms of calculus, and at the same time, as far as possible, to give him a broad outlook on the great classical problems of analysis. The student needs a background before entering on any detailed study of the fundamental properties of continuous functions, for example, and it is especially within the province of this second course in calculus to provide that background. A study of those fundamental properties should follow this course instead of being a part of it. Moreover, it seems to me important that the student should at this time get some notion of a considerable number of topics rather than go more extensively into some one subject, like differential equations, for instance. The student will presumably go more deeply into each of the topics later, and the advantage of having some preliminary notion of the subject in each case and of

a number of related subjects should much more than make up for the time apparently lost in repetitions.

It seems, then, that we have three things that we should aim to do: first, we should aim to develop the student's technique in using with some freedom as many as possible of the great algorisms we naturally associate with the calculus; second, we should aim to present as far as possible problems which the student will recognize as applications of mathematics; and third, we should aim to give as broad an outlook as possible on the classical problems of analysis.

Granting that these should be our aims, there arise a number of questions as to methods of procedure and questions as to what should be the actual topics taken up. First of all, it is obviously undesirable to try to build up our calculus from a few simple postulates in a manner satisfactory from the point of view of formal logic, for time would not permit, even if we assumed that the student could appreciate such a method. On the contrary, it is desirable to make a great number of assumptions, subject of course to the condition that they be consistent, and further that the student will agree to their validity intuitionally.[1] The more we reflect on the fundamental postulates and elementary theorems involved in questions in limits and continuity, the more important it appears that the specialist in mathematics should eventually go into these questions deeply, but also the more important it seems that in the earlier stages of his training he shall make a free use of his geometric intuitions in such questions under the guidance of a more sophisticated intuition. I would therefore urge that we use our geometric intuitions with some freedom in this course, the instructor holding himself responsible for their correctness.

Furthermore is it wise to insist on giving proofs of all, or even of approximately all, of the theorems we may wish to state, even when those theorems do not appear intuitional to the student? The fact that mathematics is built up as a matter of logical proof from a set of axioms gives the subject a special claim for our attention, but is it advisable to require our students to go through the proofs of all, or of approximately all, of the theorems they may ever want to use?

My belief is that we would do better if we took greater freedom in stating theorems without proof, and spent more time in giving our students an idea of the topic under discussion in its entirety, incidentally requiring them to build higher through a more frequent acceptance of the dicta of others. This involves no necessary loss in the training of the student's reasoning powers, for the proofs that are given need be given no less carefully. The student may feel the lack of the completeness which is one of the charms of mathematics, and this would be serious if it came too early in his course, but it seems to me that the gains far exceed the losses if this policy is adopted in the course we are discussing.

As to the standard of rigor to be maintained in the proofs which are given, it is clear, with our expressed aims, that there should not be any radical change from the standards with which the student is already familiar. It is more

[1] See E. H. Moore: "On the Foundations of Mathematics," *Bulletin of the American Mathematical Society*, Vol. 9, 1902–1903, for remarks on the use of our intuition in elementary mathematics.

important to give arguments in a form which will be readily understood by the student and which are essentially complete and exact, even if not perfectly satisfactory from the point of view of formal logic, than to lay more stress on the logical minutiæ of the proofs. Geometrical intuitional arguments which are essentially conclusive should be used with freedom whenever they will aid in clearness of presentation. But arguments which involve actual misstatements, or which the best students may easily see to be illogical or fundamentally incomplete, should of course be avoided. Furthermore, carelessness in such things as dividing by zero, neglecting one of the square roots of a quantity, or assuming the converse of a proposition should not be permitted. The last error particularly is to be avoided, for this is more than a mathematical error—it is an element of daily reasoning which it is perhaps as important for the student to appreciate as any of the mathematical theory which he will learn.

Another thing which should perhaps be mentioned is the importance to be attached to problems in the course, and the character of the problems. Professor Osgood has said:[1] "The process by which the youth actually acquires the ideas of the calculus is to a large extent and essentially through formal work of substantial character. In order to attain this end however the formal work must appear to him as having for its direct object the power to solve some of the real problems of pure and applied mathematics, and those problems must be kept before his eye." My own experience as student and teacher bears out these statements. In the presentation of the course the solution of problems should constitute a vital part, a superficially more important part than the development of the theory. In this way the student's technique is developed, he obtains an appreciation of the utility of the subject, and he may be led to a consideration of many of the great old problems.

It is my purpose to take up in a moment an explicit list of topics suitable for the course under consideration, chosen in an attempt to meet the aims we have stated. Owing to differences in the first course as given at different times and places it is found necessary to include some topics which are often adequately treated in the first course in the calculus. One would naturally wish to avoid repetition except in the case of the most fundamental notions, where a review is worth while, but these topics are included so that they will not be overlooked in both courses in calculus. Recalling that we assume as a prerequisite only a three-hour year course in calculus, we see why there will be a considerable overlapping of topics with those often considered in a first course or in other courses of about the same grade.

The topics will be taken up in the order in which they might well be given to the class, dealing with partially known and simpler topics first, and leading to new and rather more difficult topics last,—following this out as far as seems compatible with a natural desire to associate closely related topics.

As an introductory chapter we need a brief review of the ideas of function,

[1] In his presidential address to the American Mathematical Society, "The Calculus in Our Colleges and Technical Schools," April 27, 1907; see the *Bulletin of the American Mathematical Society*, Vol. 13, 1906–1907, pp. 449–467.

limit, continuity, derivative, and integral. It is more important here to call attention again to the fundamental importance of the idea of function in applications of mathematics than to attempt any refinements in the definitions of limits and continuity. It is desirable to include a classification of the elementary functions, a discussion of their discontinuities, and a working test for locating discontinuities of a function of a function.[1] By introducing hyperbolic functions here we can both make it possible to use these important functions later and at the same time provide some new material for practice in the old methods of differentiation and integration.

A chapter on the simplest types of differential equations of the first and second orders can be given as a simple application of differentiation and integration. A discussion of families of curves, orthogonal trajectories, and the solution of numerous problems in geometry and mechanics will show the utility of the subject to the student. The importance and significance of the constants of integration should receive emphasis as well as the formal solution of equations.

This may be followed by a chapter leading up to and including Taylor's Theorem with Remainder. Starting from Rolle's Theorem and the simpler Law of the Mean as geometrically evident, we may prove the extension of the latter theorem to the quotient of two functions, and establish l'Hospital's Rule for the evaluation of indeterminate forms.[2] These theorems may be applied to the geometrical problem of contact of plane curves and to the problem of finding polynomial approximations to functions. This leads directly to Taylor's Theorem with Remainder, which is illustrated with special functions and is applied in computations and applications to extremes and points of inflection.

The consideration of R_n, the Remainder Term in Taylor's Theorem, as a function of n leads to infinite series. Convergence tests, including the comparison test, Cauchy's integral test and d'Alembert's ratio test, the simplest theorems on alternating series and absolute convergence, and a statement without proof of theorems concerning power series and uniform convergence, constitute the theory of series. They may be applied to various computations and to the solution of differential equations.

The consideration of a definite integral as the limit of a sum is closely related to infinite series, and though the subject is treated in the first course in calculus, its fundamental importance makes it merit further emphasis. Taking this up now, we may include approximation formulas, evaluations by ingenious devices and by infinite series as well as by the simple use of the indefinite integral method. Comparison theorems and mean value theorems for definite integrals may be given and applied to the approximate evaluation of difficult integrals as well as to certain theoretical questions.

Improper integrals, with tests for convergence, follow the preceding. Im-

[1] The last point is one quite often overlooked and has in my experience caused many good students perplexity, finding as they do the assumption of continuity in nearly every theorem, without being able to tell whether the functions with which they deal are continuous.

[2] It is very questionable if the proof of the latter theorem for the form ∞/∞ is worth giving, considering its difficulty for the student.

portant special integrals, including the Eulerian integrals, provide applications for this topic.

A brief treatment of elliptic integrals and functions may now be introduced. Only the simplest properties and uses of these functions can be given, time not permitting a treatment of the general methods of reducing elliptic integrals to standard types, computation of the standard integrals, or a systematic development of the theory of elliptic functions.

Thus far we have considered only functions of a single variable. To introduce functions of two or more variables I find it necessary to devote some time first to solid analytics. Then partial derivatives, the total differential, change of variable, differentiation of implicit functions, Jacobians and Taylor's Series may be treated. Applications are made to small errors and to geometry of space, including tangent planes and lines, osculating planes, maxima and minima, curves on a sphere, cylinder and cone. A part of the preceding will be a review of what has been studied in the first course in calculus. Also the next topic, multiple integrals, will be partly a review. Here we may include the definition as a limit of a sum, a restatement, if necessary, of Duhamel's Theorem (or a substitute), and a variety of applications to problems in areas, volumes, mass, centroids, pressure, moments of inertia, and attraction. Line and surface integrals, Green's and Stokes's Theorem and applications form an important and interesting extension of the study of integrals, and naturally follow the preceding.

There remain a number of topics one would wish to discuss, but time will not suffice to present all of them, and it is not quite clear which should have preference. Of these, probability, calculus of variations, Fourier's Series, and a further study of differential equations could be given in the spirit of the preceding, and all have a strong claim for recognition on account of their importance in various realms of applied mathematics. Vector analysis and functions of a complex variable are less closely related to what has preceded, but are perhaps of even more fundamental importance to the physicist and engineer. On the other hand, the pure mathematician may feel a strong need for a thorough discussion of certain existence theorems or of transformation of infinite series, or of questions of double limits that arise in the calculus; but my belief is, in keeping with general aims we have stated, that these topics must be sacrificed. It is quite obvious that there will be a lack of time for an extensive treatment of any of these topics, and it would doubtless be best to attempt to discuss only two or three of them. As to which these should be, I have at present no decided opinion. Perhaps we should, on general principles, allow for that much latitude in difference of opinion, and not even try to particularize further.

A SYSTEM OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS.[1]

By G. N. BAUER and H. L. SLOBIN, University of Minnesota.

The equations under consideration are of the form

$$(1) \qquad P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad T(x, y) = 0,$$

where $P_1(x, y)$ and $P_2(x, y)$ are polynomials in x and y with algebraic coefficients, and where $T(x, y) = 0$ denotes an equation which, in general, is not satisfied if both x and y are algebraic numbers. The equation $T(x, y) = 0$ may be satisfied by a finite number of pairs of algebraic numbers. Thus, the equation $x - e^y = 0$ is satisfied for $x = 1$, $y = 0$ and for no other pair of algebraic numbers. Other equations of this type are

$$P_1(x, y) + P_3(x, y) \tan P_2(x, y) = 0, \qquad P_1(x, y)e^{P_2(x, y)} + P_3(x, y) = 0,$$

where $P_1(x, y)$ and $P_3(x, y)$ have no common factor.

It is readily seen that there exists a non-enumerable set of systems of the type (1), for $T(x, y) = 0$ may be of the form $T(x, y) \equiv P_1(x, y) + t = 0$ where t is a transcendental number. This equation is not satisfied for a single pair of algebraic numbers.[2]

Throughout this paper $P_i(x, y)$ represents a polynomial in x and y with algebraic coefficients. The curve represented by an equation of the form $P_i(x, y) = 0$ will be called a P curve. The letter t always represents a transcendental number, and $p_i(t)$ represents a polynomial in t with algebraic coefficients. The curves represented by equations of the form $T(x, y) = 0$ are called T curves.

Before considering the system (1) attention is directed to a few preliminary considerations.

THEOREM 1. *The elimination of x or y from the equations*

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0,$$

produces a polynomial equation in one unknown, with algebraic coefficients.

The elimination of either x or y gives a polynomial equation whose coefficients are derived from the coefficients of the two given polynomials by the operations of addition and multiplication. Hence the theorem is evident.[3, 4]

THEOREM 2. *The equations*

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0,$$

are satisfied simultaneously by algebraic numbers only.

[1] Read before the Chicago Section of the American Mathematical Society, December 22, 1916. *Bull. Am. Math. Soc.*, Vol. 23, p. 256.

[2] Bauer, G. N., and Slobin, H. L., "Some Transcendental Curves and Numbers," *Rendiconti del Circolo Matematico di Palermo*, Vol. 36, 1916, pp. 327–332.

[3] Burnside and Panton, *Theory of Equations*, p. 349.

[4] Bachmann, *Vorlesungen über die Natur der Zahlen*, 1892, p. 20.

This follows directly from theorem 1, for upon eliminating y, for example, the resulting equation can be satisfied by algebraic numbers only.[1]

THEOREM 3. *The system*

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad T(x, y) = 0,$$

can have no simultaneous solution except possibly one or more of the pairs of algebraic numbers which satisfy $T(x, y) = 0$.

This is apparent since, by theorem 2, the equations $P_1(x, y) = 0$, $P_2(x, y) = 0$ can be satisfied only by pairs of algebraic numbers, while, in general, $T(x, y) = 0$ cannot be satisfied by such a pair of numbers.

Thus, the system

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad y - \sin x = 0,$$

can have no simultaneous solution excepting possibly (0, 0).

Likewise, the system

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad P_3(x, y) - t = 0,$$

can have no simultaneous solution.

It is evident that no two P curves can intersect on a T curve unless, by way of exception, the isolated points on the curve represented by $T(x, y) = 0$ whose coördinates are both algebraic numbers, happen to be among the intersections of the two P curves.

If a P curve cut a T curve, no other P curve can pass through any transcendental point of intersection; hence any curve which passes through a transcendental point of intersection of a P curve and a T curve is not a P curve. It does not follow from this that the curve is a T curve since it has not been shown that the P and T curves exhaust all possibilities. In fact, the equation

$$P_1(x, y)P_2(x, y) + tP_1(x, y) = 0$$

represents a curve that is neither a P curve nor a T curve.

THEOREM 4. *The equation* $P_1(x, y) = 0$ *can be satisfied only by a pair of numbers both of which are algebraic or both of which are transcendental.*

Let x be an algebraic number, then $P_1(x, y) = 0$ may be considered a polynomial with algebraic coefficients; hence y must be an algebraic number. If on the other hand x is a transcendental number, y cannot be algebraic, otherwise a transcendental number would satisfy an algebraic equation with algebraic coefficients.

THEOREM 5. *If a P curve (not a straight line) intersect a T curve (not in its isolated algebraic points), then the slope of any line through any algebraic point of the plane and through any point of intersection is a transcendental number.* (Any point of the plane whose coördinates are algebraic numbers is called an algebraic point.)

[1] Bachmann, *ibid.*, p. 21.

Let A be any intersection of a P curve and a T curve. Its coördinates are both transcendental numbers (t_1, t_2) since by hypothesis they cannot both be algebraic, and by theorem 3 one cannot be algebraic while the other is transcendental. Let (a_1, a_2) be the coördinates of any algebraic point of the plane. Let us assume that the slope is algebraic; then the equation is of the form

$$y - a_2 = m(x - a_1),$$

where m is an algebraic number. Then we have three equations, two representing P curves and one representing a T curve, intersecting in a common point, both of whose coördinates are transcendental numbers. But this is impossible by theorem 3. Hence m must be a transcendental number.

If the P curve is a straight line the theorem is true for all algebraic points of the plane not on the line. The theorem obviously does not apply to the algebraic points on the line, if P be a line, since the equation of the P curve is an algebraic equation, and hence the slope is an algebraic number.

Also, if the given P curve is a line, and if the slope in question is an algebraic number, then any algebraic point through which the line may be drawn must lie on the given curve, *i. e.*, the line so constructed must coincide with the given P line.

Corollary 1. Any line with an algebraic slope, passing through the transcendental intersections of a P and a T curve, does not pass through any algebraic point.

If the line pass through an algebraic point, it would be possible to write the equation in terms of the coördinates of the point and the slope, and hence the line would be a P curve. Then two P curves and a T curve would intersect in a transcendental point, which is impossible.

Corollary 2. If the equation of a P curve is of the form

$$P_1(x, y) - kP_2(x, y) = 0$$

where k is any algebraic number, the multiple points of the P curve due to the intersection of $P_1(x, y) = 0$ and $P_2(x, y) = 0$ cannot be among the transcendental points on a T curve.

THEOREM 6. *If a P curve (not a circle) cut a T curve (not in its algebraic points), then the distance between a point of intersection and any algebraic point of the plane is a transcendental number.*

Let the coördinates of any algebraic point of the plane be given by (a_1, a_2), and let d be the distance from this point to a point of intersection of a P curve and a T curve. Let us assume that d is an algebraic number. Then, with (a_1, a_2) as a center pass a circle through the point of intersection. The radius of the circle is d. Also the equation of the circle is given in terms of a_1, a_2 and d, and hence it is a P curve. We then have two P curves intersecting a T curve in a transcendental point which, by theorem 3, is impossible. Hence d is not an algebraic number, and must therefore be transcendental.

If the given P curve is a circle, the theorem is still true for all algebraic points of the plane with the exception of the center of the given circle. The theorem evidently does not apply to the center of the circle.

Corollary. If a circle which is a P curve cut a T curve (not in its isolated algebraic points), then the center of the circle is the only algebraic point in the plane whose distance from a point of intersection is an algebraic number.

THEOREM 7. *No circle with an algebraic radius, r, whose center is an intersection of a P curve and a T curve, passes through an algebraic point of the plane* (where r is not equal to the radius of the P curve, in case P happens to be a circle).

For, if the circle passes through an algebraic point (a_1, a_2), it is possible to construct a circle with this point as a center, and r as a radius. It would therefore be a P curve, and two P curves would intersect a T curve, which is impossible.

THEOREM 8. *If a tangent or a normal to a P curve passes through an algebraic point (a_1, a_2), it cannot pass through the intersection of the P curve and a T curve, unless the intersection happens to be one of the finite number of isolated algebraic points of the T curve.*

The equations of the tangents and normals to the P curve, passing through the point (a_1, a_2) can be expressed in terms of a_1, a_2 and the partial derivatives of $P(x, y)$. Hence these equations represent P curves. But it is impossible for two P curves and a T curve to intersect; hence the theorem.

Corollary. A line tangent to a P curve at an algebraic point is a P curve.

THEOREM 9. *Any normal [tangent] line to a P curve, through any algebraic point (a_1, a_2) of the plane is normal [tangent] at an algebraic point.*

This follows directly from the fact that under the stated conditions, the tangent and normal are P curves.

THEOREM 10. *Any line drawn tangent [normal] to a P curve at a transcendental point is a T curve.*

For, if it contained an algebraic point, we could set up

$$(1) \qquad a_1 \frac{\partial P(x, y)}{\partial x} + a_2 \frac{\partial P(x, y)}{\partial y} = 0,$$

$$(2) \qquad (a_1 - x) \frac{\partial P(x, y)}{\partial y} - (a_2 - y) \frac{\partial P(x, y)}{\partial x} = 0,$$

respectively the curve passing through the points of contact of the tangents drawn from (a_1, a_2) to $P(x, y) = 0$, and the curve passing through the feet of all normals drawn from (a_1, a_2) to $P(x, y) = 0$, and find two P curves intersecting in a transcendental point. But this is impossible. Hence it is a T curve.

If the P curve is a straight line, the first member of (1) is *constant*, and the theorem does not apply. If the P curve is a circle and (a_1, a_2) the center of circle (2) is identically 0, and the theorem does not apply in that case.

THEOREM 11. *No transcendental point of a P curve is a singular point.*

For the singular points demand the simultaneous existence of

$$P(x, y) = 0, \qquad \frac{\partial P(x, y)}{\partial x} = 0, \qquad \frac{\partial P(x, y)}{\partial y} = 0,$$

which represent P curves and can be satisfied simultaneously only by algebraic

numbers. If one of the derivatives reduces identically to 0, the remaining equation must be satisfied simultaneously with the given equation $P(x, y) = 0$. This demands that the solution be algebraic numbers.

Corollary. A T curve cannot intersect a P curve in the singular points of the P curve, unless the singular point happens to be one of a finite number of algebraic points of the T curve.

THEOREM 12. *All of the singular points of*

$$R(x, y) \equiv P_1(x, y) + p(t)P_2(x, y) = 0, \tag{1}$$

which require

$$\frac{\partial R(x, y)}{\partial x} = 0 \tag{2}$$

and

$$\frac{\partial R(x, y)}{\partial y} = 0, \tag{3}$$

are among the intersections of $P_1(x, y) = 0$ and $P_2(x, y) = 0$, if there are any at all.

The equations [2] and [3], in general, contain t. Eliminating t between [1] and [2] and also between [2] and [3] we have two equations representing P curves, which, taken together with [1] give a system which cannot be satisfied simultaneously excepting for the values common to $P_1(x, y) = 0$ and $P_2(x, y) = 0$.

Corollary. The curves represented by the equations of the type $P_1(x, y) + p(t) = 0$ have no singular points.

THEOREM 13. *All the singularities of*

$$R(x, y) \equiv p_1(t)P_1(x, y) + p_2(t)P_2(x, y) = 0$$

are found among the intersections of $P_1(x, y) = 0$ and $P_2(x, y) = 0$, if any exist, provided $p_1(t) \not\equiv \lambda p_2(t)$.

By setting up the partial derivatives of $R(x, y)$ with respect to x and y, the argument is seen to follow the lines used in the last demonstration.

THEOREM 14. *The singularities of*

$$[1] \qquad R(x, y) \equiv p_0(t)P_0(x, y) + p_1(t)P_1(x, y) + \cdots + p_n(t)P_n(x, y) = 0,$$

if any exist, are among the simultaneous solutions of

$$P_0(x, y) = 0, \qquad P_1(x, y) = 0, \qquad \cdots, \qquad P_n(x, y) = 0$$

provided

$$[2] \qquad \| a_0, b_1, c_2, \cdots, k_n \| \neq 0$$

where the elements entering the matrix are the coefficients of the polynomials $p_0(t)$, $p_1(t)$, $\cdots$, $p_n(t)$.

Let us write

$$p_i(t) = a_i + b_i t + c_i t^2 + \cdots \quad [i = 0, 1, 2, \cdots n]$$

and let the highest power of t occurring in any polynomial be designated by j.

Also, representing $P_i(x, y)$ by A_i it is seen that the following $j + 1$ equations must subsist in order that there may be a solution:

$$\begin{aligned} a_0A_0 + a_1A_1 + \cdots + a_nA_n &= 0, \\ b_0A_0 + b_1A_1 + \cdots + b_nA_n &= 0, \\ &\cdots \\ k_0A_0 + k_1A_1 + \cdots + k_nA_n &= 0; \end{aligned}$$

for the coefficients of the individual powers of t must reduce to 0, since by placing

$$\frac{\partial R(x, y)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial R(x, y)}{\partial y} = 0$$

it is seen that the singular points must be algebraic points. This requires that the matrix [2] be equal to 0 or that $A_0 = A_1 = \cdots = A_n = 0$. But by hypothesis the matrix is not equal to 0 and hence $A_0, A_1, \cdots, A_n$ must reduce to 0, i. e., the singularities are among the simultaneous solutions of $P_0 = 0$, $P_1 = 0$, $\cdots$, $P_n = 0$.

Corollary. The total number of singular points, which require

$$R(x, y) = 0, \qquad \frac{\partial R(x, y)}{\partial x} = 0, \qquad \frac{\partial R(x, y)}{\partial y} = 0,$$

cannot exceed jk where j and k are the degrees of the two polynomials of lowest degree, which enter in [1].

It is evident that many such equations [1] may occur where jk would be much less than the maximum number of such points possible as determined from the table of Plückerian Characteristics. Thus for example

$$p_1(t)x + p_2(t)y + p_3(t)P_n(x, y) = 0$$

where $P_n(x, y)$ is a polynomial of the nth degree with the absolute constant not 0, has no such singularities, for $x = 0$, $y = 0$ do not satisfy $P_n(x, y) = 0$.

The various theorems pertaining to singularities may easily be extended to geometry of higher dimensions.

PRACTICAL SOLUTION OF LINEAR EQUATIONS.

By H. T. BURGESS, University of Wisconsin.

If we attempt to find a Fundamental System of Solutions[1] of a system of linear equations by the methods found in the literature,[2] we meet with considerable difficulty in the application.

It is the purpose of this brief paper to explain a scheme which works with remarkable ease and simplicity in practice, and which is likewise available for theoretical purposes, if we so desire to use it.

1. **Preliminary Matrix Theory.** If the rectangular matrix

$$A' = \left\| \begin{matrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{matrix} \right\| .$$

is of rank r,[3] we can reduce A' by elementary transformations[4] on the rows only to the form $\left\| \begin{matrix} R \\ Z \end{matrix} \right\|$, where Z consists of $(n - r)$ rows of zeros.

One method of procedure is as follows: (1) if $a_{11} = 0$, some $a_{1j} \neq 0$, interchange rows 1 and j; (2) divide row 1 by a_{11}; (3) multiply row 1 successively by $a_{12}, a_{13}, \cdots, a_{1n}$ and subtract the successive products from rows 2, 3, $\cdots$, n respectively; (4) continue this process until either (a) r columns have been so reduced or (b) some column, say the kth, $k < r$, is found where every element below the $(k - 1)$st row is zero. In case (a), the resulting matrix has the form $\left\| \begin{matrix} I & S \\ Z_1 & Z_2 \end{matrix} \right\|$, where I is the unit matrix[5] of order r, and $(n - r)$ rows of zeros appear in Z_1 and Z_2. This must be the case, for the presence of a non-zero element in Z_2 would increase the rank of A' by one. In case (b), pass on to the first column, say the pth, in which a non-zero element appears in or below the kth row. Interchange rows, if necessary, so that $a_{kp} \neq 0$. Divide the kth row by a_{kp} and reduce all other elements in the pth column to zero as in (3). Continue this process until r columns, exclusive of those passed over, are reduced as in (3). The result is in the form

$$\left\| \begin{matrix} J & P \\ Z_1 & Z_2 \end{matrix} \right\| ,$$

where J is of rank r and has r rows, and $(n - r)$ rows of zeros appear in Z_1 and Z_2. That the rank of J is r is apparent if we omit the columns which were passed over

[1] See Bôcher's *Introduction to Higher Algebra*, p. 50.

[2] See Bôcher, loc. cit.

[3] See Bôcher, pp. 20–22.

[4] See Bôcher, p. 55.

[5] See Bôcher, p. 74.

and observe that the remaining matrix is the unit matrix I of order r. As before, a non-zero element can not appear in Z_2, for this would increase the rank of A' by one.

2. **Solution of a System of Linear Homogeneous Equations.** The system

$$(a)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0, \end{aligned}$$

has a matrix A, and if we interchange the rows and columns in A, we get the matrix A' discussed in section 1. Write the matrix $\| A' \mid I \|$ and reduce this matrix by elementary transformations on its rows only until the part indicated by A' assumes either the form (a) or (b) of section 1. The entire matrix $\| A' \mid I \|$ will then be reduced to the form

$$\left\| \begin{array}{c|c} R & K \\ Z & S \end{array} \right\|,$$

where there are r rows in K and R, r being the rank of A, and $(n - r)$ rows in S and Z, the rows of Z consisting of zeros.

If $r = n$, $x_1 = x_2 = \cdots = x_n = 0$ is the only solution.

If $r < n$, there are $(n - r)$ linearly independent solutions of (a); and these form a fundamental system of solutions. These solutions appear as the rows in S.

That each row of S is a solution follows from the fact that each row of S gives a linear combination of the rows of A' which is zero and hence a linear combination of the columns of A which is zero.

That the $(n - r)$ rows of S are linearly independent follows from the fact that the determinant $\begin{vmatrix} K \\ S \end{vmatrix} \neq 0$ by reason of its derivation from the non-zero determinant $\mid I \mid$ by elementary transformations.

That the system of solutions comprised in the rows of S is a fundamental system follows from the fact that they are linearly independent and $(n - r)$ in number.[1]

Illustration. Solve the system

$$\begin{aligned} x + 11y - 2z + 8w + 3t &= 0, \\ x + 2y - z + 3w + 2t &= 0, \\ -2x + 7y + z - w - 3t &= 0, \\ -7x + 4y + 5z - 11w - 12t &= 0. \end{aligned}$$

[1] See Bôcher, p. 52, Th. 3.

We write

$$\left\|\begin{array}{rrrr|ccccc} 1 & 1 & -2 & -7 & 1 & 0 & 0 & 0 & 0 \\ 11 & 2 & 7 & 4 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 8 & 3 & -1 & -11 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & -3 & -12 & 0 & 0 & 0 & 0 & 1 \end{array}\right\|.$$

(1) Add rows 4 and 5 and subtract from row 2;
(2) Multiply row 3 by 4 and add to row 4;
(3) Add row 3 to row 5 and subtract row 1 from the result;
(4) Multiply row 1 by 2 and add to row 3;
(5) Add row 3 to row 4;
(6) Multiply row 3 by 3 and add to row 2;
(7) Interchange rows 2 and 3.

The result is

$$\left\|\begin{array}{rrrr|rrrrr} 1 & 1 & -2 & -7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & -9 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 6 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right\|$$

and there are two solutions.

3. **Non-homogeneous Linear Equations.** If we set $x_n = 1$ in the system (a), we have a non-homogeneous system. The process of solution is the same except the last column of S must be reduced to a column of 1's by elementary transformations on the rows of S.

A necessary and sufficient condition for such a solution is that at least one element in the last column of S shall be different from zero. This is equivalent to the condition that the matrix and the augmented matrix shall have the same rank.[1]

Illustration. Solve the system

$$\begin{aligned} 4x - y + 5z + 1 &= 0, \\ 2x - 3y + z + 5 &= 0, \\ x + y + 2z - 2 &= 0, \\ 5x \phantom{{}-3y} + 2z - 1 &= 0. \end{aligned}$$

We write

$$\left\|\begin{array}{rrrr|cccc} 4 & 2 & 1 & 5 & 1 & 0 & 0 & 0 \\ -1 & -3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 5 & 1 & 2 & 2 & 0 & 0 & 1 & 0 \\ 1 & 5 & -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right\|.$$

[1] See Bôcher, pp. 44–46.

(1) Interchange rows 1 and 4;
(2) Multiply row 2 by 4 and add to row 4;
(3) Multiply row 2 by 5 and add to row 3;
(4) Add row 1 to row 2;
(5) Multiply row 2 by 5 and add to row 4;
(6) Multiply row 2 by 7 and add to row 3.

The result is

$$\left\|\begin{array}{cccc|cccc} 1 & 5 & -2 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -5 & 0 & 12 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 & 0 & 5 \end{array}\right\|.$$

and the solution is $x = \frac{1}{5}$, $y = \frac{9}{5}$, $z = 0$.

If the "5" in the last row and column had come out a zero, the equations would have been inconsistent.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Missouri.

2732. Proposed by PAUL CAPRON, U. S. Naval Academy.

A conical cup, filled with a fluid, stands with the vertex on a smooth horizontal surface. The inner and outer surfaces of the cup are similar cones of revolution, having altitudes h and $h(1 + x)$; the ratio of the specific weights of the material of the cone and the fluid is σ; the height of a barometer column of the fluid is h_0. Show that for equilibrium

$$h_0/h(1 + x)^2 + \sigma x(1 + x + x^2/3) \nless 2/3.$$

2733. Proposed by J. L. RILEY, Stephenville, Texas.

An ellipse of constant eccentricity passes through the focus of a parabola and has its foci on the curve. Find the envelopes of its axes.

2734. Proposed by E. L. REES, The University of Kentucky.

Given two circles tangent to each other externally. From the extremity of a diameter through the point of tangency, draw a secant such that the segment between the circles shall be equal to a given segment.

2735. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If two lines AE and BD, drawn from the vertices, A and B, of a triangle to the opposite sides, divide the angles A and B so that the parts of A are respectively less than the corresponding parts of B, then AE is greater than BD.

2736. Proposed by M. COHEN, Freshman, Johns Hopkins University.

Prove by elementary geometry that the orthocenter, the centroid, and the circumcenter of a triangle lie on a line (the Euler line), and that the centroid lies between the other two and is twice as far from the orthocenter as from the circumcenter.

SOLUTIONS OF PROBLEMS.

2669. Proposed by S. A. COREY, Albia, Iowa.

Let $A_1, A_2, \cdots, A_8$, and $-(A_1 + A_2 + \cdots + A_8)$ be the vector sides of an enneagon, plane or gauche. Also let $B_1, B_2, \cdots, B_8$, and $-(B_1 + B_2 + \cdots + B_8)$ be the vector sides of a second enneagon, where $B_1 = C_1A_1 - C_2C_5A_3 - C_3C_6A_5 + C_4C_5C_6A_7$, $B_2 = C_1A_2 - C_2C_5A_4 - C_3C_6A_6 + C_4C_5C_6A_8$, $B_3 = C_2A_1 + C_1A_3 - C_4C_6A_5 - C_3C_6A_7$, $B_4 = C_2A_2 + C_1A_4 - C_4C_6A_6 - C_3C_6A_8$, $B_5 = C_3A_1 + C_4C_5A_3 + C_1A_5 + C_2C_5A_7$, $B_6 = C_3A_2 + C_4C_5A_4 + C_1A_6 + C_2C_5A_8$, $B_7 = C_4A_1 - C_3A_3 + C_2A_5 - C_1A_7$, and $B_8 = C_4A_2 - C_3A_4 + C_2A_6 - C_1A_8$, C_1, C_2, C_3, C_4, C_5, and C_6 being scalars.

Then, if a_s = tensor A_s, b_s = tensor B_s, and $\cos(A_rA_s)$ = cosine of the angle included between A_r and A_s and $\cos(B_rB_s)$ = cosine of the angle included between B_r and B_s, establish the following relation between the sides and angles of the two enneagons:

$$[C_1^2 + C_5C_2^2 + C_6C_3^2 + C_5C_6C_4^2][a_1a_2 \cos(A_1A_2) + C_5a_3a_4 \cos(A_3A_4) + C_6a_5a_6 \cos(A_5A_6) + C_5C_6a_7a_8 \cos(A_7A_8)]$$
$$= b_1b_2 \cos(B_1B_2) + C_5b_3b_4 \cos(B_3B_4) + C_6b_5b_6 \cos(B_5B_6) + C_5C_6b_7b_8 \cos(B_7B_8).$$

Show that Geometry problem 506 is a special case of the foregoing. Give illustrative example, using triangle or other simple geometric figure, by assuming that some of the sides of the first enneagon are zero.

Solution by Proposer.

Whenever $A_1, A_2, \cdots$ and A_8 are scalar (or algebraic) quantities, and $B_1, B_2, \cdots$ and B_8 have scalar values corresponding in form to those given in the problem, we have the algebraic identity,

$$(C_1^2 + C_5C_2^2 + C_6C_3^2 + C_5C_6C_4^2)(A_1A_2 + C_5A_3A_4 + C_6A_5A_6 + C_5C_6A_7A_8)$$
$$= B_1B_2 + C_5B_3B_4 + C_6B_5B_6 + C_5C_6B_7B_8.$$

Inasmuch as all the terms in $A_1, A_2, \cdots A_8$, and $B_1, B_2, \cdots B_8$ in this algebraic identity are of the second degree, a geometric interpretation may be obtained by assuming that $A_1, A_2, \cdots A_8$ and $B_1, B_2, \cdots B_8$, are vectors. This follows immediately from the fact that vector multiplication is commutative in so far as the scalar part of the product is concerned whenever all the vector terms employed are of the second degree. But the scalar part of the vector product B_rB_s is $-b_rb_s \cos(B_rB_s)$. Substituting this scalar part of the vector product in both members of the above algebraic equation and changing signs we obtain at once the equation contained in the problem. If $A_1 = A_2$, $A_3 = A_4$, $A_5 = A_6$, and $A_7 = A_8$, the problem becomes identical with Geometry problem 506.

Example. As long as a vector maintains a constant length and direction in space, its origin in space may be altered at will. Hence we need not confine our attention to closed geometric figures in interpreting the given identity. Let DEF be a given triangle. Bisect DE in G. Draw GF and extend EF to F'. Draw GH intersecting DF in I and EF' in J. Let $A_1 = GF$, $A_2 = GI$, and $A_3 = GD$. Also let $A_4 = A_5 = A_6 = A_7 = A_8 = 0$, $C_1 = C_2 = C_3 = C_4 = C_5 = 1$, and $C_6 = 0$. Substituting in the given identity, paying strict attention to the direction of the vectors employed, and dividing by the constant factor, we readily get $2GF \cos FGI = DF \cos GID + EF \cos EJG$, a known result.

2670. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A telegraph wire, weighing one tenth pound per yard, is stretched between poles on level ground, so that the greatest dip of the wire is three feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

Solution by Elbert H. Clarke, Hiram College, Ohio.

It is a well-known property of wires hanging freely from two supports that the tension at

any point is equal to that which would be produced by a wire of the same weight per unit length, hanging vertically and being of the same length as the ordinate of the point where the tension is measured, the equation of the curve being of the form,

$$y = \frac{a}{2}(e^{x/a} + e^{-x/a}).$$

Hence, the equation of the catenary curve assumed by the wire in the above problem will be

$$y = 700(e^{x/1400} + e^{-x/1400}),$$

the unit of distance being one yard.

The amount of dip or sag in a symmetrical segment of horizontal projection, $2h$, will be

$$d = \frac{a}{2}(e^{h/a} + e^{-h/a}) - a$$

$$= \frac{h^2}{2a} + \frac{h^4}{24a^3} + \frac{h^6}{720a^5} + \cdots.$$

In our problem $d = 1$, $a = 1400$, and a very close first approximation is given by neglecting all terms in the series after the first. Hence, $h = 52.915$, approximately. Using more terms the result correct to three places is $h = 52.912$. The distance between the poles, correct to the nearest tenth of a foot is 317.5 feet.

Also solved by ROGER A. JOHNSON.

2671. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find two rectangular parallelopipedons whose edges are rational whole numbers and whose solid diagonals are also rational whole numbers and equal.

SOLUTION BY S. A. COREY, Albia, Iowa.

We have the identities

$$\begin{aligned}(x^2 + y^2 + u^2 + v^2)^2 &= (x^2 - y^2 - u^2 + v^2)^2 + (2xy - 2uv)^2 + (2ux + 2vy)^2 \\ &= (x^2 - y^2 + u^2 - v^2)^2 + (2xy + 2uv)^2 + (2vx - 2uy)^2 \\ &= (x^2 + y^2 - u^2 - v^2)^2 + (2ux - 2vy)^2 + (2vx + 2uy)^2.\end{aligned}$$

By letting x, y, u, and v represent rational whole numbers any number of solutions of the problem may be obtained. One such solution is obtained by letting $x = 1$, $y = 2$, $u = 5$, and $v = 7$, and we find *three* parallelopipedons fulfilling the requirements of the problem, with edges 21, 66, 38; 27, 74, 6; and 69, 18, 34, respectively, the solid diagonal of each being 79. The three smallest have edges, 1, 2, 2; 2, 4, 4; and 2, 3, 6, respectively.

Also solved by L. E. LUNN, W. P. DURFEE, and H. L. OLSON.

2673. Proposed by WILLIAM O. BEAL, University of Minnesota.

A plane through the center of an oblate spheroid makes an angle, i, with the plane of its equator. Express the eccentricity, e', of this section in terms of the eccentricity, e, of a meridian section and the angle, i.

SOLUTION BY C. A. BARNHART, Colorado College.

Let the equation of the oblate spheroid be $x^2/b^2 + y^2/b^2 + z^2/a^2 = 1$, $(b > a)$, of which the z-axis is the axis of revolution. Let $OP = b'$ be the semi-minor axis of the given plane section of which b is the semi-major axis. Then P will be a point on the meridian section, which may be assumed to lie in the yz-plane and to have the parametric equations:

$$z = a \sin \theta, \qquad y = b \cos \theta;$$

and the eccentricity,

$$e = \frac{\sqrt{b^2 - a^2}}{b}.$$

Since

$$e' = \frac{\sqrt{b^2 - b'^2}}{b}$$

and

$$b' = \sqrt{y^2 + z^2} = \sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta},$$

$$e' = \frac{\sqrt{b^2 - b^2 \cos^2 \theta - a^2 \sin^2 \theta}}{b} = \frac{\sqrt{(b^2 - a^2)(1 - \cos^2 \theta)}}{b} = e \sin \theta.$$

But $z/y = \tan i = (a/b) \tan \theta$, or $\tan \theta = (b/a) \tan i$. Therefore,

$$\sin \theta = \frac{b \tan i}{\sqrt{a^2 + b^2 \tan^2 i}} = \frac{b \tan i}{\sqrt{b^2 - b^2e^2 + b^2 \tan^2 i}} = \frac{\tan i}{\sqrt{\sec^2 i - e^2}}.$$

Substituting for $\sin \theta$, we have

$$e' = \frac{e \tan i}{\sqrt{\sec^2 i - e^2}}.$$

Also solved by Roger A. Johnson, Elbert H. Clarke, H. L. Olson, and the Proposer.

2674. Proposed by J. O. MAHONEY, Dallas, Texas.

If two sides of a triangle differ by less than a certain length, e, the two opposite angles will differ by less than a certain quantity λ, expressed in degrees, such that $\lambda < 61e/a$ where a expresses, with a possible error e, the length of the apparently equal sides of the triangle.

Solution by Roger A. Johnson, Hamline University.

This theorem as stated, is not true. Consider, for instance, the triangle

$$a = 1001, \qquad b = 1000, \qquad c = 99.$$

Here,

$$A = 87° \, 44' \, 32'', \qquad B = 86° \, 35' \, 10'', \qquad C = 5° \, 40' \, 18''.$$

Now, considering the nearly equal angles A and B, we have, in fact, $\lambda = 1.157$, whereas by the formula given, we should have $\lambda = .061$ or less. This is an extreme case, but it will be found that in any triangle in which the nearly equal angles are greater than about 50°, the formula does not hold.

As a matter of fact, the correct expression is

$$\lambda = \frac{180}{\pi} \frac{e}{a} \tan A,$$

where A represents the larger of the nearly equal angles. We will not consider the case that either of these two angles exceeds or equals 90°.

If a and b are two sides of a triangle, α, β, the opposite angles, we have

$$\sin \beta = \frac{\sin \alpha}{a} \cdot b;$$

whence

$$\cos \beta d\beta = \frac{\sin \alpha}{a} db,$$

in radians, or

$$d\beta = \frac{180°}{\pi} \cdot \frac{\sin \alpha}{\cos \beta} \frac{db}{a}$$

For our problem, $d\beta = \lambda$, $db = e$, and $(\sin \alpha)/(\cos \beta)$ may be replaced by the greater of the values $\tan \alpha$, $\tan \beta$, yielding the inequality given above.

Also solved by Elijah Swift and Paul Capron.

2675. Proposed by E. B. ESCOTT, Kansas City, Mo.

Sum the series

$$-\frac{1}{2}\cdot\frac{1^3x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{2^3x^5}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{3^3x^7}{7}+\cdots.$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Differentiate the given series (permissible, as a power series, convergent $|x| < 1$). Dividing by x^2, we have the series,

$$P(x) = -\frac{1}{2}\cdot 1^3+\frac{1\cdot 3}{2\cdot 4}2^3(x^2)-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}3^3(x^2)^2+\cdots.$$

If we substitute z for x^2 and then integrate this from 0 to z, divide by z and integrate again, and repeat this process, we arrive at a familiar series. (We are integrating a function which is obviously continuous at $z = 0$.) The resulting series is

$$-\frac{1}{2}z+\frac{1\cdot 3}{2\cdot 4}z^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^3+\cdots = (1+z)^{-1/2}-1.$$

Reversing the processes we have carried out on the series, we finally obtain the original series in closed form. Thus, differentiating $(1 + z)^{-1/2} - 1$ and multiplying by z, repeating this process, and differentiating the result, we have, putting $z = x^2$,

$$P(x) = \frac{-x^4+10x^2-4}{8(1+x^2)^{7/2}}.$$

Multiplying this by x^2, integrating and determining the constant of integration, the value of the given series is

$$\frac{1}{8}\left\{\frac{3x}{(1+x^2)^{5/2}}-\frac{5x}{(1+x^2)^{3/2}}+\frac{3x}{(1+x^2)^{1/2}}-\log(x+\sqrt{1+x^2})\right\}$$

$$=\frac{x+x^3+3x^5}{8\sqrt{(1+x^2)^5}}-\tfrac{1}{8}\log(x+\sqrt{1+x^2}).$$

Also solved by H. L. OLSON and the PROPOSER.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

CLUB ACTIVITIES.

THE JUNIOR MATHEMATICAL CLUB OF THE UNIVERSITY OF CHICAGO, Chicago, Ill. [*1918*, 34–5].[1]

November 15, 1916: "Newton" by Gail F. Moulton '19 and Thomas McN. Simpson Gr.

November 29: "Elementary notions of line complexes and congruences" by Levi S. Shively Gr.

December 13: "Notes in the history of the theory of functions of a complex variable" by Joseph L. Walsh Gr.

January 14, 1917: "Quadratic forms" by James E. McAtee Gr.

January 28: "Transformation of Coördinates" by Kenneth Lamson Gr.

[1] This abbreviation indicates that on pages 34–35 of this MONTHLY, 1918, there may be found further information concerning The Junior Mathematical Club.

February 14: "An Application of Bayes's theorem[1] in the theory of probabilities" by Professor William D. Cairns of Oberlin College.

February 28: "Cauchy" by George H. Cresse Gr.; "Poncelet" by Ernest P. Lane Gr.; "Gauss" by Mary C. Suffa Gr.; "Weierstrass" by Webster G. Simon Gr.

April 11: "Elementary notions of continuous groups" by Israel A. Barnett Gr.

April 25: "Note on a set of postulates" by William P. Ott Gr.

May 9: "Plücker" by Orrin W. Albert Gr.

May 23: "The velocity of a planet" by George H. Cresse Gr.; "The density of a sphere" by Ernest P. Lane Gr.; "Riemann" by Cornelius Gouwens Gr.

June 6: "Infinite determinants" by William G. Simon Gr.

January 16, 1918: "Twisted curves in vector analysis" by Ernest P. Lane Gr.

January 30: "Gauss's reciprocity theorem" by Horace Alson Gr.

February 13: "The limaçon" by Gladys Gibbens Gr.

February 27: "Lagrange" by Charles C. Spooner Gr.

March 13: "Statistical interpretation of entropy" by Edward S. Akeley Gr.

April 17: "The fundamental theorem of algebra" by Ernest Zeisler '19.

May 1: "Some elementary geometric concepts of the complex variable" by Tel C. Kimball Gr.

May 15: "The discovery of Neptune" by Frederick C. Leonard '18.

May 29: "Reflected curves" by Louis Shapotken '18.

THE GRINNELL COLLEGE MATHEMATICS CLUB, Grinnell, Iowa.

This club was founded in March, 1917. At the five meetings during the remainder of the year the programs included the following papers: "The problem of Apollonius" by Professor Raymond B. McClennon; "Planimeters" by Professor William J. Rusk; "Geometry of four dimensions" by Lloyd W. Taylor, Jr., instructor; "Applications of Mathematics to gunnery" by Earl Kilgore '18.

During 1917–18 the club functioned for the first semester only. Linn Smith '19, and Ethel Rivers '17, were respectively president and vice-president. The topics discussed included, "Mathematical fallacies" by Linn Smith '19, and "Squaring thc circle" by Fay Breckenridge '18.

THE MATHEMATICAL CLUB, Harvard University, Cambridge, Mass. [*1918*, 186–7].

October 11, 1916: "Geometrical constructions with special instruments" by Professor Charles L. Bouton.

October 25: "Functions of infinitely many variables" by Dr. William L. Hart, Benjamin Peirce instructor.

November 8: "A classification of closed curves on a surface of finite connectivity by means of a canonical set of closed curves" by Harold C. M. Morse Gr.

[1] Cf. I. Todhunter, *A History of the Mathematical Theory of Probability*, Cambridge and London, 1865, pp. 294 ff.

November 22: "Primary quadratic forms" by Dr. Tracey A. Pierce, instructor.
December 6: "Some intimately related linear equations" by Professor Roland G. D. Richardson of Brown University, Providence, R. I.
January 23, 1917: "Determinants of many dimensions" by Lepine H. Rice Gr.
February 21: "Minkowski's contributions to pure mathematics" by Dr. Edward A. Kircher, Benjamin Peirce instructor.
March 7: "A theory of rectangular matrices" by Professor Clarence L. E. Moore of Massachusetts Institute of Technology.
March 21: "Certain heresies in the teaching of elementary dynamics" by Professor Edward V. Huntington.
April 4: "The fundamental theorem of algebra" by Forrest H. Murray Gr.
May 2: "What is real algebra" by Dr. Henry M. Sheffer, instructor in philosophy.
May 16: "Independent postulates for groups and fields" by Ronald M. Foster '17.

The average attendance at Harvard meetings during 1916–17 was about 26, during 1917–18 about 20.

March 6, 1918: "Calculating machines" by Mr. Modern of the Monroe Calculating Machine Company.
March 20: "Bernhard Riemann" by Professor William F. Osgood.
April 3: "Pencils of Lines" by Dr. Alton L. Miller, instructor.
May 1: "Areal derivatives" by Dr. Gabriel M. Green, instructor.
May 15: "Slide rules" by Professor William R. Ransom of Tufts College, Tufts College, Mass. Professor G. D. Birkhoff was elected faculty adviser, during 1918–19, with power to call a meeting for election of officers.

The Mathematics Club of the University of Kansas, Lawrence, Kansas. [*1918*, 35–6].

This club was organized in December, 1911. The following programs supplementing those already given were issued annually in printed form:

October 5, 1914: Election of officers.
October 12: "Mathematics in astronomy" by Professor Ellis B. Stouffer.
October 26: "Addition and subtraction applied to geometry according to the principles of Grassmann" by Edward H. Carus '12, instructor.
November 8: "Who's who in mathematics" by Charles F. Green '14.
November 22: "The fourth dimension" by Laurens E. Whittemore Gr.
January 11, 1915: "The three problems of antiquity" by Cyril A. Nelson Gr.
January 25: "What is mathematics" by Professor Ulysses G. Mitchell.
February 15: "A discussion of the gyroscope" by Wendell M. Latimer '15.
March 8: "Current events in mathematics" by Eva M. Coors '16.
March 22: "The polar planimeter" by Austin Bailey '16.
April 12: "Insurance mathematics" by Professor Charles H. Ashton.
April 26: "Napier and the invention of logarithms" by Ethel W. Mallonee Gr.
May 10: "Quadric surfaces" by Ottilia W. Ducker Gr.
May 24: "A trip to infinity" by Professor John N. Van der Vries.

September 27: Election of officers.
October 11: "Fermat's theorem and allied topics" by Dr. Solomon Lefschetz, instructor.
October 25: "Non-Euclidean geometry" by Jessie Jacobs '15.
November 8: "Line construction" by Ada H. West Gr.
November 22: "Who's who in mathematics in America" by Professor Mitchell; "Mathematical reference books" by Professor Stouffer.
December 13: "Curve tracing" by Paul W. Harnley '15.
January 10, 1916: "Methods of computing errors" by Professor Herbert E. Jordan.
February 14: "Quadratic forms in number theory" by Mabel W. Arnett '15.
February 28: "Elements of orbits of heavenly bodies and Kepler's laws" by Cora J. Shinn '17.
March 3: "Mathematical Fallacies" by James B. Ramsey '16.
March 27: "Some definite integrals" by Arthur W. Larsen, instructor.
April 10: "The origin of the calculus" by Laura J. McKay '16.
April 24: "The mathematics of the calendar" by Leonard L. Steimley, instructor.
May 8: "Finite geometry" by Cyril A. Nelson '14.
May 28: "Review of Memorabilia Mathematica" [Edited by R. E. Moritz] by Florence R. Scheidenberger '15.
September 25: "Methods and customs in German Universities" by Professor Ashton.
October 9: "The mysteries of the fourth dimension" by Margaret Coleman '17.
October 23: "Mathematical games" by Professor Van der Vries.
November 6: "Three famous problems of antiquity" by Hazel E. Parkinson '18.
November 20: "Some simple applications of vector analysis" by Arthur W. Larsen, instructor.
December 11: "The slide rule" by Hobart F. Lutz '19.
January 8, 1917: "Magic squares and cubes" by Cora J. Shinn '18.
January 30: "Japanese and Chinese mathematics" by Frances E. Adams '18.
February 12: "Adding and multiplying machines" by Professor John J. Wheeler.
February 26: "Properties of the number 9" by Marie O. Graff '17.
March 12: "How to draw a straight line" by Earle B. Miller, instructor.
March 26: "Paper folding" by Bernice Boyles '18.
April 9: "Review of De Morgan's Budget of Paradoxes" by Helen R. Garman '18.
April 23: "The planimeter and rectifier" by Lewis M. Hull '17.
May 14: "Mathematical recreations" by Mignonette Uhl '18.
May 25: Picnic.

THE WHITE MATHEMATICS CLUB AT THE UNIVERSITY OF KENTUCKY, Lexington, Ky. [*1918*, 90].

The following list of programs is supplementary to that previously given.
October 11, 1916: "How to draw a straight line" by Professor Paul P. Boyd.

October 18: "Exceptions to the laws of radicals" by Professor Elijah L. Rees.
October 25: "On the contraction of homogeneous spheroids" by Professor Harold H. Downing.
November 1: "Intrinsic equations" by Professor Joseph W. Davis.
November 8: "Theorems relating to the three normals, through a point, to a parabola" by Vernon C. Grove Gr.
November 15: "Elliptic integrals in the problems of the inverse fifth power" by Harry R. Allen Gr.
November 22: "Eclipses" by Homer L. Reid Gr.
December 6: "Graphical construction for a function of a function and for a function given by a pair of parametric equations" [review of W. H. Roever's article in this MONTHLY, 1917] by Clarence W. Harney '17.
December 13: "Linkages" (continued) by Professor Boyd.
December 20: "Trisection of an angle by means of conics" by Professor Rees.
January 6, 1917: "Areas of pedal curves" by Professor Davis.
January 10: "Helmholtz's contraction theory" by Professor Downing.
February 7: "History of Egyptian and Phœnician Mathematics" by Vernon G. Grove Gr.
February 14: "History of Grecian Mathematics" by Harry R. Allen Gr.
February 28: "History of Hindoo and Arabic mathematics" by Homer L. Reid Gr.
March 14: "History of Italian mathematics" by Clarence W. Harney '17.
March 23: "History of English mathematics" by H. L. Reid Gr.
March 28: "History of French mathematics" by V. G. Grove Gr.
April 4: "History of German mathematics" by H. R. Allen Gr.
April 17: "A chart of mathematical history" by Professor Rees.
April 25: "Non-Euclidean geometry" by Myrtle R. Smith '17.
May 1: "The general linear transformation by linkages" by Professor Boyd.
May 9: "Perturbations treated geometrically" by Professor Downing.
May 18: "Pythagoras and the Pythagorean theorem" by Mary Beall '19.
January 16, 1918: "Integrals related to the Lebesgue integrals" by Professor Davis.
January 23: "A discussion of the Rochester plan" [review of W. Betz's article, on "The Teaching of Mathematics in the Junior High School," in *The Mathematics Teacher*, December, 1917] by Professor Davis; "Mathematical requirements in certain high schools" by Miss Mamie Schmidt of the Lexington High School.
February 7: "Review of Townsend and Goodenough's Calculus" by Professor Rees.
February 14: "The cyclo-harmonograph"[1] by Professor Boyd.
February 20: "Arithmetical progression of the nth order" by Professor Downing.
February 27: "Introduction to infinitesimal analysis" by Professor Davis.

[1] See R. E. Moritz, (1) "The Cyclo-harmonograph; an instrument for drawing large classes of important higher plane curves," *Scientific American Supplement*, August 5, 1916; (2) On the construction of certain curves given in polar coördinates," in this MONTHLY, May, 1917.

March 6: "A theorem of mechanics proved by vector analysis" by Professor Rees.
March 27: "Point sets" by Professor Downing.
April 3: "Point sets" (continued) by Professor Davis.
April 10: "Point sets" (continued) and "Shortest distance between two points" by Professor Downing.
April 17: "Point sets" (continued) by Professor Davis.
April 24: "Solution of two geometrical problems" by Professor Rees; "Point sets" (continued) by Professor Downing.
May 8: "Photogrammetry" by V. G. Grove Gr.
May 15: "Some applications of vector analysis to kinematics" by H. R. Allen Gr.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF MAINE, Orono, Me. [*1918*, 132].

The following programs are supplementary to those already given:

March 8, 1916: "Simple method of constructing normals to the parabola" [review of S. G. Barton's paper in this MONTHLY, June, 1914] by Maynard F. Jordan '17: "Life of the late Simon Newcomb" by Charles I. Emery '17.
April 5: "Mathematics in the secondary school" by Hoyt D. Foster '16; "Addition formulas for the trigonometric functions" by Marie F. Foster '16; "Fallacies in geometry" by Professor Lowell J. Reed.
April 19: "Methods of teaching high school mathematics" by James A. Hamlin, principal of Old Town High School.
May 31: "Systems of ovals and the ellipse as a special case" by Zella Colvin '16; "Rolling curves" by Raymond D. Douglass '15.
October 11: Social meeting at the home of Professor James N. Hart; there was a short program consisting of competition between several teams in solving problems and mathematical puzzles.
October 25: "Applications of partial derivatives in relation to the study of conics" by Sumner C. Cobb '17; "Life and work of Charles P. Steinmetz"[1] by Clarence H. Drisko '18.
November 8: "On the solutions of linear equations having small determinants" [review of F. R. Moulton's article in this MONTHLY, October, 1913] by Charles I. Emery '17.
November 22: "Curve fitting" by John I. Miner, computer in the Agricultural Experiment Station; "Life of Percival Lowell, astronomer" by Edith De Beck '18.
December 6: "The fourth dimension" by Dr. Norbert Wiener, instructor.
January 17, 1917: "Solution of certain problems prepared in Granville's and Smith's calculus text books with erroneous answers" by Lester C. Swicker '19; "Methods and symbols used in mathematics before the sixteenth century" by Professor Truman L. Hamlin.

[1] Professor of electro-physics at Union College, Schenectady, N. Y.

February 14: "Solution of problems that appeared in the AMERICAN MATHEMATICAL MONTHLY and in *School Science and Mathematics* by Samuel Wiseman '19.

April 18: "Some astronomical topics of current interest" by Professor Hart.

May 16: "A geometrical interpretation of Taylor's formula" by Professor Harley R. Willard.

December 19, 1917: "History of trigonometry" by Edith L. Deering '21; "The nine-point circle" by Edith DeBeck '18.

February 27, 1918: "The eclipse of June, 1918" by Professor Hart; "Comets" by Samuel Guptill '20.

March 27: "Applications of trigonometry to railway curves" by Alpheus C. Lyon, professor of civil engineering; "Ptolemy's theorem and its applications" by Albert J. Bedard '21.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NORTH CAROLINA, Chapel Hill, N. C. [*1918*, 90–91].

The club members are drawn not only from the department of mathematics but also from the departments of chemistry, physics, and civil and electrical engineering. It has been the aim of the club "to bring teacher and student into closer and more sympathetic contact, to call to the student's attention matters for which opportunity did not present itself in class, to correlate more closely the work of allied departments, to arouse the interest and to foster the development of superior men while lending help and courage to the weaker. The programs have been limited both as to number of papers and as to the time occupied by them, so that problems of interest could be proposed and discussed." These have been conducted by the club's secretary, John W. Lasley, instructor.

The following list of programs supplements that previously given:

November 28, 1916: "The differential coefficient viewed as a singular form" by Professor William Cain; "Mathematical requirements for electrical engineering students" by Elden I. Staples, instructor in electrical engineering.

January 9, 1917: "The logic of mathematics" by Henry H. Williams, professor of philosophy.

February 13: "The inscription of a regular 17-gon in a circle" by Sherman B. Smithey '17; "Some characteristic theorems in the foundations of geometry" by Professor Archibald Henderson.

March 6: "A note on linear equations" by Mr. Lasley; "The teaching of mathematics in the high school" by Lester A. Williams, professor of school administration.

January 7, 1918: "Mathematics historically considered" by Professor Cain. (Lecture open to the public.)

February 6: "A method for finding the complex roots of a cubic equation" by Mr. Lasley.

February 18: "Some aspects of modern geometry" by Professor Henderson (Public lecture).

March 4: "The origin and development of number" by Mr. Lasley (Public lecture).

March 18: "War maps and the use of scales" by Thomas F. Nickerson, associate professor of civil engineering. (Public lecture.)

April 1: "Common sense in mathematics" by Dr. A. Wilson Hobbs, instructor in mathematics. (Public lecture.)

April 22: "Surveying" by Marvin H. Stacy, professor of civil engineering. (Public lecture.)

April 30: "Complex numbers" by Houston Everette '20.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, Eugene, Oregon. [*1918*, 134–135].

The remaining programs in 1917–18 were as follows:

February 27: "Solution of the fifth degree equation" by Mary Mottley '19; "The Leibnitz-Newton controversy over the discovery of the calculus" by Glen Langdon '20.

May 15: "Theory of numbers, especially prime numbers, congruences, and Pythagorean numbers" by Olga Soderstrom '18; "Septic curves" by Dr. Roy M. Winger, instructor.

VINCULUM, University of Pennsylvania, Philadelphia, Pa.

This club was organized in May, 1917, "to further interest in mathematics, to provide a seminar for undergraduate students, and to promote sociability among those interested in the subject. Membership is limited to women students, in any department, in any class, who are majoring in mathematics.[1] During the past year there were 20 members and the average attendance at meetings, including visitors, was 25.

Officers 1917–18: President, Edith P. Levinson '18; vice-president, Anna L. Kuhn '20; secretary, Marion George '20; treasurer, Anna Corson '19.

The following is a complete record of meetings (apart from those of a business nature only) from the foundation of the club to the end of the year 1917–18.

November 12, 1917: "Human side of mathematics" by John H. Minnick, assistant professor in education (formerly instructor in mathematics).

December 10: "Early development of arithmetic" by Maurice J. Babb, assistant professor of mathematics.

January 14, 1918: "Application of elementary arithmetic in physics and astronomy" by Professor Edwin S. Crawley.

April 15: "Infinity" by Professor George H. Hallett.

THE MATHEMATICAL CLUB OF SMITH COLLEGE, Northampton, Mass. [*1918*, 91].

October 23, 1916: Social meeting.

[1] At the University of Pennsylvania there were in 1917–18 approximately 900 women—graduate 200, in college courses for teachers 300, in arts 50, and in education 350—all (except graduates) being candidates for bachelor degrees.

November 13: Review of G. A. Miller's *Historical introduction to mathematical literature* by Florence Hatch '17; Review of *A summary of mathematics in the secondary school of tomorrow* by the Mathematics Club of Chicago High School Teachers, by Gertrude Schloo '17.

December 4: Social at the home of Professor Ruth Wood.

January 8, 1917: Review of "On the nature of mathematical reasoning" from Poincaré's *Science and hypothesis* by Muriel Irving '17.

February 12: Review of "Space and Geometry" from Poincaré's *Science and hypothesis* by Lillian Miller '17; Review of "A non-Euclidean world" from J. W. A. Young's *Lectures on fundamental concepts of algebra and geometry* by May E. Owen '17; [title of paper missing in club's records] by Eleanor E. Stearns '17.

March 5: "Lobachevsky's geometry" by Katherine Baxter '17; "Riemann's geometry" by Anna Hebel '18.

March 26: "Bolyai's geometry" by Ruby M. Burt '17;[1] Review of "Experiment and Geometry" from Poincaré's *Science and hypothesis* by Janie B. Bartlett '17.

April 30: "The fourth dimension" by Aline Hueston '17; "The slide rule" by Winifred L. Chase '17; "The integraph" by Professor Pauline Sperry.

May 21: Social meeting at Professor Eleanor P. Cushing's home.

February 11, 1918: Professor Harriet R. Cobb spoke of work at Columbia during two summers; Martha Chadbourne '14 described her graduate work at Harvard and Radcliffe.

March 4: "Pole and polar relations" by Florence B. Putnam '18; "Plane homologies" by Cornelia D. Hopkins '19.

March 25: Professor Ruth Wood spoke on her graduate study at the University of Göttingen; Professor Suzan R. Benedict told of graduate work at the University of Michigan.

April 29: "Brilliant points" by Martha Chadbourne Gr.

May 20: Social meeting.

THE MATHEMATICS CLUB OF VASSAR COLLEGE, Poughkeepsie, N. Y. [*1918*, 136].

February 17, 1916: Organization meeting.

March 15: Election of officers; "Demonstration of Simpson's Rule" by Constance Andrews '16; "Explanation and discussion of the planimeter" by Mildred Allen '16.

April 20: "The equilateral hyperbola" by Frances Atwater '16; "Pascal's hexagon" by Mary McManus '16; "The path and range of projectiles used in modern warfare" by Ruth Core '16; "The construction of shells used in modern warfare" by Helen Moore '16.

May 12: "Some early stages in the evolution of algebra" by Professor Elizabeth B. Cowley.

October 24: Election of Officers; "Professor Mittag-Leffler's last will and testa-

[1] The last three papers were based mainly on Chapter 3 of Poincaré's *Science and hypothesis*.

ment" by Caroline Bacon '17; "Explanation and discussion of the limit" by Roberta Pickering '17.

November 28: "The construction of some curves by linkages" by Helen Moulton '19 and Rachel Franklin '19: "The slide rule" by Katrina Jaggard '17.

December 13: "Lewis Carroll, author and mathematician" by Margaret Finck '17; "Mathematical recreations" by Beatrice Boyden '18.

February 20, 1917: "History of the adding machine" by Maxime Harrison-Berlitz '19; "Explanation and demonstration of the Burroughs adding machine" by Mr. Madison of the Burroughs Machine Co.

February 22: Election of officers.

March 1: "Mathematics in the secondary schools" by Dean Frederick C. Ferry of Williams College.

April 17: "The fourth dimension" by Mary Moher '16; Eleanor Hussey '16, and Mary Appelgate '17.

February 26, 1918: Election of officers for the second semester: President, Martha Braun '18; vice-president, Helen Thompson '19; secretary-treasurer, Louise Stuerm; Professor Cowley (faculty member of executive committee); Christie White '19 (member at large of executive committee); "Planimeters" by Marjorie Wheatley '18 [see *1918*, 136].

April 23: "A mathematical problem of warfare" by Elizabeth B. Conklin '18; "Mathematical fallacies" by Susan Barr '20.

May 7: Picnic.

The number of meetings of the club was smaller in the year 1917–18 than in the previous year because most of its members participated in red-cross work, in farming activities, or in a preparedness course in mechanical drawing. "Those students who wanted to go into government work or to enter offices of electrical companies found this course particularly useful."

The Junior Mathematical Club, University of Wisconsin, Madison, Wis. [*1918*, 188–9].

October 19, 1916: "American mathematicians" by Professor Edward B. Van Vleck.

November 2: "Graphical representation" by Professor Arnold Dresden.

November 16: "Pappus of Alexandria" by Mary Henry '17; "Models" by Professor Linnaeus W. Dowling.

December 7: "Three breakdowns in mathematics" by Professor Ernest B. Skinner; [title of paper missing from club's records] by Rachel McKannan '17.

January 18, 1917: "Work of Archimedes" by Professor Charles S. Slichter; "Life of Archimedes" by Margaret Chapman '17.

March 1: "Peano's theory of natural numbers" by Frederick Wood, instructor in mathematics.

March 15: [Details of program missing].

March 29: "Squaring the circle" by Professor Walter W. Hart; [title of paper missing] by Stella Andrews '17.

April 19: "Importance of notation" by Dr. Thomas M. Simpson, instructor in mathematics.

May 4: "Geometrical fallacies" by Mary Dillman '17; "Arithmetical fallacies" by Joseph L. Walsh Gr., travelling fellow from Harvard University.

May 18: "Relativity" by Warren Weaver Gr.

April 3, 1918: "Integers" by Professor Skinner.

April 17: "Progress in notation" by Meta Wood '17; "Fractions" by Professor Hart.

May 8: "Lantern slides on mathematicians" by Professor Dresden.

CLUBS REPORTING ON SESSIONS IN 1917–18.

The names of these 37 clubs are arranged chronologically according to the dates of their organization, and pages of this MONTHLY, 1918 where their reports are published are indicated.

The Mathematical and Physical Society of the University of Toronto, Toronto, Ontario.—Founded January, 1882. Membership (men and women) 75; Meetings 9; Average attendance 50. [Pages 229–231.]

The Mathematical Club, Harvard University, Cambridge, Mass.—Founded about 1898. Membership (men only). Meetings 12. Average attendance 20. [Pages 186–187, 449–450.]

The Mathematical Club of Smith College, Northampton, Mass.—Founded October, 1899. Membership (women only) 35. Meetings 11. Average attendance 24. [Pages 91, 455–456.]

Undergraduate Mathematics Club, University of Illinois, Urbana, Ill.—Founded December, 1899. Membership 36. Meetings 8. Average attendance 8–12. [Pages 404–405.]

Pi Mu Epsilon Fraternity, Syracuse University, Syracuse, N. Y.—Founded November, 1903. Membership (men and women) 44. Meetings 9. Average attendance 30. [Pages 271–273.]

Mathematics Club, The Western College for Women, Oxford, Ohio.—Founded 1905. Membership (women only) 28. Meetings 6. [Pages 231–232.]

The Junior Mathematical Club of the University of Chicago, Chicago, Ill.—Founded November, 1905. Membership (men and women) 15–20. Meetings 14. [Pages 34–35, 448–449.]

The Mathematical and Astronomical Club of Swarthmore College, Swarthmore, Pa.—Founded March, 1907. Membership 42. Meetings 16. Average attendance 30. [Page 135.]

Mathematics Club of Mount Holyoke College, South Hadley, Mass.—Founded November, 1907. Membership (women only) 53. Meetings 5. Average attendance 20. [Pages 312–313.]

The White Mathematics Club at the University of Kentucky, Lexington, Ky.—Founded November, 1908. Membership 7. Meetings 22. [Pages 90, 451–453.]

Barnard College Mathematics Club, Columbia University, New York.—Founded 1909. Membership (women only) 40. Meetings 6. Average attendance 25. [Pages 226–227.]

Mathematics Club of Iowa State Teachers College, Cedar Falls, Iowa—Founded December, 1909. Meetings 3. [Pages 311–312.]

The Mathematics Club of Hunter College, New York.—Founded 1910. Membership (women only). Meetings 8. [Pages 187–188.]

Mathematics Club of Columbia University, New York.—Founded November, 1910. Meetings 11. Average attendance 14. [Pages 227–228.]

The Mathematics Club of Albion College, Albion, Mich.—Founded January, 1911. Membership (men and women) 19. Meetings 15. [Pages 354–357.]

The Newtonian Society of the State College of Washington, Pullman, Wash.—Founded November, 1911. Membership (men and women) 14. Meetings 10. Average attendance 10. [Pages 410–411.]

The Mathematics Club of the University of Kansas, Lawrence, Kansas.—Founded December, 1911. Average attendance 19. Meetings 16. [Pages 35–36, 450–451.]

The Junior Mathematical Club, University of Wisconsin, Madison, Wis.—Founded March, 1912. Membership (men and women) 25. Meetings 7. Average attendance 18. [Pages 188–189, 457–458.]

The Mathematics Club of Goucher College, Baltimore, Md.—Founded November, 1913. Membership (women only) 22. Meetings 10. Average attendance 15. [Pages 357–358.]

Denison Mathematics Club, Denison University, Granville, Ohio.) Founded November, 1914. Meetings 11. Average attendance 25. [Pages 403–404.]

Junior Mathematics Club, University of Minnesota, Minneapolis, Minn.—Founded December, 1914. Membership (men and women) 20. Meetings 10. Average attendance 15. [Page 312.]

The Mathematics Club of Brown University, Providence, R. I.—Founded February, 1915. Membership (men and women) 60. Meetings 7. Average attendance 42. [Pages 33–34.]

The Mathematics Club, University of Colorado, Boulder, Colo.—Founded October, 1915. Membership (men and women) 41. Meetings 12. Average attendance 30. [Page 185.]

The Mathematical Club of the University of Nebraska, Lincoln, Neb.—Founded October, 1915. Membership about 60. Monthly meetings 7:30 to 9 P. M. Average attendance 35. [Pages 313–315.]

The Mathematics Club of Northwestern University, Evanston, Ill.—Founded January, 1916. Membership (men and women) 20. Meetings 10. [Pages 132–134, 409.]

The Mathematics Club of the University of Maine, Orono, Me.—Founded February, 1916. Membership (men and women) 18. Meetings 7. Average attendance 8. [Pages 132, 453–454.]

The Mathematics Club of Vassar College, Poughkeepsie, N. Y.—Founded January, 1916. Membership (women only) 37. Meetings 6. [Pages 136, 456–457.]

The Mathematics Club of the University of North Carolina, Chapel Hill, N. C.—

Founded October, 1916. Membership 25. Meetings 10. Average attendance 15–20. [Pages 90–91, 454–455.]

The Mathematics Club of the University of Oregon, Eugene, Oregon.—Founded October, 1916. Membership 30. Meetings 4. Average attendance 20. [Pages 134–135, 455.]

The Pentagram, University of Texas, Austin, Texas.—Founded October, 1916. Membership (men and women) about 30. Meetings 15. [Pages 273–276.]

The Mathematics and Physics Club of the University of Alabama, University, Ala.—Founded November, 1916. Meetings 4. Average attendance 12. [Page 226.]

The Mathematics Club of Greenville College, Greenville, Ill.—Founded September, 1916. Membership (men and women) 25. Meetings 6. Average attendance 20. [Pages 89–90.]

The University of Saskatchewan Mathematical Society, Saskatoon, Sask.—Founded November, 1916. Meetings 6. Average attendance (men and women) 14. [Pages 270–271.]

The Grinnell College Mathematics Club, Grinnell, Iowa.—Founded March, 1917. Membership (men and women). Meetings 6. [Page 449.]

Vinculum, University of Pennsylvania, Philadelphia, Pa.—Founded May, 1917. Membership (women only) 20. Meetings 5. [Page 455.]

Mathematical Club of Rockford College, Rockford, Ill.—Founded October, 1917. Membership (women only) 24. Meetings 9. [Pages 188, 409.]

The Mathematics Club of Connecticut College, New London, Conn.—Founded December, 1917. Membership (women only) 7. Meetings 6. [Page 270.]

Mathematics Club of the University of Montana, Missoula, Montana.—Founded March, 1918. Membership (men and women) 22. Meetings 6. Average attendance 18. [Pages 408–409.]

It may be remarked that more than half of the Clubs were founded within the past four years: that nine were organized in 1916, four in 1917, and one in 1918.

Nearly one quarter of the clubs are for women only: Connecticut, Goucher Hunter, Mt. Holyoke, Pennsylvania, Rockford, Smith, Vassar, and Western.

The number of members in the various clubs varies from less than 10 in such places as Connecticut and Kentucky, to more than 50 at such places as Mt. Holyoke, Brown, Nebraska and Toronto. The number of meetings has varied from 3–5 (for example at Iowa State, Alabama, and Pennsylvania) to 15–22 (for example at Albion, Swarthmore, and Kentucky).

1918—SUMMARY NOTES.

During the year an attempt has been made clearly to indicate the ideals of (1) every American undergraduate mathematical club which held meetings during 1917–18, as well as of (2) the following three clubs which deemed it wise to suspend operation for that year:

The Euclidean Circle of Indiana University, Bloomington, Ind.—Founded September, 1907. [Page 228.]

The Mathematical Club of the Kansas State Agricultural College, Manhattan, Kansas.—Founded September, 1913. [Pages 405–408.]

The Mathematics Club of the University of Oklahoma, Norman, Okla.—Founded October, 1916. [Pages 315–316.]

To this end, in so far as information was procurable each club's organization has been described, and its programs for one or more years have been published.

It is hoped that this material has served, and will serve for some time to come, as a fruitful source of suggestion as to possible methods of club organization and conduct, and as to ideals which may be successfully tried out. A sample constitution (at Oklahoma) has been published, and it has been observed that most clubs favor formal organization with student officers, and a member of the faculty serving in advisory capacity on either the program or executive committee. At such places as Barnard, Saskatchewan, and Toronto, an honorary president is chosen by the students from the faculty, while all other officers are students. Six clubs have student officers only; Iowa State, Kentucky, North Carolina and Syracuse favor faculty control of practically all offices; informality in organizations is characteristic of Brown, Colorado, Goucher and Western. Just what type of control should be maintained by the faculty must often be determined by local conditions.

The facsimile of the certificate of membership at Texas given on page 275, the printed programs of Greenville, Kansas, Kansas State, and Toronto, and the club photographs arranged for at Brown, Greenville, and Indiana, will serve as interesting suggestions.

The address in our October issue, by an experienced teacher who has, for many years, met with great success in his conduct of clubs, sets forth the extreme value of the club and the great importance of considerations involved in the preparation of its programs. During the year more than 650 published programs must have offered helpful suggestions in this connection and made clear to many readers how vital a factor the club may be made in the mathematical, intellectual, and social life of a college.

The Albion program scheme which allots some task to every member at frequent intervals, such plans as those at Goucher, Kansas State, Kentucky, and Wisconsin, for working up portions of the history of mathematics in connection with club programs, and the very general introduction into club programs of biographies of mathematicians, are also noteworthy.

While titles of appropriate program topics are suggestive there are few clubs which do not feel the need of indications of possibilities of the topics, and of exact references to the literature. It was with this idea in view that a beginning was made in meeting this need and the following subjects were treated during the year: 1. The oldest mathematical work extant; 2. Geometrography and other methods of measurement of geometrical constructions; 3. Arithmetical prodigies; 4. Ptolemy's theorem and formulæ of trigonometry; 5. Paper folding; 6. Women as mathematicians and astronomers; 7. The binary scale of notation, a Russian peasant method of multiplication, the game of mim, and Cardan's rings; 8. The

logarithmic spiral; 9. Golden section; 10. A Fibonacci series; 11. Euler integrals and Euler's spiral—sometimes called Fresnel integrals, and the clothoïde, or Cornu's spiral; 12. Geometry of four dimensions; 13. Constructions with a double-edged ruler; 14. The cattle problem of Archimedes.

With one exception (because calculus is involved) all of these topics are suitable for programs of every one of the clubs mentioned above. In most cases far more was suggested than was regarded as possible for consideration in a single evening. The bibliographies were purposely made pretty full, partly because the synopses were contributions to mathematical history, and partly that the resources of each library might be laid under contribution when possible. It was not intended that anything of real importance and interest should have been overlooked; nevertheless it was not noted till too late that (1) Campanus, in the thirteenth century, proved the irrationality of golden section and that his argument (by mathematical induction) was reproduced by Genocchi and Cantor;[1] that (2) discussion connected with Fibonacci's series occurs in the writings of Daniel Bernorilli as early as 1728 and of Euler by 1726 (facts to which Mr. G. Eneström of Stockholm has kindly drawn my attention); and that (3) Euler's spiral has played an interesting and important rôle in connection with the history of railroad curves. The literature of this subject has been surveyed by the editor and it is possible that at a later date the results of his inquiry may be thrown into the form of a club topic.

* * *

The following passage of a personal letter to the editor, from a distinguished professor in a Scottish University, will be of general interest.

"But what I want most to write about today is to thank you for, and to offer you a word of appreciation of, your bulletin of the 'Undergraduate Clubs.' We have nothing like this over here, and the whole thing strikes me as admirable. It seems to me that it might with immense advantage be extended (as perhaps it already is on your side) to other sciences as well as to mathematics. Our better students either drop their work altogether on graduation, or either insist on attempting, or perhaps are tempted (for instance by the Carnegie Scholarships) to attempt, 'original investigation' before they are fit for it. There is no temptation and little opportunity to prolong their own studies, to engage in wider reading or to make acquaintance with the historical aspect of their science. Often a comparatively raw youth or girl comes to me and wants to 'do original research.' I say 'Why, you have read nothing but a text-book or two; you have read nothing worth speaking of. Why not read for a year or two; make yourself master of what has been done in this field or that, and widen your horizon. Epitomize the literature of some theme, or of some historical period.' But the reply is always the same. *'I want to do "original research"*; and the Carnegie

[1] *Annali di scienze matematiche e fisiche* (Tortolini), tomo 6, pp. 307–308; Cantor's *Vorlesungen über Geschichte der Mathematik*, Band 2, 2. Auflage, 1900, p. 105. See also this MONTHLY, 1918, page 197.

Trust will give me a good scholarship for doing so, and for nothing else in the world.' Now your plan seems to me to precisely meet the case. You encourage your young people to do *work*—not necessarily 'original work'—which is more than any reasonable man can expect of them; but at least work which involves just so much originality or at least independence as can fairly be expected. And it is work by which they learn something; while heaps of so-called original work, as I see it done, teaches nothing, for it is too often confined to some tiny problem, and only means watching the spot of a galvanometer, or making endless and all but identical titrations or measurements."